An Introduction to Smooth Manifolds

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Lecture 19

Derivative of Inclusion Map

So, hello and welcome to the 19th lecture in the series and last time I had to stop somewhat abruptly.

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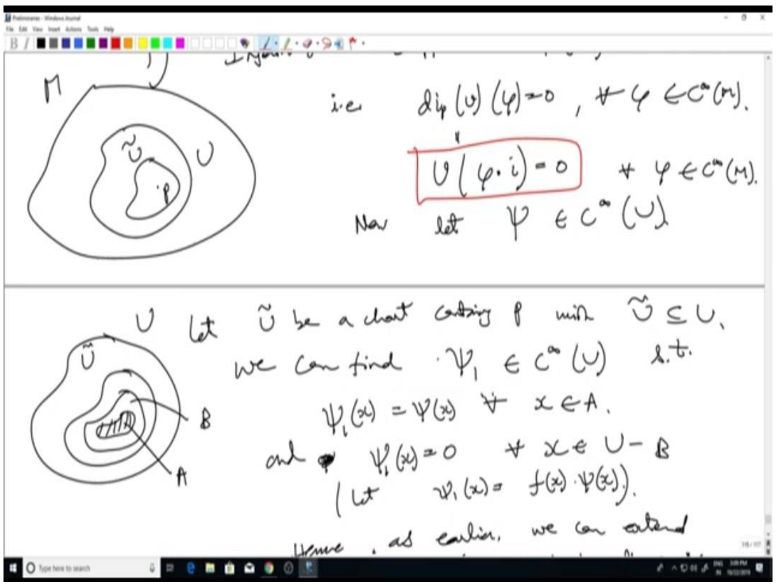

Let me resume from that, precisely that point in the argument. So, what we were trying to prove was that well, that if you have an open set then the derivative of the inclusion map is an isomorphism and we are in the process of proving that this map is injective. So, I have to prove to the kernel of this linear map is 0 and start with the vectors at the d i p v 0. This by definition means that for all (())(01:16). So, if I take a c infinity function in m then this v of e compose with i is 0.

So, as I said earlier this amounts to saying that the action of this vector. So, we want to prove that this vector v 0, this derivation v 0. So, that would amount to saying that v of any c infinity function on u should be 0 but what we have here is not quite that we have that v of the restriction of a c infinity function on m to u is 0. So, when I start with v in c infinity m and restrict it to u and then act by v I should get 0.

So, but what I want is, I should start with an arbitrary c infinity function in u and I would like to claim that v of c is 0. So, to do this you will use this small lemma that we have proved earlier. So, again, I go back to the previous in the proof of the lemma had this u tilda a and b, I will use the same notation as in the previous lecture we can then…

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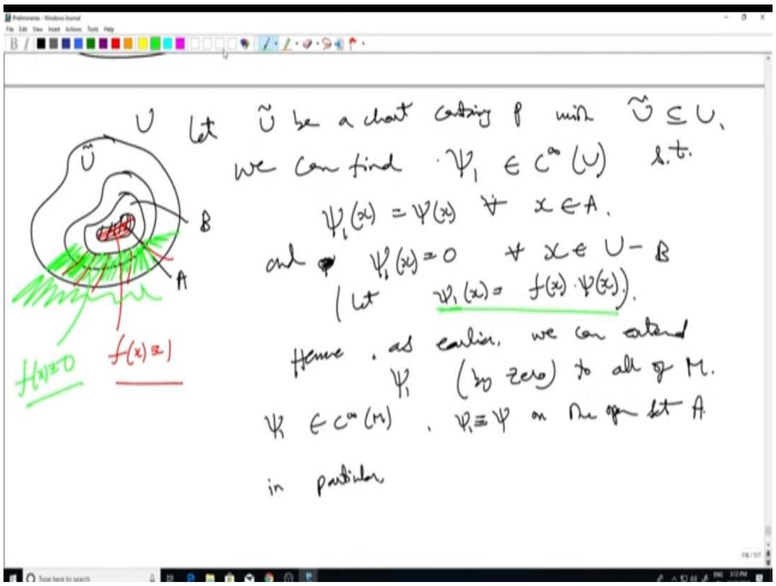

