An Introduction to Smooth Manifolds

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Lecture No 12

Examples of Smooth Maps

Hello and welcome to lecture number 12. So let me resume from this diagram that I had started last time.

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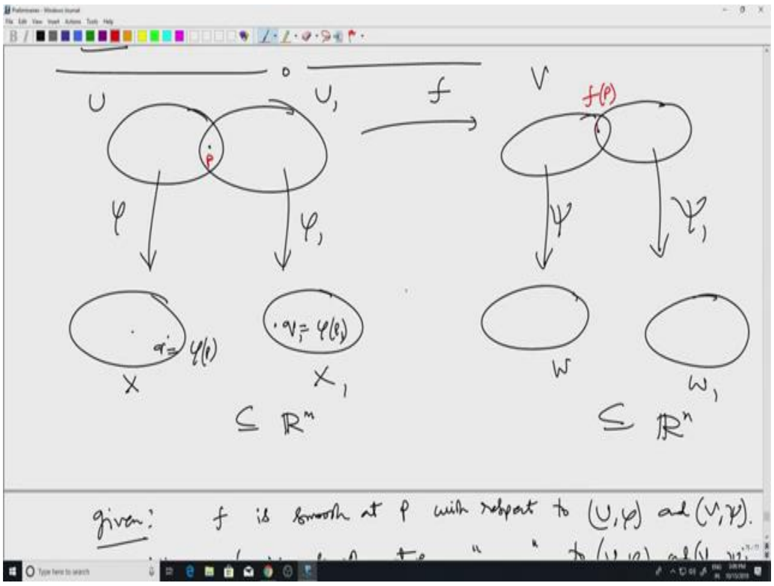

So we were talking about smoothness of maps between manifolds and issue was the following, that the very definition of a smooth map involved the choice of charts and we wanted to check that if it is smooth with respect to one pair of charts then it is smooth with respect to any other pair and this diagram is supposed to illustrate the picture.

So the same point P can be contained in two different charts U phi (U1) phi 1, similarly f of P can be contained in two different charts. So let us assume that we are given that f is smooth at P with respect to U phi and V, VC and we want to check that f is smooth at P with respect to this new, this other pair of chart.

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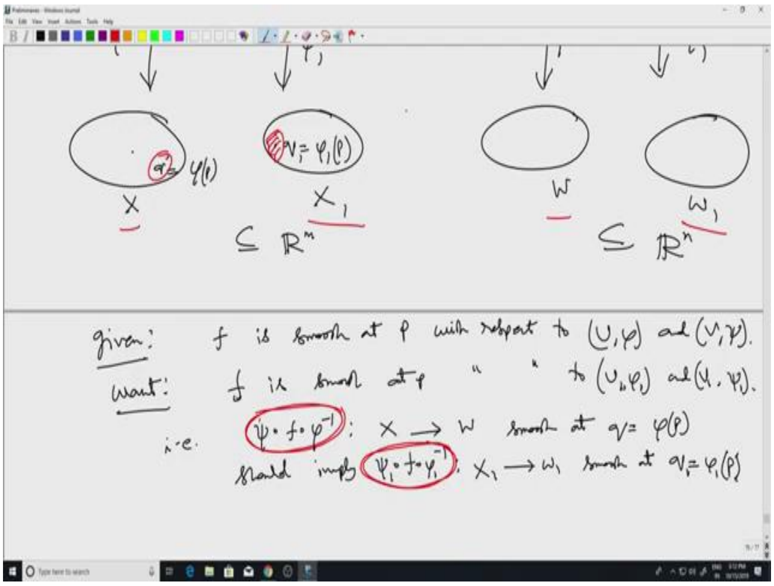

So, well the way we defined it smoothness at P with respect to the first pair would amount to saying that phi inverse called this composed map from X this domain X to W. So from this to this would be smooth at the corresponding point q. So this is what we are given and smoothness with respect to the second pair of charts would amount to saying that this new composed map which is from X1 to W1 should be a smooth at the corresponding point q1 equal to there is a small change here it is phi 1 of P of this.

