**An Introduction to Smooth Manifolds**
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**Lecture 13**
**Tangent Spaces**

Hello and welcome to the 13th lecture in the series. And we were discussing some examples of smooth maps between manifolds and I was dealing with a case of product manifolds towards the end of last lecture. So, let me resume that.

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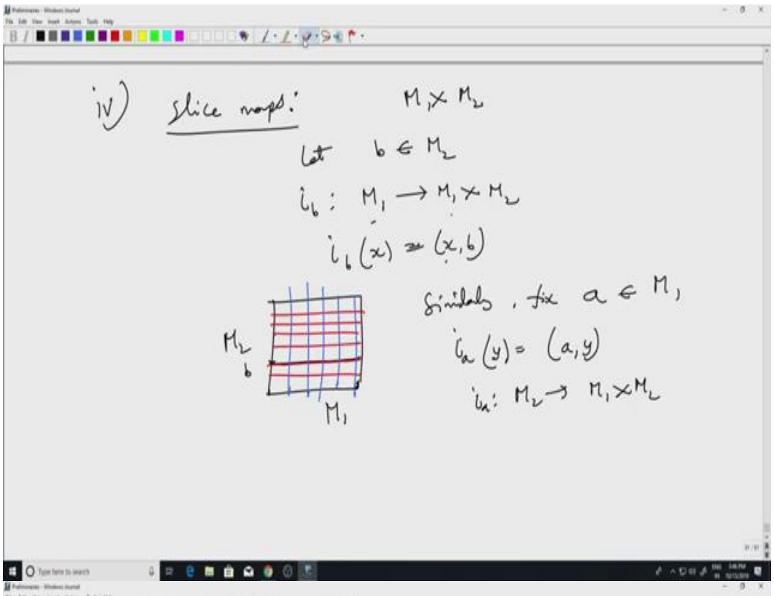


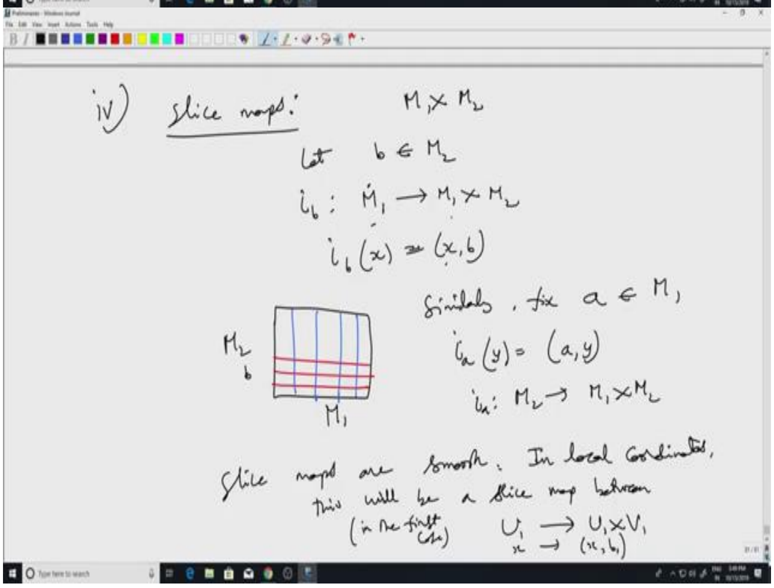


So, I have a product manifold M1cross M2, and let us fix a point b in M2, corresponding to that b run map of M1 into M1cross M2. Namely, I just take x and put it the y coordinate, the second coordinate to be b. So, schematically, what are we doing? Well, let us say this is M1, this is M. So, if I take a point b, for every b in M2, I have a copy of M1, so I have a copy of M1 sitting in M1 cross M2, so this is, the second coordinate is fixed to be b, and the first coordinate varies over, this x varies over all points of M1.

So, the picture sort of illustrates why this is called as slice. All these are slices, horizontal slices. Similarly, I have vertical slices as well. Similarly, fix a in M1 and define ia of y to be a comma y. This ia is a map from M2, M1 cross M2.  So, it puts sort of, just puts M2 inside this product.

