cAn Introduction to Smooth Manifolds
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Lecture 15
Derivative of Smooth Maps

Hello and welcome to the 15th lecture in the series. So, in last class we talked about interpreting vectors in Rn as derivations on C infinity functions. Let us see how this can be used to define the tangent space of a manifold.

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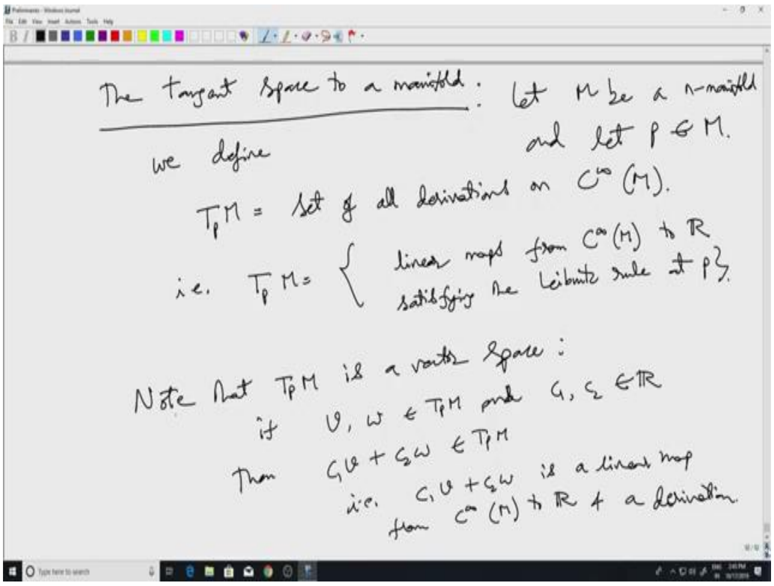

So, we begin with so let me state the, give the definition straight away. But it will not be clear in the beginning that we will get a finite dimensional vector space, so you have to do a bit more work. The tangent space to a manifold. So let M be a n manifold and let P be a point on M. We define TPM, the tangent space to M at the point P to be the set equal to set of all derivations on C infinity M i.e. TPM equal to set of all T, set of all linear maps from C infinity M to R satisfying the Leibnitz rule at P. Note that TPM a vector space.

In other words, if I have, if T1 rather let us see a different notation for elements of TPM. So if v and w belong to TPM then and C1, C2 belongs a real numbers then C1 v plus C2 w is also an element of TPM i.e. C1 v plus C2 w is a linear map from C infinity M to R and a derivation. This is immediate from the defining property of a derivation after all. What do we have to check? The linearity is clear enough. If you have two linear maps, then their linear combination is also a linear map. And checking that it is a derivation is not any harder.

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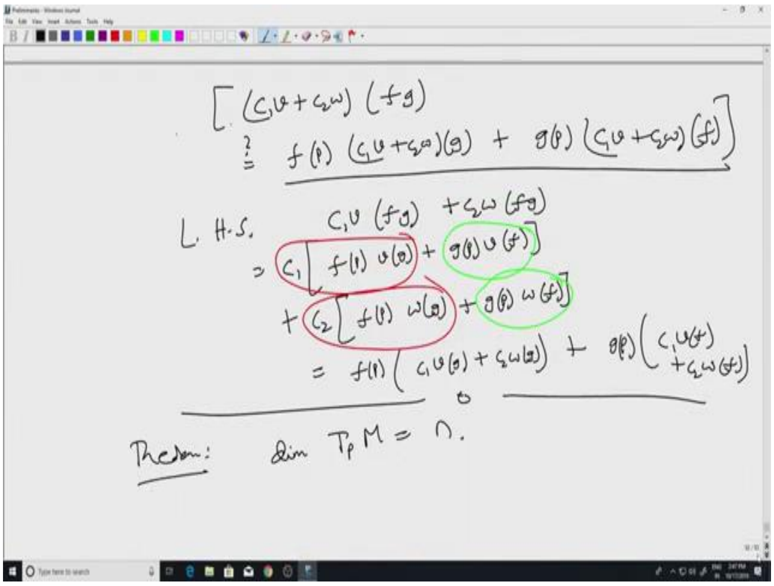