So let us do that. Now I am supposed to start with this map, start with this map and from this I am supposed to get some information about this map. Well first of all let us observe that the smoothness of a map at a point is a purely local issue in the sense that if I can prove that this new map is smooth in some neighbourhood of q1 then that is good enough, I mean the domain I do not have to consider this map on the whole domain. Some neighbourhood of this, the main point is q1 should be inside that domain and if I ensure that this map is smooth in that smaller domain that is smooth at q1 that is good enough.

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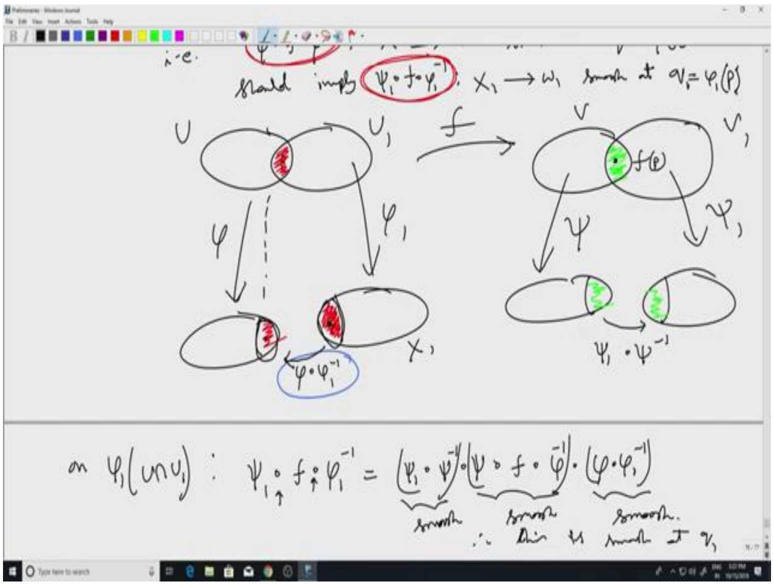

So let us proceed. The idea is to bring transition functions into play. So what I do is? Let me draw the picture again. U, U1 and there is my point P. I have f, V, V1, this is f of P at this point here in the intersection. That is f of P, and f and then I have this chart maps phi, phi 1, C, C1. Now let us see where the shaded region goes to, that goes via phi that goes to something here and under this it goes to something similarly here as well.

So this shaded region will go to these two. We have our transition the point is that these shaded regions here can be related by the transition maps. So the transition map here is, so I can start with phi 1 inverse. So this is phi 1 inverse, I do phi 1 inverse and then phi we will take me from this so this transition maps here will take me from this red region to the this red region. And similarly here I have another transition map which takes me from first I do C inverse composed with C1. So C inverse will land me back in the manifold and then C1 will give me this part.

Now what I do is this once I have this picture the proof is quite straightforward because all I do is I am interested in V1 composed with f composed with C inverse. The so this is a map which is to begin with it is defined on this whole open set X1 but I confine my attention to this red shaded region here. So I will just confine my attention to this part here and on that part what I can do is instead of going via, actually there is a small so the inverses I put wrong so actually it is so I wanted first to phi 1 inverse and then to C1.

Now as I said this map is defined on all of X, X1 but I am interested only in smoothness at this point q1, q1 is somewhere here in this red shaded region. So I will confine my attention to just this smaller open set, the red open set. And then on that I will instead of just going by a phi 1 inverse, I will first go like this. I will first act by phi 1 inverse composed with phi. So this is on all the action is taking place on this shaded region here.

So that shaded region is precisely phi 1 of U intersection U1, this I can write, so first I come here then that is why I need more space. So let me just write it more to the right. So first I do phi composed with phi 1 inverse and then do phi inverse, so that after the first map I will be in this part here. Now I will go via phi inverse and then do f that will land me inside the target manifold and then I will apply the map C which will land me inside this. And then and then what I should do is, I should then yeah so then I will finally use this transition map to come back here.

So C inverse composed with C1. And of course this is explained in terms of pictures but in actually as composition of maps this phi inverse and phi just give me a identity here. And the C inverse and C give me identity here. So all essentially all I have done is started with this three term composition in between the this here I just put phi inverse and phi. Here I put C1 inverse and C. Only thing to keep in mind is that we must keep track of that domain, we have shunt the domain here.