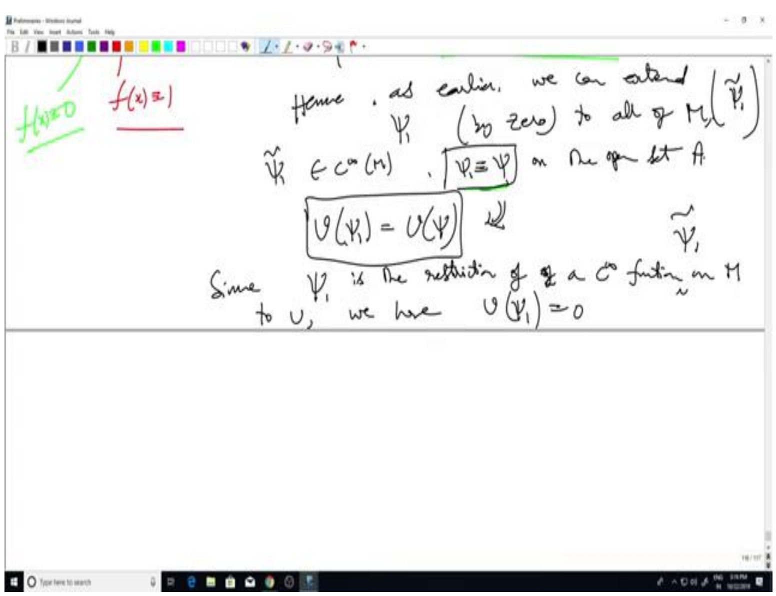

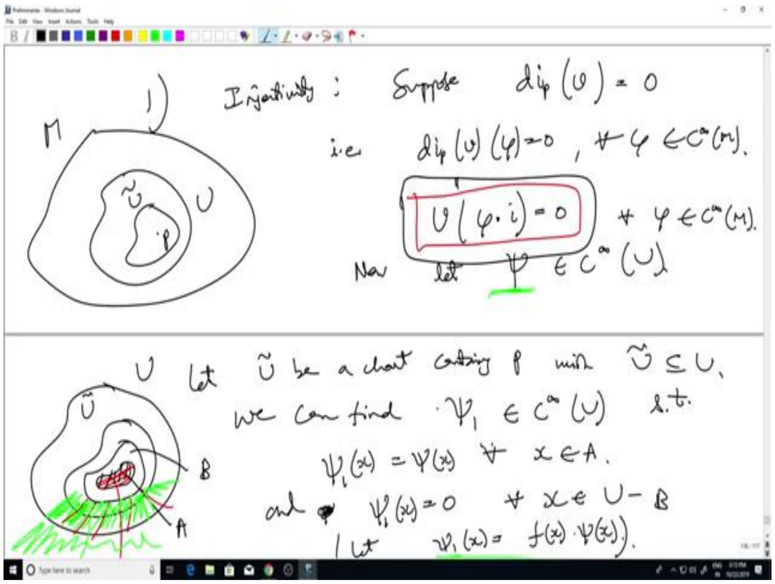

Now, what I will do is, I will just take the c of x which is defined on all of u and then I multiply it by this cutoff function f of x as f of x as defined earlier. Recall that f of x was a c infinity function on all of m such that f of x was identically on this portion f of x is identically 1 and outside that on this green portion outside b inside a, it is 1 and outside b f of x is identically 0. I will take such a function and multiply it with the function c that I am starting with here and I will call this psi 1.

Well, what is the main property we want that psi 1 is c of x. If x is in a because f of x is, as I said here, f of x is 1 and if I am outside d then f of x is 0. So, therefore, the psi 1 would be 0 as well and to begin with since this c was only defined on u, the psi 1 is also only defined on u. However, it is going to be 0 outside b anyway.