It does not really do anything to, the point except that it puts it sort of we can imagine it as, in the case of the horizontal slice, it puts M1 at different heights. And in the case of this vertical slice, it puts M2 in different horizontal locations. These are all, the, of course the picture should not exceed. So, these are the horizontal slices and vertical slices will be like this. So, all you doing is just putting M2 at different sort of locations. Now, these slice maps are smooth, so ib is smooth at all points of M1, ia is smooth at all points of M2.

And in fact, so here again, one can just take some coordinate chart. So, here one has to start with a coordinate chart around a point in M1. Just use the same coordinate chart, as a part of a product coordinate chart in M1 cross M2. It does not matter what coordinate chart you use for M, for this b in M2, but for M1 just use the same one. And one is in local coordinates, this will be a slice map between, so in U1 cross V1.

So, this will be in Euclidean spaces now. So, this U1 and U1 cross V1. So, it will be able to form again, x, x comma. So, this is in the local coordinates, this will be a slice map between. So, in the first case, first case meaning, this maps of the type ib where the second coordinate is fixed, it will be of this form, so, b1. And we know that between domains and Euclidean space this such a map is trivially smooth. So, it is a corresponding map between, manifolds is smooth as well.

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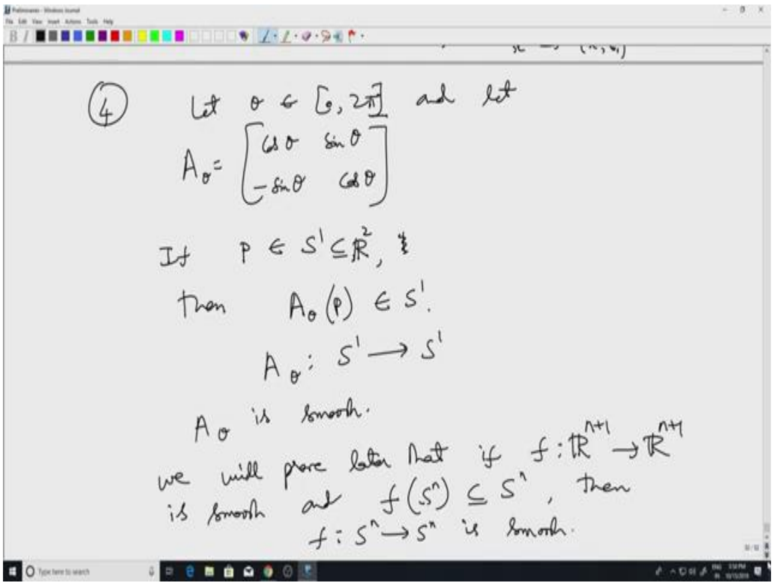


Now, let us look at something more specific. Let us look at a rotation map, on S1. Let theta belong to 0 to 2 pi, and let A theta be, cos theta, sin theta, minus sin theta, cos theta. Now, if P belongs to S1, then S1 of course regarded as a subset of R2. So, then A theta of P also belongs to S1. Well, this is a rotation matrix. So, if P belongs to S1 just means that norm P is 1, since it is a rotation matrix, A theta P also has norm equal to 1. So, A theta is map from S1 to, and this A theta rotation map is smooth. Of course, one can use either or, we have constructed 2 coordinate systems on each sphere, one is the projection where the charts are just projection maps, and the other one is stereographic projection.

One can use either of them to check that this is smooth. But more efficient way of doing things would be, later on we will prove that, we will prove later that if you have any smooth function from Rn plus 1 to Rn plus 1, is smooth and f takes the n dimensional sphere back to itself. Then, if we regard f as a map from the sphere back to itself, forgetting Euclidean space, this map is smooth.

So, the map we are, if you know that the map between the bigger Euclidean space is smooth and of course, if it takes the sphere back to itself, otherwise this does not even make sense. So, we need the sphere to go back to itself, then f as a map from Sn to Sn is also smooth. And in fact, this is just a very special, this will be a very special case of map between sub manifolds and so on.

So, but the thing is that it is a rather than work with some coordinate charts on S1 and trying to prove that this map is smooth. It will be, we will write, will give a much neater proof that this for instance, that this rotation is smooth.