So that is why on this I have this equation and the manifold condition implies that this is smooth. Now since we already know that f is smooth with respect to the this original pair of charts. This is smooth and this is smooth again with the transition function. So we have a composition of three smooth at the points corresponding to P. For instance, this map the first map (phi 1) phi composed with phi 1 inverse, this map is anyway this is smooth everywhere, this is a transition function. The second one is smooth at the point q the middle one is smooth at q and finally it is a transition smooth everywhere.

So overall this is smooth therefore which is this is smooth at q1 which was the point here. This was q1 right. So that, that shows that this explicitly brings out the role of the transition function. So because of the smoothness of transition functions we do not have to worry about whether some condition holds only with respect to some chart or the moment you have an atlas in other words system of charts where transition functions are smooth then one can work with given a point one can work with any chart containing that point.

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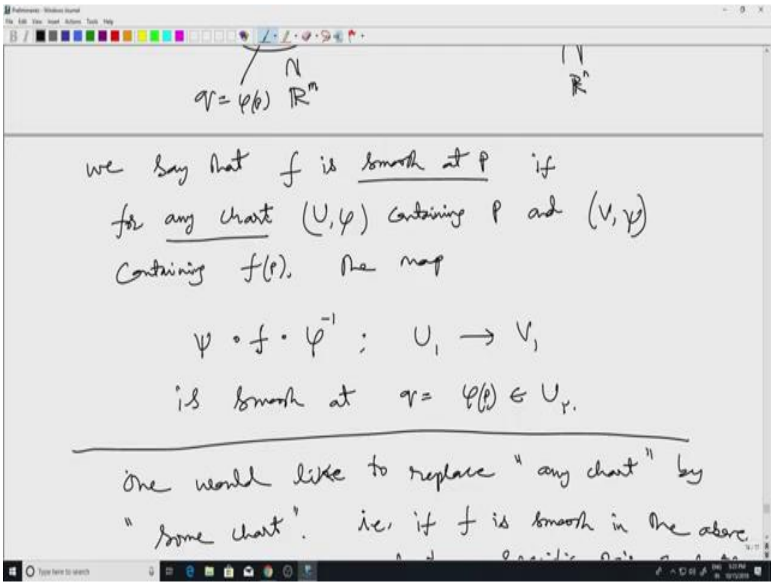

Now there is one important thing that should be mentioned when I defined, let us go back a few slides and when I talk about smoothness of, smoothness at P, I have to mention the following I mean for this composition to make sense the image of since I am doing C of f of something I want to ensure that f of this U1 rather actually this the definition of smooth maps.

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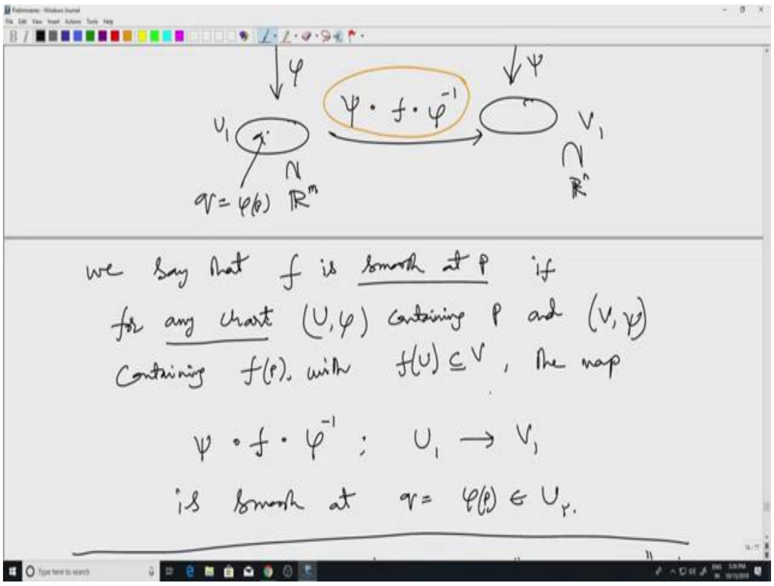

So I want to ensure that this map C of f of phi inverse make sense in other words I want containing P and we say that f is smooth before any chart containing P and containing f of P. Before I say talk about smoothness I just want to ensure the map is well defined. So I would say containing such with f of U contained in V the map of this. So I will choose the charts U and V such that f of U contained, we should start with charts U and V such that f of U contained in V. Of course this can always be done if f is continuous at P. So but I put this as a condition here rather than adding it, mentioning continuity this map this happens right.

