An Introduction to Smooth Manifolds
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Lecture No 10
Higher Dimensional Spheres as Smooth Manifolds

Hello and welcome to the 10th lecture in the series. So I will continue with the where I left off last time which was we explicitly talked about the, we explicitly showed that the circle S1 is a smooth manifold.

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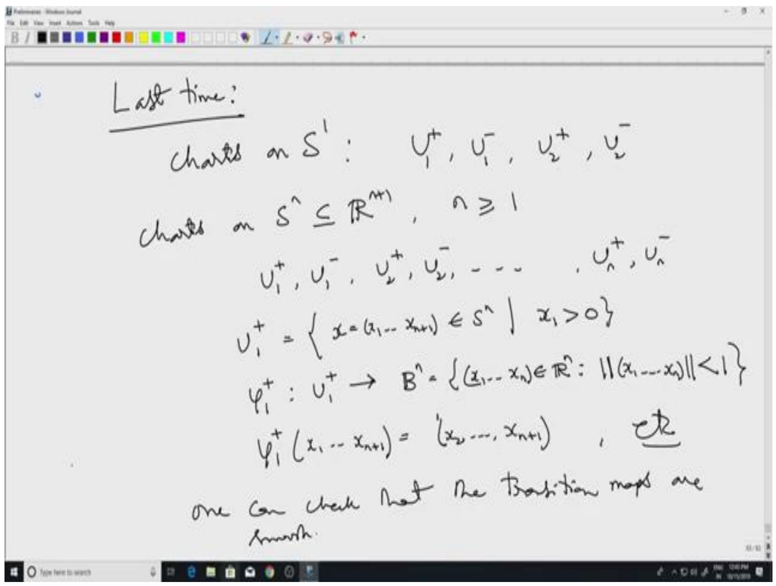

Now let me start from there. So we actually in order to show that this was a smooth manifold we had to work with four charts which I called U1 plus, U1 minus, U2 plus, U2 minus. And in fact the same one can generalize these charts on Sn contained in Rn plus 1, n greater than or equal to 1. Well for instance if one's one can think of n equals 2 so I will be looking at the two dimensional sphere at R3.

But the point is that in all of these cases one will end up so for each coordinate one will end up getting two charts. So U1 plus, U1 minus, U2 plus, U2 minus and so on all the way up to Un plus, Un minus. So and the chart maps as in the S1 case just projections onto whatever coordinate one is considering. For instance, U1 plus would be set of all x equal to x1, xn plus 1 in Sn such that the first coordinate the x1 is positive. And the corresponding chart map in this case would be phi 1 plus from U1 plus and it would, it will go to n dimensional ball, the open n dimensional ball.

All points in Rn with norm less than strictly less than 1. And the map as I said this just a projection, so phi 1 plus would be x1 up to xn plus 1, you would be just projecting on to the so here so x2, xn plus 1. So this would be the map and similarly (phi 2 minus) phi 1 minus would be the same map. The domain would be different; you would be looking at all points. So what and one can again check the transition maps.

So I will just put etcetera here a transition maps are smooth. However, this is a somewhat inefficient way of doing things. Ideally one would like to have a chart which covers as much of the manifolds as possible. So that one would like to work with as few charts as possible. And the task of checking the transition functions are smooth also the number of checks one has to do will reduce.

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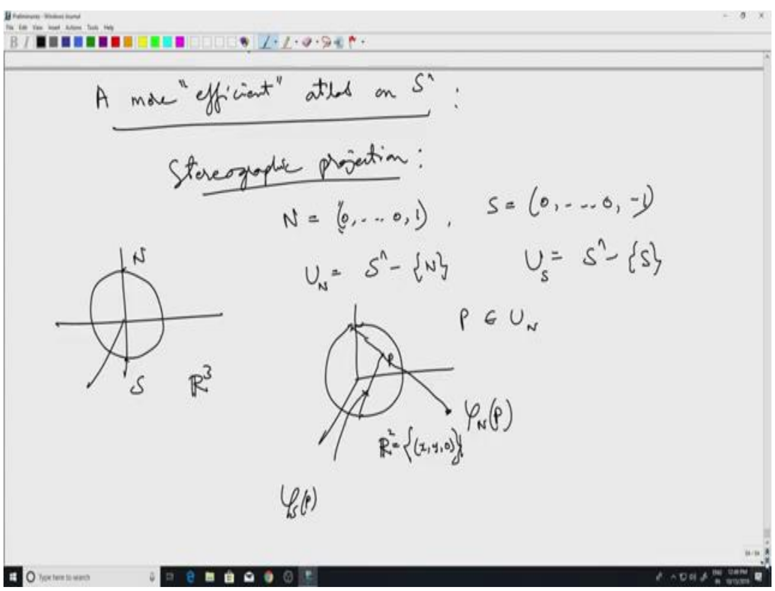

Fortunately, in the case of Sn one has a more efficient, efficient in a I do not mean efficient in any precise mathematical sense, more efficient atlas on Sn. We can cover it turns out that we can cover Sn with just two charts independent of what n is and this is the famous stereographic projection. I will need two points the north pole which is n equals 0, 0, 0, 1 last coordinate is 1 and the south pole which is 0, 0, 0, minus 1.