So, if I extend psi 1to all of m by declaring it to be 0 everywhere psi 1 is psi infinity function on m and psi 1 equal to psi on the open set a. How is this psi 1 going to help us? Well, since psi 1 and c these 2 agree on the open set a, we know that v of psi 1 equal to v of psi by that lemma. So, this thing here implies this.

Also, since psi 1 is the restriction of a c infinity function on m to u, we have v of psi 1 is equal to 0 by a hypothesis. That is what we started with. This thing here tells us that the restriction is 0.

Psi 1 has globally defined on all of m and when we restrict it to, actually to be clear one should give it a different name. We can extend psi 1 by 0 to all of m and let us call the extension as psi

1, maybe put a tilda. Psi 1 tilda of x is 0, if x does not belong to u anyway, and inside u would agrees so it is psi 1.

So, psi 1 tilda belongs to c infinity m and psi 1 tilda is identically equal to c on the open set a. These two agree on the open set. Here I do not need a tilda. So, this extension property is used only in this last step. The fact that when I want to claim that v of psi 1 tilda is 0. For this part I need this.

So then and here I actually should use. So, just to claim that this psi 1 equal to that this to get this, I do not really need the extension to all of m, just on u is good enough but in the last part I noticed that psi 1 is the restriction of a c infinity function. So, what I called psi 1 tilda on m, we have v of psi 1 is 0 and that is it. So, that proves injectivity.

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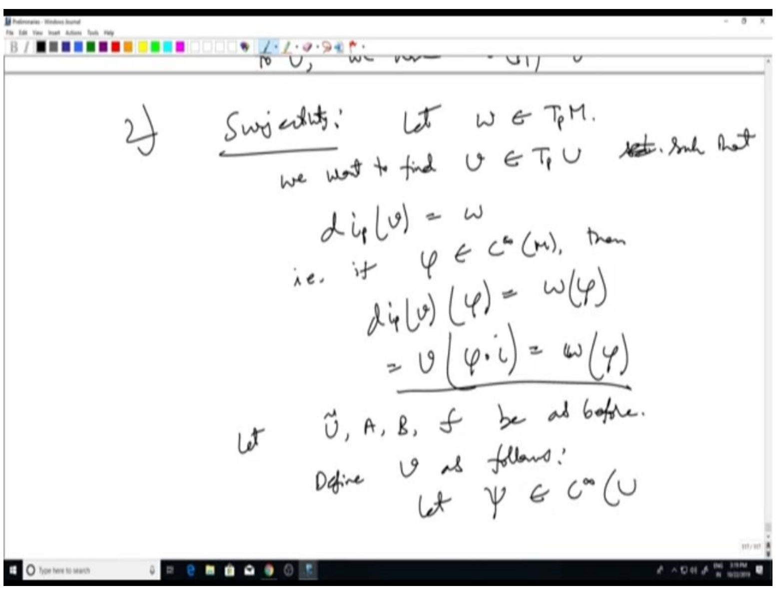

Now, as per surjectivity one would like to say that so let us start with let w belong to t p m. We want to find v in t p u such that d i p of v equal to w. In other words, again what this really means is that i.e start if v is in c infinity m then d i p of v acting on phi equal to w of phi. And this we know is the same as v of as usual so I will just take the phi some m and then phi of i.

So, what I want is, I want to, so this is the equation which should be satisfied by this vector v that I am looking for. Well, the natural thing to do is just use w itself and the point is that w acts on c infinity functions on, defined on all of m. While a vector v d p u can act on a month different class of functions. Those functions which are just defined on u, not necessarily on the all of m.

So, but still I can use w to define a v as follows. So, we again do the same old trick of this. Let us work with this a b and this f. This cutoff function f which is identically 1 in a neighborhood of p and 0 outside this b.

So, do the same thing. Let u tilda a b f be as before. Define v as follows. In fact this equation that I have here suggest that what the definition of v should be. So, let, so the point is v should act on the c infinity function which is just defined on u, not necessarily all of m.