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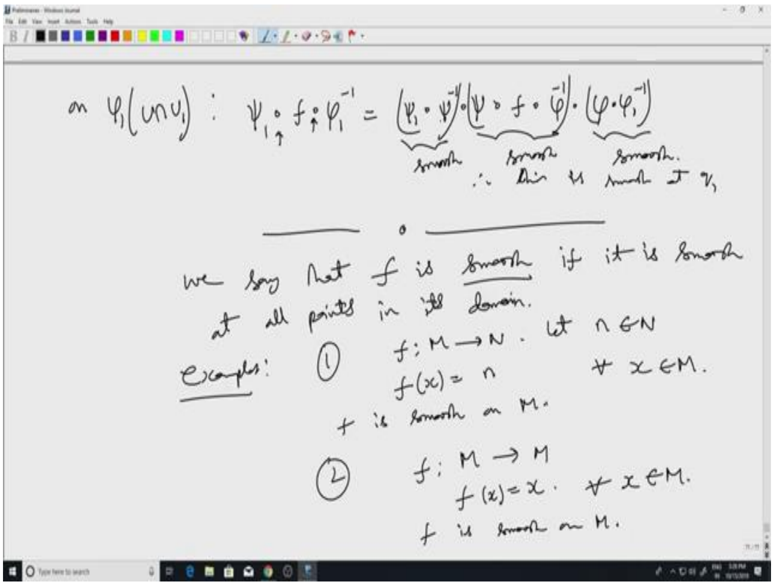

So then as usual so now that we have a notion of continuous smoothness at a point. As usual we say that, we say that f is smooth if it is smooth at all points in its domain. Now let us see some examples, start with the most trivial ones. The most trivial map one can think of is just a constant map so let us take f of x equals some let n belong to N and let f of x equal to n for all x in M. So in this case of course the so as usual one has to invoke their definition at whenever we want to say so here if you want to check smoothness.

So here f is smooth everywhere, f is smooth on M and how which are so the question becomes, which charts do we choose to see this very clearly. We have just proved that it really does not it does not matter which charts one work with but in actual computations working in a specific pair of charts will make things very clear but in this case it does not matter. So if I take any each, so if I take P in N and f of P is going to be N irrespective of what P is, so I just take any chart around P any chart around N and this composition map will be again a constant map, so it will be smooth.

The next trivial map is f from M to M, f of x equals x the identity map. Again it does not matter which charts we take. So we just in fact the easiest thing to do would be, so you take a point P here, f of P is again going to be P. So you can take the same chart for the domain and the image and when we do this composition we will get the identity map on between open sets in Rn which is of course smooth.

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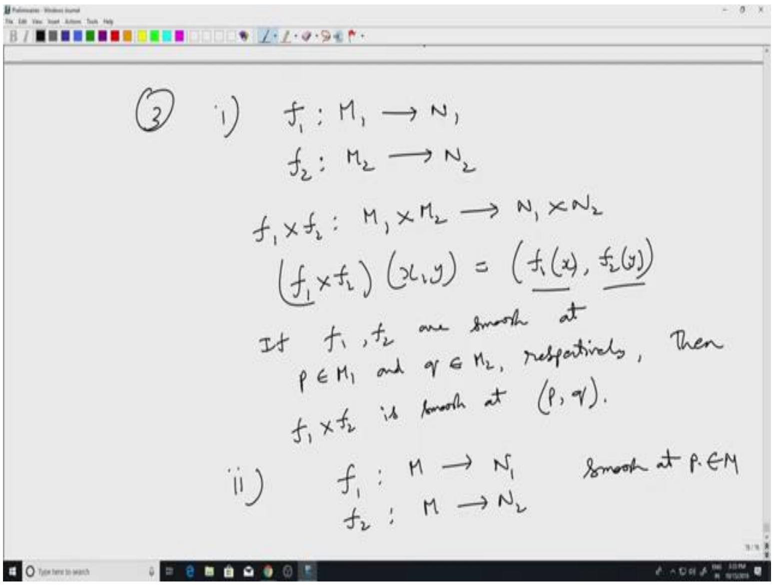