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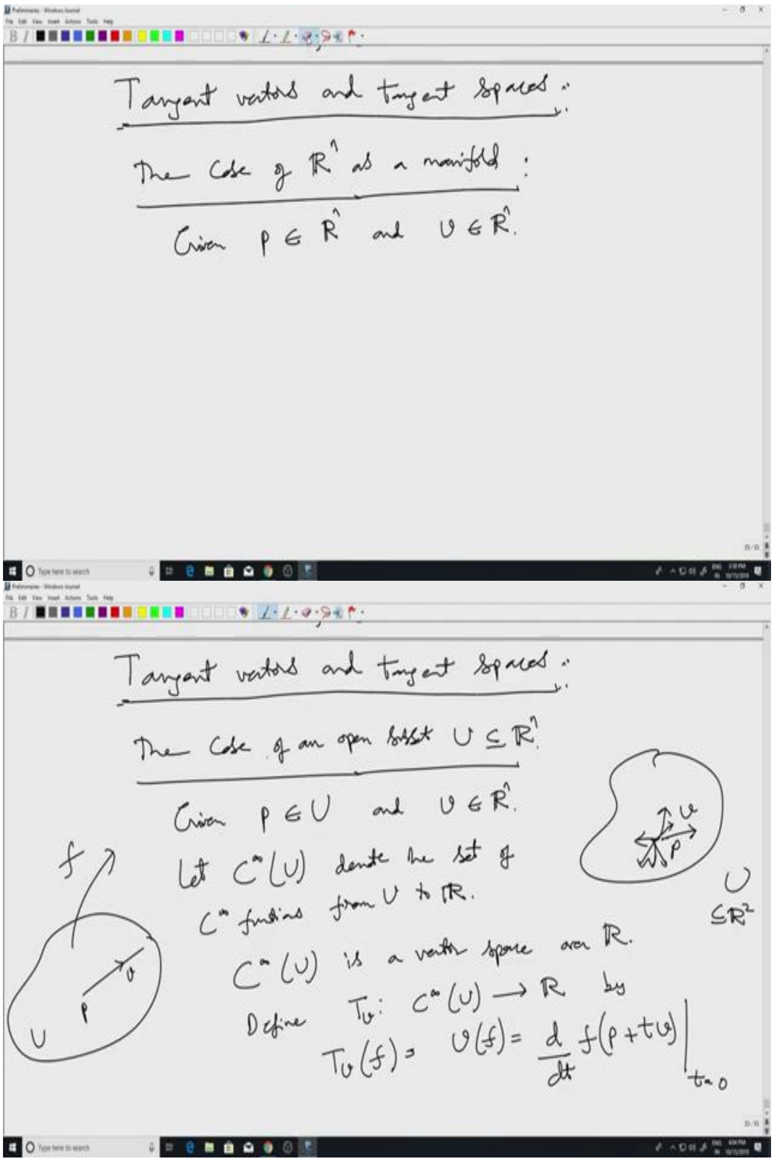


So, now that we have notion of smooth maps, let us, we want to talk about another very important, in fact the fundamental notion sort of associated to a smooth manifold, is that of a tangent vectors and tangent spaces. So, the moment you have is one has a smooth manifold. By its very definition, one has at each point to the smooth manifold, we will define a finite dimensional vector space called the tangent space.

This notion of tangent space is going to play pretty much whenever one deals with smooth manifolds, this will automatically make an appearance. So let us, now, the way a tangent vector is defined, there are several possible ways of defining this. But whatever way one chooses, whatever way, whatever sort of definition one works with, the certain amount of abstraction is necessary.

Because the whole point is that the, when we say something is a manifold that definition does not assume that, whatever we have is a subset of Euclidean space. Locally, of course, we know that it is homeomorphic to Euclidean space, but overall the whole set may not be a subset of any Euclidean space. So, one will have, one cannot use the, the Euclidean space in any essential way, even if I am talking about tangent vectors. So, some amount of abstraction is inevitable.

So, to motivate this the definition of a tangent vector let us in fact, start with the simplest manifold namely Euclidean space. So, let us start with, we already know how, the case of Rn as a manifold. So, you have an idea of how to talk about tangent vectors and, tangent vector at a point, in, tangent vector to Rn at a point. So, given P in Rn, v in Rn. In fact, let me just modify this slightly, it is a bit confusing if one works directly with Rn, more clearer thing is the case of an open subset U contained in Rn. So, I would like to, what I am trying to do here is my manifold is a open subset U in Rn.