So, let start with the c. C belong to c infinity u and I am going to use w to define the action of v. Well as directly I cannot let w act on c because it is defined only on u not on all of m. However, we know that all that really matters is to see the action of a derivation we just need to know what it does close to in an open neighborhood of the point, not necessarily in the big open set u.

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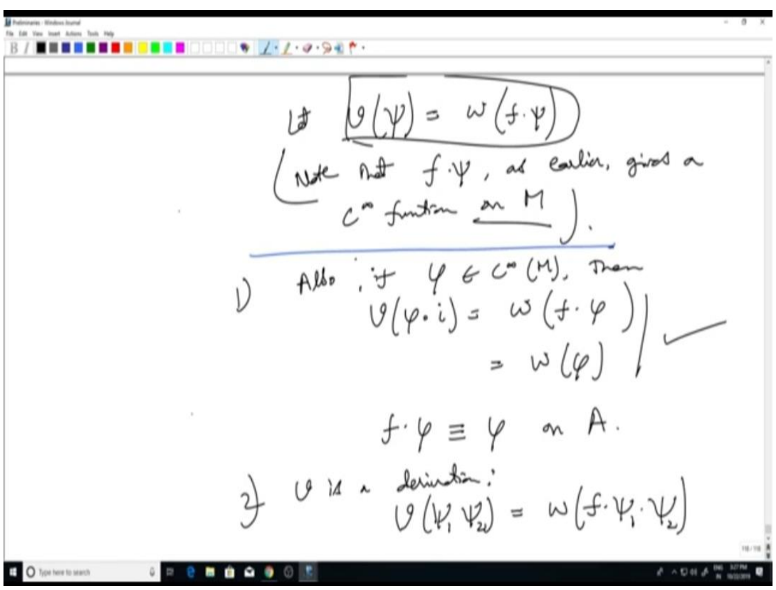

And so what I do is let v of c I just extend the c to all of I starting with c, I get a c infinity function on all of m. I really do not care whether that new function agrees with this old c on u. It is good enough that they agree on this much smaller open set a. Some open set containing p is good enough.

So, I just define it to be w acting on f composed with f multiplied by c. Note that f composed with c as before as earlier gives up c infinity function on m. Again extension by 0 will give us infinity function and moreover inside the open set a f multiplied by c is equal to c. So, and I define something like this. This let v of c is equal to this.

Then of course, I defined something but whether it is v I want to check that this equation holds. After all, that is the meaning of a defined something. So, let me check that the equation holds. That equation says that v of also v of so to check that this equation holds I have to start with some c infinity function in m.

Also if v belongs to c infinity m, then this now that I have a v, v of so what I have here is v compose with i. In other words, the restriction of phi to u. So, this according, now I have to use the definition of v. So, this is supposed to be w of f multiplied by phi composed with i. That is going by this here and well, what are we supposed to check and this is yeah, so this is supposed to be question is, is this equal to w of phi itself.

Well, actually not quite this. So, here I should change it a bit. So, here I started with psi. So, here the i does not occur anymore. I mean there is some slight notational change because the point is that the psi is anyway defined only on but a question of putting an i occurs only when you have a globally defined function.

So, let us leave it like this. Well, so the only question becomes whether w of f multiplied by phi equal to w of phi and the answer is yes. Again this equation holds because f multiplied by phi is identically equal to phi on a. Both are c infinity functions on m and they agree on A. Therefore, the action of w on that is the same. Again by that lemma.

So, this is the equation is satisfied and one more thing is that of course one has to check that this is actually a derivation. The way I have defined it that v of c. Let this so that one has to check that this is indeed a derivation and that not difficult to check so let me quickly do that. V is a derivation. So, what do I have to check? I have to start with v of psi 1 psi 2 that is supposed to be equal to w of f psi 1 psi 2.