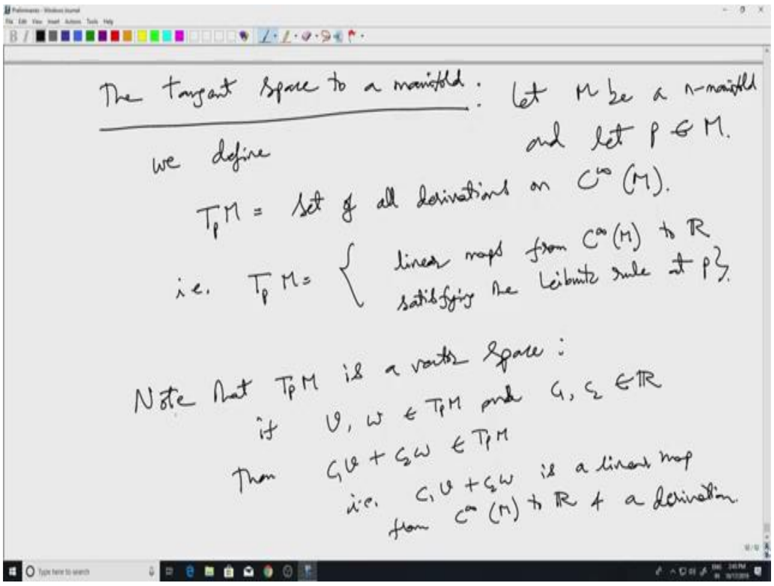

So just a matter of saying that C1 v plus C2 w suppose I want to check it is a derivation. All I have to do is act it on a product of two functions fg. Then I want to know whether this is equal to C1 v plus C2 w. This is equal to f of P times C1 v plus C2 w of g plus g of P, C1 v plus C2 w of f. And the way to see it is just start with the left hand side C1 v plus C2 w acting on fg is, by definition it is equal to C1 v acting on fg plus C2 w acting on fg. Now just use the derivation property for v and w. So I will get C1 times f of P, v of g plus g of P, oops so I should change it slightly g of P and then I have f of v.

Similarly, the second term is f of P W of g plus g of P w of f. Then I just combine the corresponding terms. So if I combine the, so I will combine this and this and I will combine this and this along with the C1 and C2. So I will end up getting f of P times C1 v of g plus C2

W of g plus g of P times C1 v of f plus C2 W of f which is essentially what I want here. So this is what I wanted and that is what I have here. So it is quite straightforward to check that it is indeed a derivation. So the point is that TPM was a vector space. However, it is one has to still so what one would expect is that it is actually just again an n dimensional vector space since we are dealing with the n manifold and that is the theorem. So, let state it as a theorem that it has dimension n.

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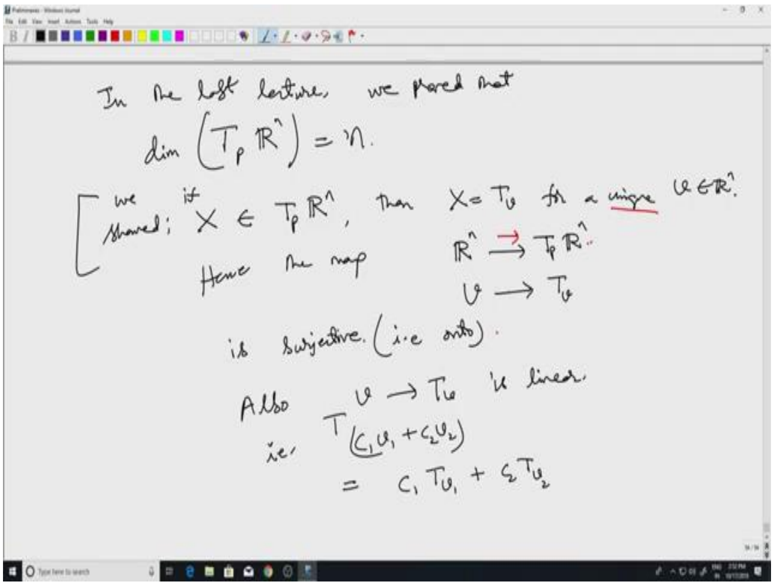