And we already know, that the given, so this is U and have a point P. For instance, if U is in R2, let us take the simple case. Well, tangent vector at any point. Normally, one would think of, one things of an arrow based at P and pointing in some direction. This of course, does not make sense. But, in effect, what one is saying is that all, one is looking at all possible directions in which one can move and moving yet remain within the manifold.

If you already, if you have an open set, of course, whatever direction you move in for short distance you will end up inside U. So, from naively, every possible direction is a tangent vector. So, any v in R2 would be a tangent vector at P. So, at the end of the day, the tangent space, a tangent vector would be just a point in R2 this v. However, this would not do, when we talk about manifolds. So, let us look at it from a different point of view. So, let us start with P. Again, just take U, let us take a point P and a vector v in Rn. As we just discussed, any point in, any vector in Rn can be sort of intuitively regarded as a tangent vector at P. Any element of Rn can be regarded as a tangent vector at, but I want to get hold of this v in more abstract way.

So, instead of this v as a point in Rn, let me define a map. So, first let C infinity U, denote the set of C infinity functions from U to R, real valued C infinity functions on U. Here, I said that set of C infinity functions of course, since the target is R, this is actually a vector space I can add two such functions, multiply them by scalar, and so on. Is a vector space over R.

I want to define, now as, I taken a point P and a V in Rn. Corresponding to this, I am going to define a map, define a map from Tv from C infinity U to R. So, to every C infinity function, I want to get a real number, given this point P, and this vector v. Well, we know what to do, we can just take the directional derivative of the function at the point P in the direction v.

So, let us, of f is just what I have denoted earlier by v of f, which is d by dt f of P plus tv at t equals 0. So, I just look at the straight line, starting at P in the direction v for small t, when this line will be contained in the open set U and small t is all that matters for taking the derivative. So, this will be contained in set U and then I have this function f, so restrict f to this straight line and then take the usual 1 variable derivative. So, I get, I have this thing. So, in short, it is just the directional derivative but our view point is that this we have fixed v. And of course, we have fixed the point P. And for every v, we get a map.

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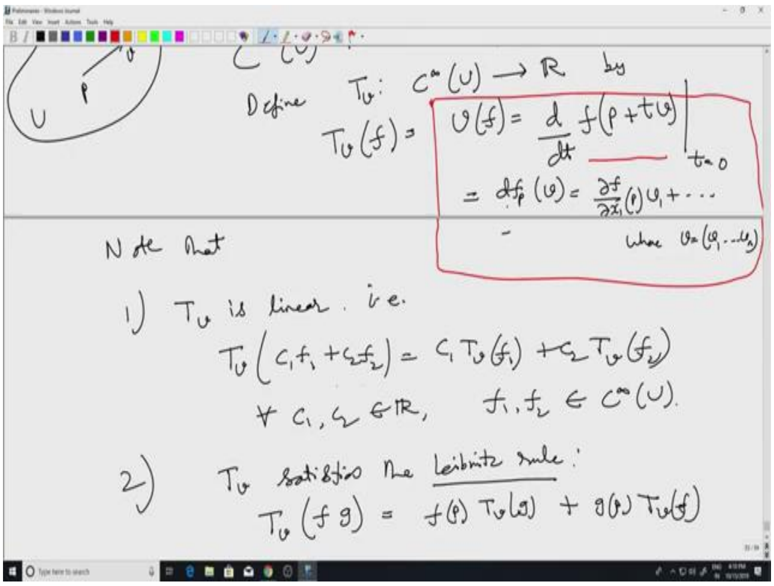

Note that, 1, Tv is linear, i.e, Tv of C1 f1 plus C2 f2 it is Tv C1. For all C1 C2, real numbers and f1, f2 C infinity functions on U. So, this is, actually this is not so evident if from this formula that I wrote down here, but we know that this is actually given by, this is also equal to dfP of v.

So, recall that f was a map to R, so, I have that derivative as a linear map from this to this. So, I can act on this, and the derivative as a linear map has components given by partial derivatives. So, this is also equal to delf by del x1 at p, v etc. where v equals v1, v2, vn.