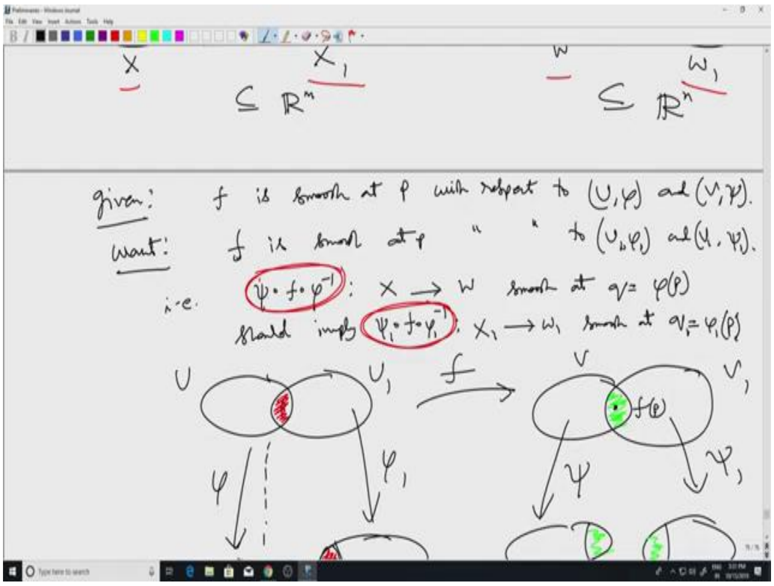

Now for something slightly more interesting. Let us look at product manifolds and here there are two natural maps associated to product match, let us check that they are smooth. The first one is suppose I have maps between M1 and N1 and f2- M2 and N2 then I can form this product map M1 cross M2 to N 1 cross N2, where this product map is defined in the obvious way which is I just take f1 x, f2 y.

So this will be again so if f1 f2 are smooth at p in M1 q in M2 respectively then a f1 cross f2 a smooth at p, q and so to prove this again it is, so here so one just works with a product chart here and then it turns out that if you do this express the map in local coordinates as we did as we have been doing so by expressing in local coordinates I just mean considering this C composed with f composed with phi inverse.

So if you use product coordinates for that then one has one will just get a corresponding map here and a corresponding map in the first coordinate here and a corresponding map in the second coordinate and one can easily check that this product map is smooth as well. Another thing where I have (M1 to N) M to N1, f2 from M to N2. Let us say both are smooth at a point P in M. So unlike the previous case I am just taking the same manifold as the domain not two different ones.

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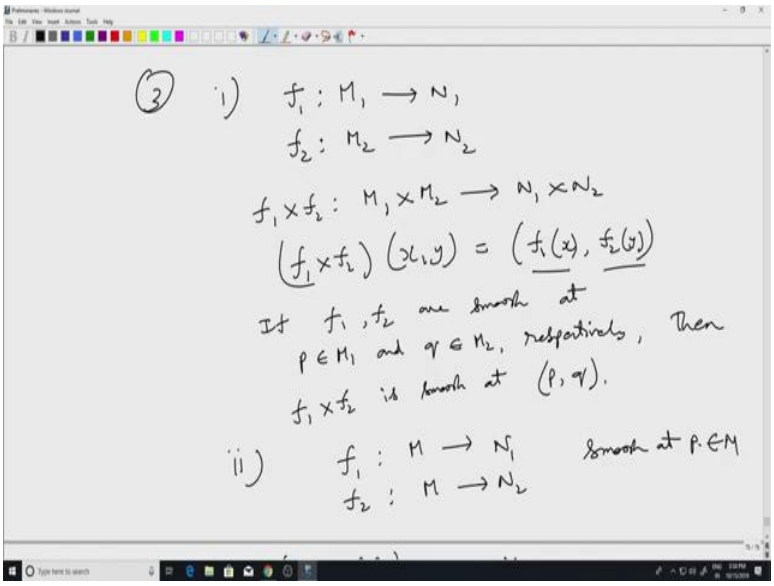