Now, what we have proved is that in the last lecture what we proved the last lecture we proved that TP, Rn when the manifold is Rn itself. I did not stated in these terms. But what I did was in the last lecture I showed that if we start with any derivation v on Rn, with the our new notation, a derivation on Rn would be an element of TP Rn. So that what I am starting with, we showed if v belongs to this, actually the notation is becoming a bit inconsistent.

So let instead of v, let us say as we showed that, if x belongs to this, then x equals Tv for a unique v in Rn so, this gives us, this enables us to set up an isomorphism between Rn and TP Rn, hence the map from Rn to Tp Rn given by v going to Tv is this map is subjective i.e. onto. So, since the previous statement says that if you start with any derivation here in the space so an element of the space is a derivation, if I start with in derivation it can be written as Tv for some v. So, that means precisely that this map here is onto, this map is onto. And not only that, it is we also seen that this map is injective that follows from the uniqueness of this.

So, two different vectors cannot go to the same derivation. So, because we have seen that, if Tv equals Tw, then v has to be equal to w. So, in other words, this map is a bijection but not only is it a bijection this map is also linear also v going to Tv is linear and this is i.e., what does this amount to C v1 plus C v2 C2 v2 the directional derivative along this linear combination is equal to C1 T v1 plus C2 T v2. And this is again immediately clear from the definition of the directional derivative. This is not so clear from the, rather than using the definition.

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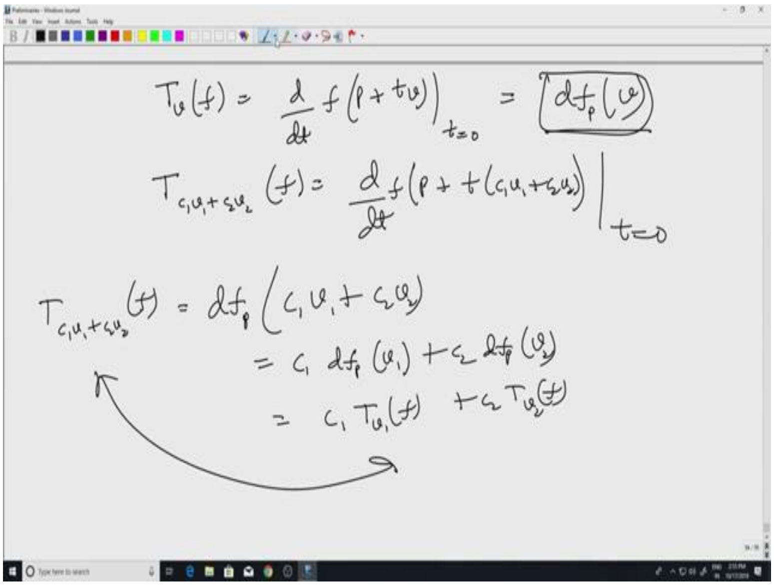