So, this, there are several ways of writing the direction derivative. The definition is this. d by dt is this, but that is the same thing as dfP acting on v, which is also the same thing as this. If we use the second one, dfP of, now, actually, for this one, it is, one can use any of these, the linearity with respect to functions even if you use this d by dt of f of P plus tv, it is clear enough.

Or I can use this here, or I can use any of these 3 things will show this linearity with respect to functions. Later on, I will be needing linearity with respect to v, then one will have to use a different one. At this stage, any of these 3 things will do. So, Tv is linear. The second crucial thing is that, Tv satisfies the Leibnitz rule.

So, what does this mean. Well, this, notice that the C infinity U is not just a vector space, but also there is a multiplication, multiplicative structure involve on this. So, in other words, if I have two C infinity functions, I can multiply them and get another C infinity function. So, let us do that. So, I have two functions f and g, and if I take Tv of fg, so I will just get the f of p, Tv g plus g of P Tv f.

So, in other words, we have a product rule for derivatives that is all we are saying. And this again, follows from, for instance, any of these 3 things. Especially, the first one is, makes it quite clear, this definition of Tv instead of f, I will be working with f times g. And then one can use the usual product rule and get this.

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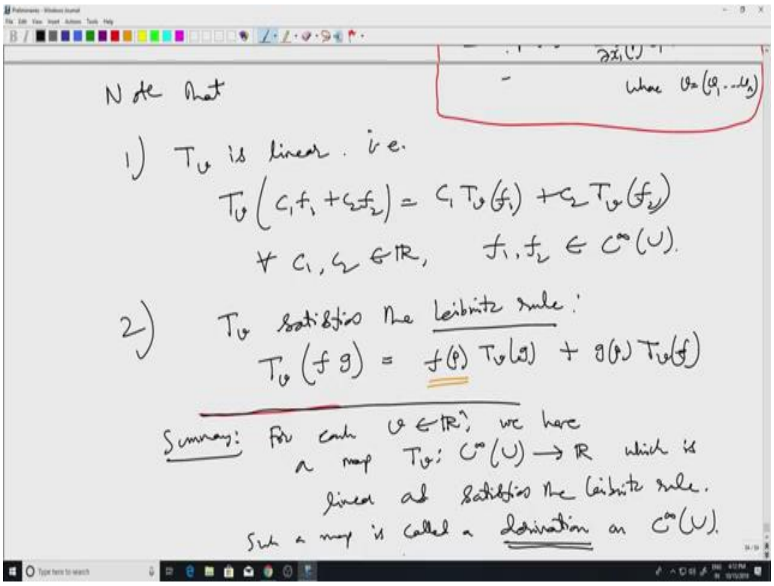


So, what we have is, so what we have done is for, so, brief summary is, for each v in Rn, we have a map Tv from C infinity U R. All this happens, all this is happening with the point P fixed, notice that this Leibnitz rule property that I have here, the point P plays a crucial role, the, this P occurs here and here. So, that is always in the background. So, we have a map Tv from this to this which is linear and satisfies Leibnitz rule.

So, we say that such a map is called a derivation on C infinity. So, a linear map satisfying the Leibnitz rule is called a derivation. Well, actually a derivation is something more general than this and defined on more general objects then C infinity U. In this context, as it turns out, giving it a new name turns out to be unnecessary because it turns out that a derivation has to be something very specific. It is more or less like it is, in fact, because our main result is that, the following, that it is going to be a directional derivative.

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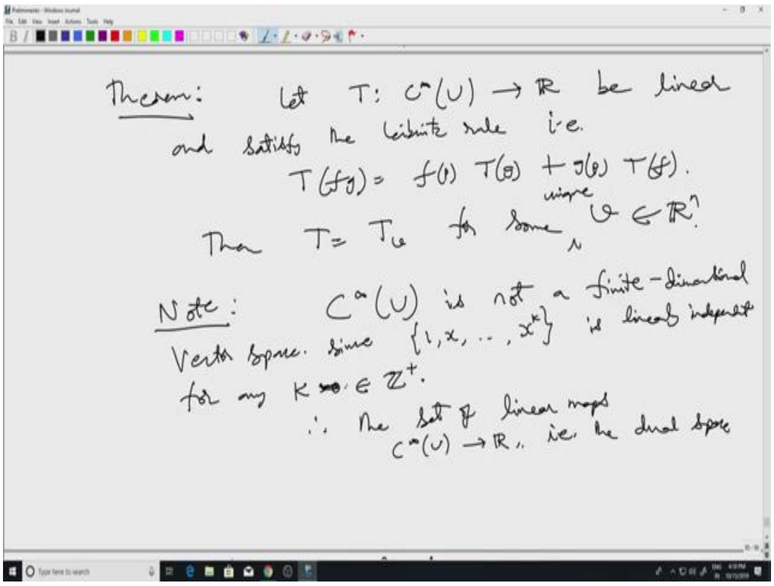