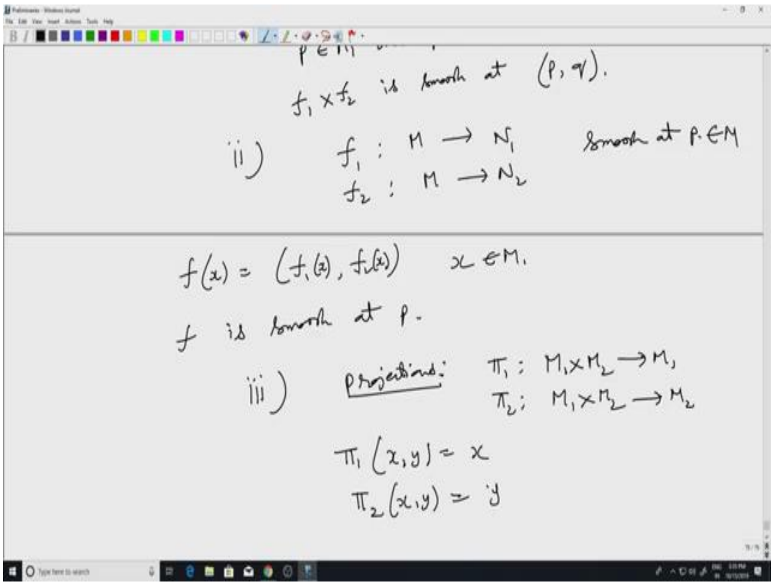

Then I look at this map f of x equal to f1x, f2x, x in M1. Again it is f is smooth at P if both of these f1 and f2 which we have assumed are smooth at P it will follow that f is smooth at P. And here so what I can do is, so the fact that f1 is smooth at P will imply that for any chart around P and in each heart around f1 p this the composition is smooth.

Now to simplify things I can just use when I work with f2 I can use the same chart as I did with f1 around the point P. Here of course N2 I will have to use a different chart and but on M1, on M I can use a single chart but it just it really does not matter it just simplifies notation a bit that is all. Even if one uses different charts it does not change anything. The other more intrinsic map associated to a product is projections. So I have the first projection pi 1, M1 cross M2 to M1 and pi 2, M1 cross M2 to M2. So pi 1 of x, y is just x; pi 1 of x, y. pi 2 is y.

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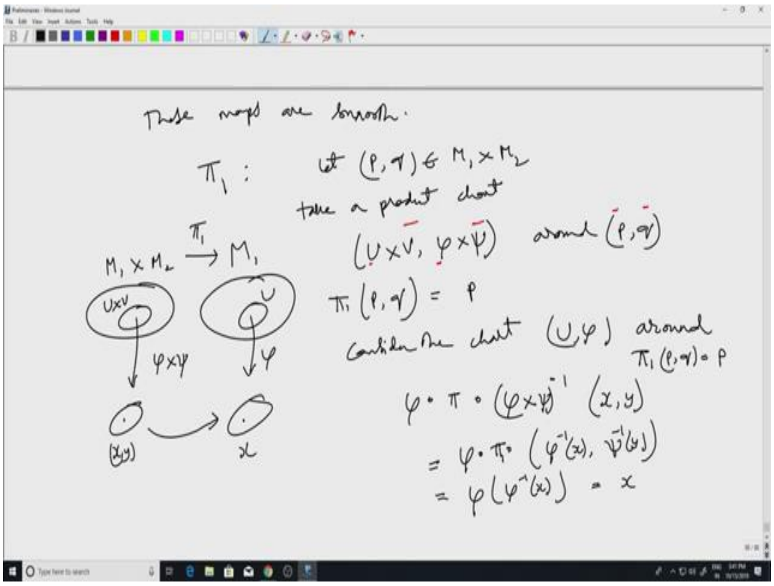