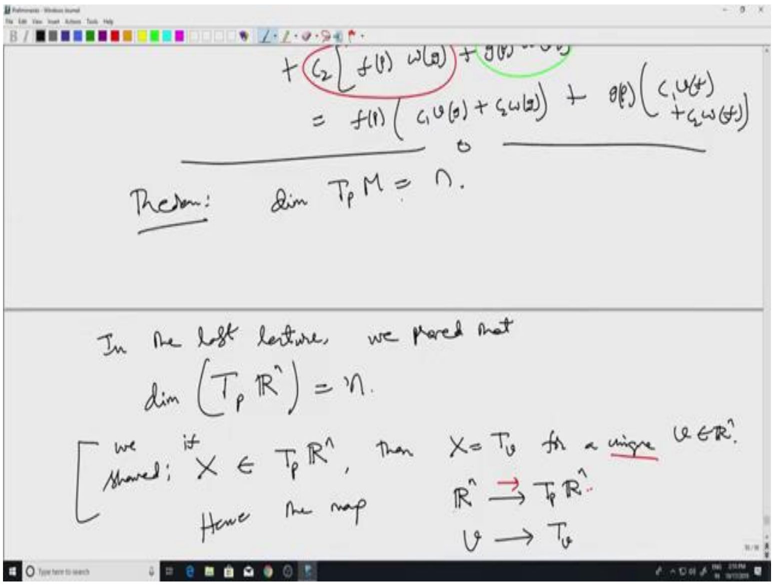

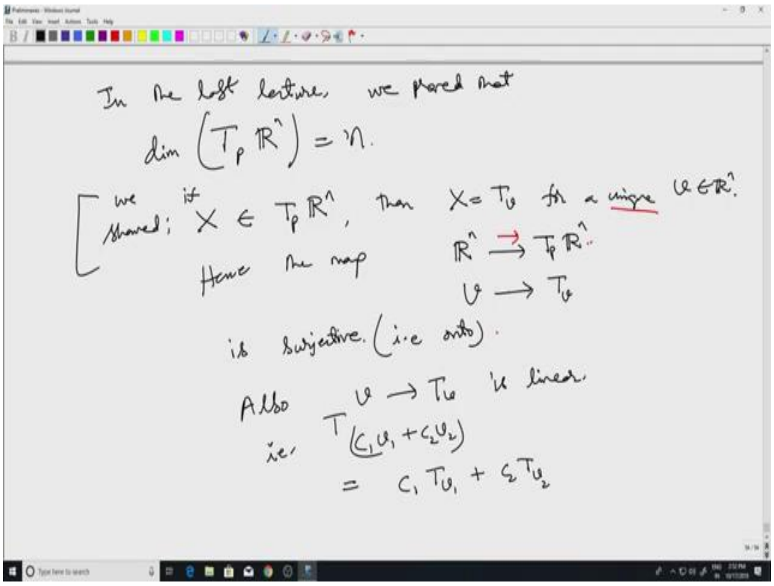

If you use the definition, then Tv f is d by dt of f of P plus Tv at t equals to 0. But if you use this, it is not immediately clear that if I put C1 v1, f of P plus t times C1 v1 plus C2 v2, this is what I would get. But then it is not immediately clear that this I can write it as something involving I can bring out the C1 outside etcetera. Rather than use this I would, it is easier to see if I (use), we know that this is the same thing as dfP of v if I take this, it is immediately clear because after all dfP is a linear map therefore when I put this, then I just get this and, one would be done. This is the same thing as C1 Tv1 of f C2 Tv2 of f this was TC1 v 1 plus C2 v 2 of f. So, using this expression here makes the linear dependence on v immediately clear.

While if I use the definition it is not immediately clear one has to go one or two steps ahead. Now so, to summarize, we are we want to prove that the dimension of the space of derivations on C Infinity functions on a manifold is n. And what we have proved in the last lecture is that the space of derivations on Rn or other on C infinity Rn has dimension n. So, how does one use this fact about Rn to get information about the whole manifold? So, there is a key missing ingredient here, which I will now talk about. Yeah, right. This is the so called local property of derivations, but before I come to that, perhaps even without knowing that this, this is dimension of TPM is n or even finite dimensional, I can still bring in the notion of a derivative of a smooth map.