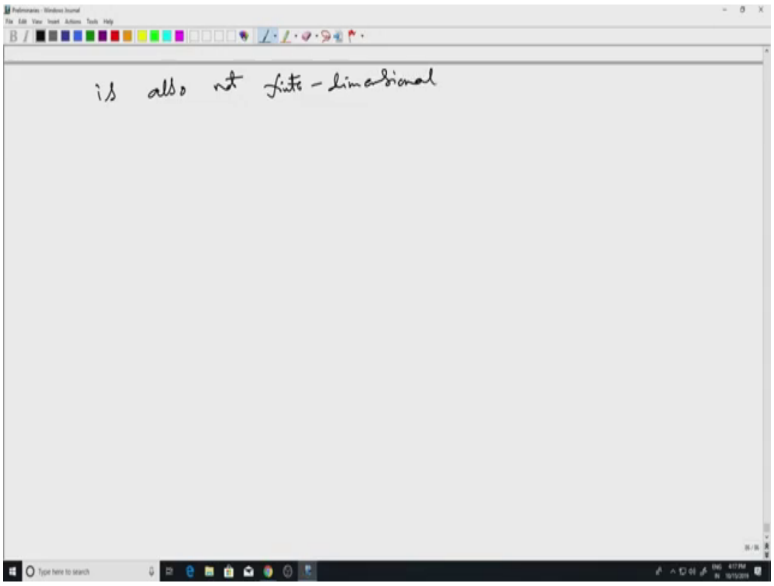


So, theorem let T from C infinity R, C infinity U, C infinity U to R be linear and satisfy the Leibnitz rule i.e Tfg is f of p, T g plus g of p, T f, then T equals Tv for some v in Rn. So, in other words, if you start with an abstract, with any map from C infinity U to R which is linear and which satisfies the Leibnitz rule.

Notice that, this Leibnitz rule for this Leibnitz rule to make sense, one does not have to really talk about derivatives. All one is doing is considering the action of T on product function. And, we are saying that T of fg has this very specific form. The point is, this second condition, this Leibnitz rule condition is so strong that this forces any such map to be necessarily a directional derivative with respect to some vector v in Rn.

And so, I should mention that, note C infinity U is not a finite dimensional vector space. This is obvious, since you have an infinite set of linearly independent vectors. So, since 1, x if I take is linearly independent, for any K, any K in Z plus, natural numbers. So, in other words, there is no bound on the size of, on how many linearly independent vectors we can have.

So, we can have arbitrarily large set of linearly independent vectors. This is not a finite dimensional vector space. And if therefore, the set of linear maps, from C infinity U to R associated to every vector space, where the dual space i.e that the dual space is also not finite dimensional.

It is easy to see that, if a vector space is not finite dimensional, the dual space is also not finite dimensional. So, if we just had, in this theorem, there are 2 conditions on T, one is it is linear and then the Leibnitz rule. If you just had linear, then the set of such maps would be very large space, in the sense that it would be a infinite dimensional vector space.

However, the moment you impose linearity plus Leibnitz rule, the theorem tells us that the set of such maps is severely restricted. In fact, what we, each map corresponds to. So, here actually, I should add that for some unique v. So, for every search map, there is exactly 1 vector, corresponding vector in Rn.

So, in other words, set of such maps is in bijective correspondence with Rn, which is a finite dimensional vector space. So, just linearity would be, given in infinite dimension vector space, while this linearity plus Leibnitz rule gives us Rn. So, let us stop here.

Next time I will prove this theorem, and then use this to talk about define tangent vectors on manifolds and tangent spaces. Then we will proceed from there. Thank you.