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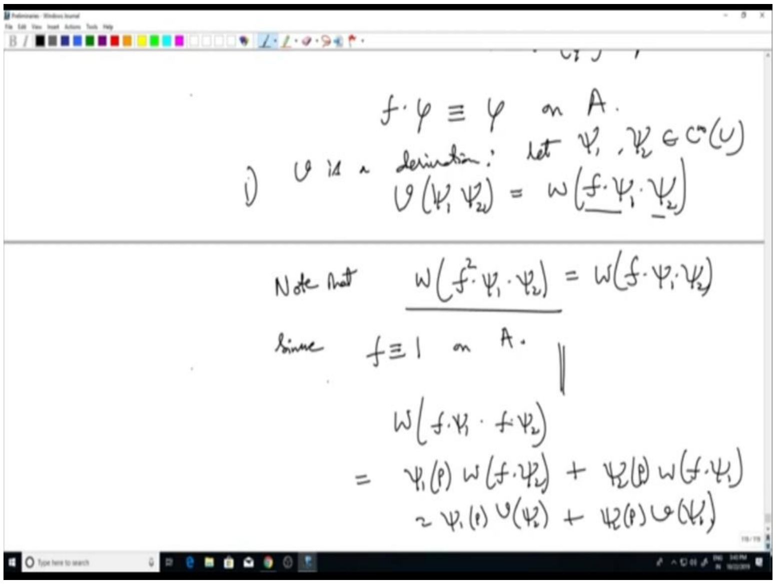

So, let us check that v is a derivation by definition v. So, if I start with 2 c infinity functions. Let psi 1 psi 2 belong to c infinity u. By definition I have this. Now, what I notice is that note that instead of having a single f suppose I have f square. W of f square psi 1 psi 2 is the same as w psi 1 psi 2.

Since f equal to f is identically 1 the open set A. So, in other words these two functions f squared multiplied by psi 1 times psi 2 this is c infinity function on m and so is this. On A both of them agree. Both f and f squared are 1. So, these two agree and therefore w is the value of w on acting on that is the same.

But writing it in this form makes it easier to check the derivation property so then I can write it as w times psi 1, f times w of f times psi 1 w multiplied by use the other f and use the derivation property of w.

So, this will give v f at the point p is 1. So, psi 1 p w pf psi f 2 plus psi 2 p w of psi 1. This is psi 1 p. By definition this is v of psi 2 and v of psi 1. So, that completes the proof in all detail. So, we have checked that. So, along the way we have also proved various other things. So, let us just note them down.

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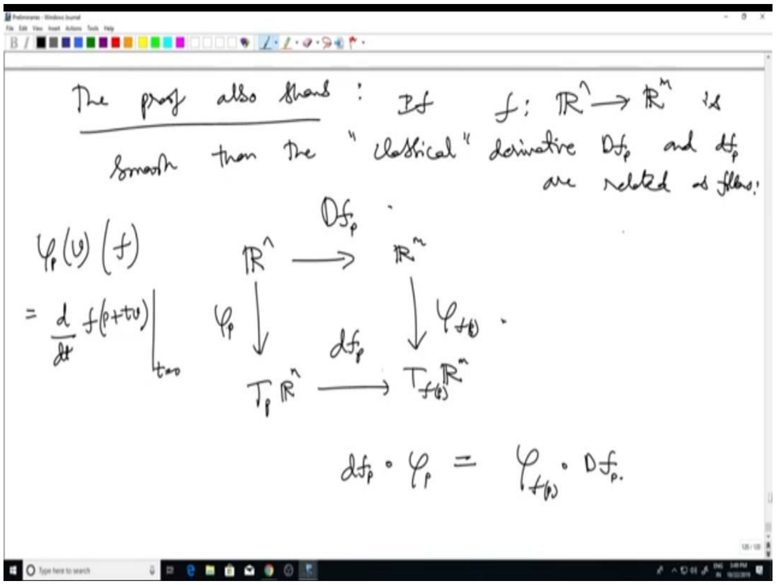

So, the proof also shows various things. Well one thing is that this if f is a smooth function, smooth, then I would like to say that the derivative of f as we have defined it in the abstract sense should in some sense coincide with the derivative of f that we started the course with, the classical sense. So, but to make sense of this statement, we have to observe that the classical derivative.