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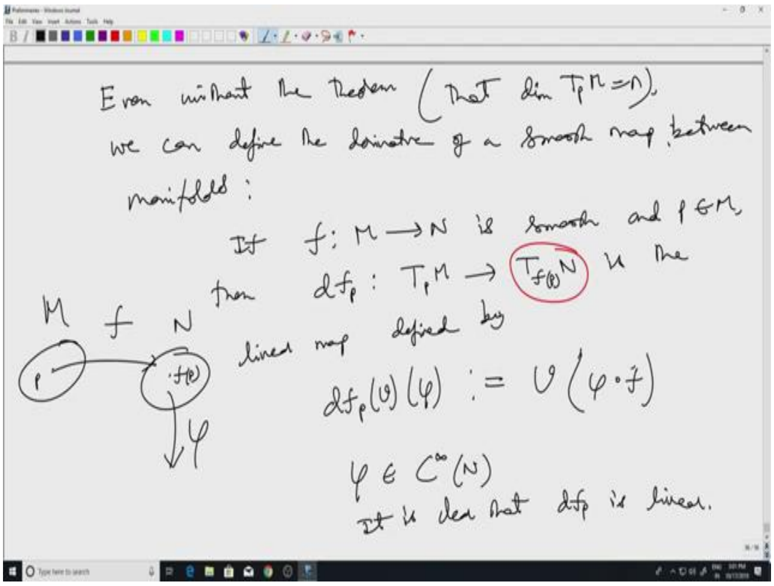

So, even without the theorem that dimension of TPM is equal to n, we can define the derivative of a smooth map between manifolds. In the (case), then when we started this course, we started by defining the derivative of a smooth map between open sets and Euclidean spaces is a linear map between Rn and Rm. But now what we are going to do is we are going to replace the Rn and Rm by this notion of a tangent space. So for us if f from M to N is smooth and p belongs to M then dfP is going to be a map from the tangent space to M at P to the tangent space at f of P. And this is the linear map defined by.

So, dfP at, the input is a vector in element of TPM. However, an element of TPM itself is a certain linear map on a different vector space namely C infinity functions on M. So, now I am supposed to get dfP of v is supposed to be an element of this space. So in other words, it is going to be a linear map on C infinity n, which is a derivation at f of P. So it has to act on some C infinity function on n. So let us call that phi, phi. So, where, here phi is, so the picture is like this M N f is a map, P is a point here f of P is a point here. Phi is a C infinity function on N, and dfP of v is going to act on that.

And my definition of this is going to be just I will use the v. So V, let us think derivation on this side. I am going to compose f and phi. So I will get a function C infinity function on M. Phi composed with f. So, this is my definition of the derivative. And one thing is certainly clear immediately from the definition that this association this is linear in v. It is clear that the dfP is linear.

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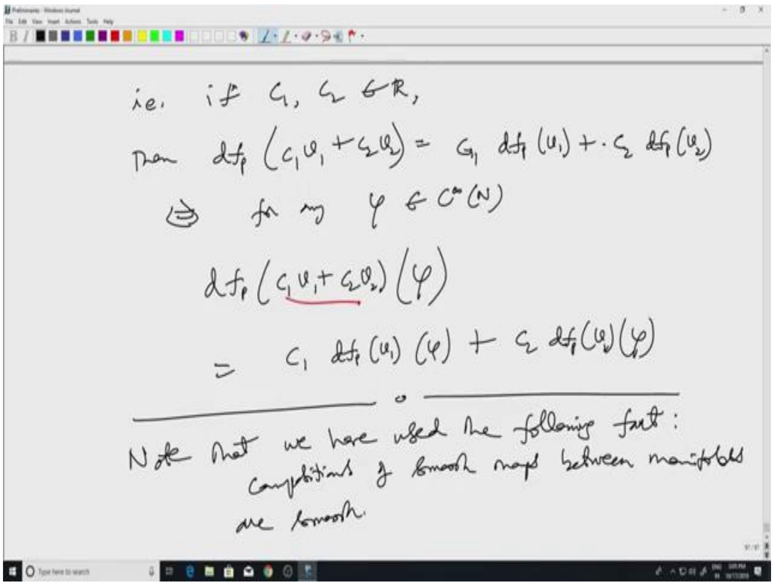