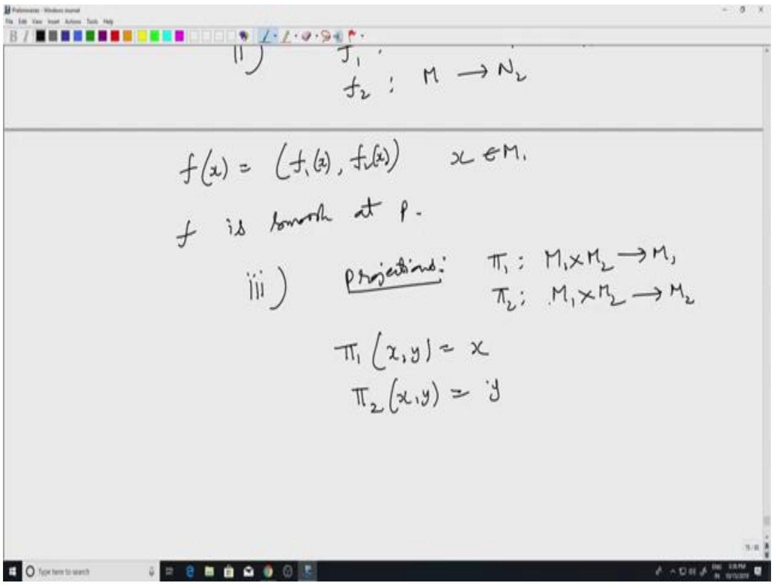

These are the projection maps and these maps are smooth at all points. So again the in fact here if one takes a the product coordinate chart here and so let us say so here for example if I work with pi 1, so let us just quickly see in the case of pi 1. Let p, q belong to M1 cross M2 take a product chart, so this we have defined to be of the following form.

U cross V, phi cross C. Take a product chart around p, q. Now the pi 1 of, I will need a, I will also need a chart around pi 1 of p. q. So this would be this is in other words a chart around P. Well in the product chart by definition this U, phi is a chart around. So in the product chart this U and this phi is by definition a chart around P, V and C is a chart around q. But I do not care about q.

I need a chart around pi 1 of p, q P. So I will just take the same U, phi which I done earlier. So consider the chart U, phi around pi 1, this became messy, around by pi 1 of p, q which is P. If I do this then, so I will have to basically so I have here have M1 cross M2 and then M2 this is the project M1 rather the projection first projection and on a portion of this I have a chart which was the product chart V cross C and on this I have well this was U cross V this was just U then I have a phi right.

So if I do phi cross C inverse composed with pi, composed with phi of x, y then I will end up with well after this first map by definition I will get phi inverse x, C inverse y, the first projection and then first projection will give me just the x the first coordinate here phi inverse x which is just x. In other words this compose the map projection in local special local coordinates like this is again a projection x, y here, I started with x, y here and I end up with x and this is true at all x, y inside this domain not just at p, q. So it is just the usual projection in between Euclidean spaces therefore this is smooth.

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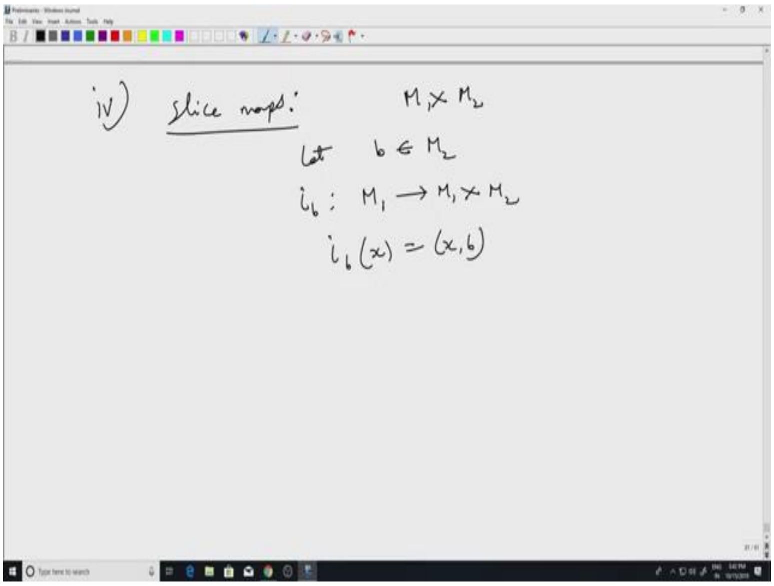

So projection maps are smooth and the other thing is that in the context of products there is other natural map is a slice map, slice maps or inclusion maps M1 cross M 2. Again I start with M1 cross M2. Let b be a fixed point inside, let b belong to M2. Corresponding to this b I have a map. ib of ib from M1 to M1 cross M2. So schematically so ib of x is x, b. We will stop here. In my next lecture I will say a few words about this and give some more examples and move on to tangent spaces okay. Thank you.