So, let us say then the “classical” derivative, I put it in quotation marks. So, this classical derivative is let us denote it by d f at the point p. This is a map from r n to r m again. The classical derivative and the abstract derivative, abstract derivative is not the right word, classical derivative and so let us just use the notation d f.

So, classical derivative d f p and this current notion of derivative small d f p related as follows. So, the classical derivative is a map from r n to r m. Well, the new derivative is a map from t p r n to t f of p r m but we know that this, there is an isomorphism here and likewise here so what the statement one wants to make is that this diagram is commutative.

In other words, if I call this isomorphism as phi p and this one as phi f of p, let me stick to phi p. Phi p phi f of p. So, if I have in this diagram it is commutative. In other words, d f of p, moreover, I need some arrows in specific directions.

So, this is let us phi p go in this way, so then d f of p composed with phi p is equal to, well, you do this first. This is just another way of saying that if we identify the tangent space to r n with r n itself while this natural isomorphisms that we have then this the new derivative coincides with the old derivative.

For this to of course make sense one need to it is not enough that t p r n is an n dimensional vector space. The fact that this derivative and this derivative are the same holds when we use specific isomorphisms from here to here. So, this isomorphism we know what it is. Phi p if I start with an element of r n v that should give me a derivation of r n and that derivation is so it should act on a function and this is the usual direction derivative.

D by d t of f of p plus t v at t equals 0. This is the likewise for this as well. So, we know we have proved that this is an isomorphism, phi p is an isomorphism. Likewise phi f of p and with under these isomorphisms the old derivative and new derivative coincide.

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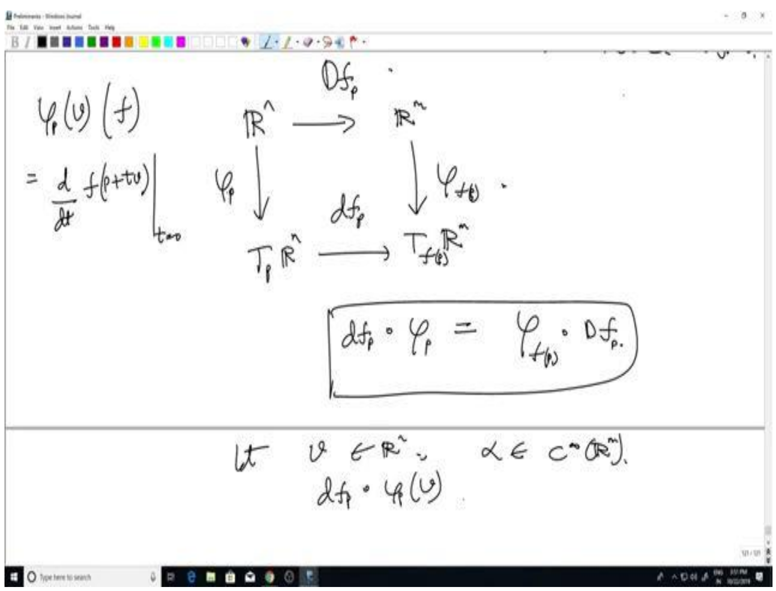

And how does the proof show this? Well, it is just a matter of the working through the, so for instance here, suppose, I want to check this equation, I would start with an element of r n. So, let v belong to r n, then I want to check if d p phi p of v, I want to this is the left hand side of this equation here and I want see what it is. Well, the thing is that after all this final output phi p is going to land here and d f p of that is going to land here. So, it is a derivation on c infinity r n so I should act it on a function on r m. So, let alpha belong to c infinity r m.

So, then it is matter of plugging in here and checking. So, I will do this next time. Again I have to stop abruptly in the middle of my proof. So, thanks for listening. We will continue from this point and then I will give some more examples of these tangent spaces. Thank you.