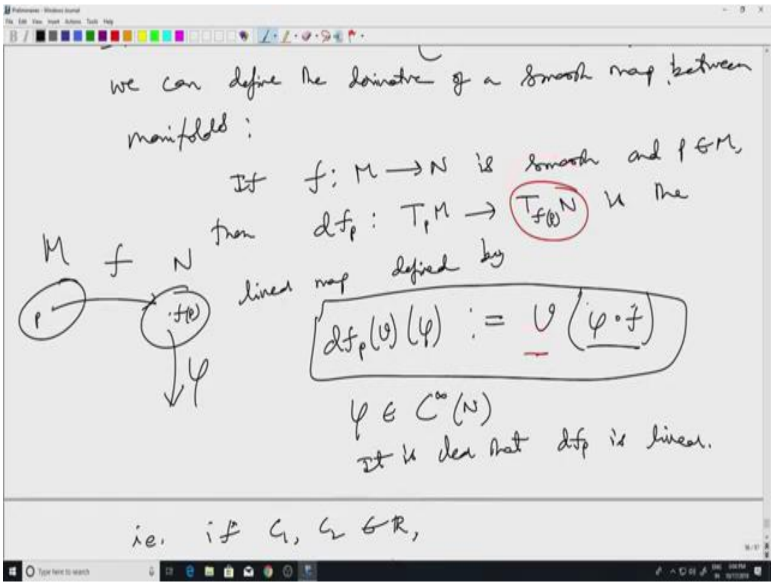

So again what does it mean to say that this is linear i.e., if C1 C2 belong to R then dfP of C1 v 1 plus C2 v2 where C1 dfP v1 plus, oops so, I do not need a bracket here plus C2 dfP v2 and what in turn I mean this is still this is clear enough, but what this really means is that this, that this equal to this amounts to saying that for any phi in C infinity N, the left hand side acting on this phi is equal to right hand side acting on phi. So this entire thing acting on phi should be equal to C1 df of P v1 acting on phi plus C2 df of P phi 2 acting on phi, at this point everything is just a number, the left hand side is a real number right hand side is a real number.

So, at this point, we just go back to this equation here and then everything becomes immediately clear because after all the left hand side is C1 v 1 plus C2 v2 so, I would end up instead of this v I would end up plugging in this thing here and so, the action of C1 v1 plus C2 v2 on this function and then just split it up and so on. So, that is it, but there is one small issue in what in this definition, namely that I implicitly used a certain fact. So, I had a smooth function phi from N to R, and f was a smooth function from M to N. I took the composition and got a function from M to R, which is this phi composed with f. But in order to act v on that, I need to know that this is still this composition is smooth.

In other words, note that we have used the following fact which I did not prove or even state because I will need to state it again when I talk about chain rule. So I will return to this, but let me just mention it here compositions of smooth maps between manifolds are smooth. So that is this phi composed with f, that I used to say that that is smooth it is, I have used this fact. But given that, it is clear that so we do have a notion of a derivative. It is defined in a very abstract but abstract way, from the way it is defined here, it is not, at first sight it is not clear where derivatives are involved at all.

Of course, if one keep remembers that vZ derivation. And we know that derivations of somehow are essentially given by derivatives, then it is clear that and I will be differentiating this phi as well phi and f both. So if one keeps the recalls that then it is clear that derivatives of phi will come into play. But that is in a somewhat vague sense, but more precisely, one would like to claim that this notion of derivative is equal interval earlier notion of derivative when we go back to Rn.

This notion the point is that this note notion of derivative or differential of a smooth map makes sense on a manifold while earlier notion of derivative of course is which involved taking ratios and norms of things and so on, so that absolutely does not make sense. This abstract definition makes sense, but we want to check that it coincides with earlier definition. So I will talk about that in the next lecture. So we will stop here. Thank you.