So in case of, this is R3, this would be this point is n this point is the south pole. But the same definition holds in for any dimension. I just put the last coordinate 1 here and the last coordinate minus 1. Now corresponding to this north pole I have a, my chart is going to be let Un equals Sn minus the north pole, Us equals to Sn minus the south pole.

So this Un is the entire sphere but I just removed the north pole and Us is entire sphere minus the south pole. And the chart maps are defined as follows. This is what is called the stereographic projection. So phi N, so first let me just describe it geometrically then I will write down an explicit formula.

Again I will do in the case n equals 3 at least the picture but definition is the same in higher dimensions. So what one wants one starts with a point P in Un, P belongs to Un. So in other words it is any point of Sn other than the north pole. Then I just connect this P to the north pole by a straight line which is well defined since P is itself not the north pole.

So I have a unique straight line connecting the north pole and P. Then I extend it all the way and see where it hits the, this plane rather this sub space in this case it is R2 which is given by the z coordinate being 0 set of all x, y, 0. So in this so I just extend this line and see where it hits this articular, the x, y plane, just copy of R2.

And this is what I defined to be phi N of P. And similarly when I start with the south pole for phi S of P so again let me take the same point P. This time I start with the straight line connecting the south pole to this point P and see where it hits the x axis. Sorry the x, y plane. And that is phi N of. So this point here would be phi S of P. Well it is not difficult to work out the, the formula for the point in terms of the coordinates of P.

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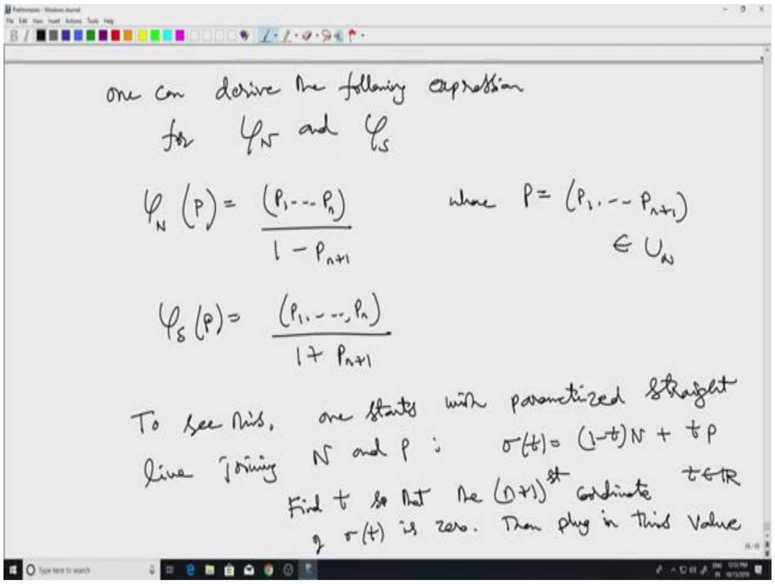

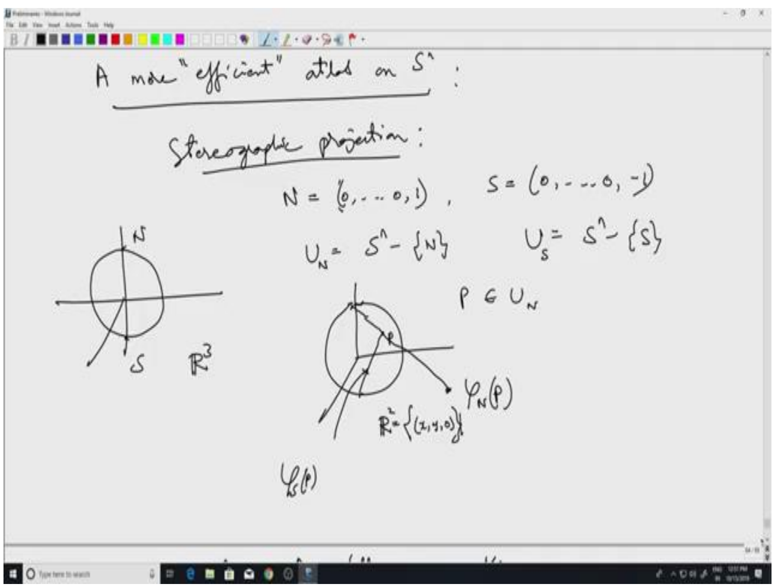

So all one has to do is one can derive the following expression for phi N and phi S. Turns out that phi N of P is just the first n coordinates of the point divided by 1 minus P n plus 1where P equal to P1, Pn plus 1 belongs to Un. And it turns out that phi S of P is P1, Pn, 1 plus Pn plus 1. And the way one gets these formula is the just by looking at to see this, one starts with the parameterised straight line joining N and P.

So this is a parameterised as I said the normal parameterisation of a straight line would be just sigma t equals well 1 minus t times N plus t times P, T belongs to R. So this is for as t varies over

R. I get all points on the straight line which passes through N and P. If I restrict to t between 0 and 1 I get the line segment between N and P.

But actually as we saw in the previous picture I would I want to go past P and see where I land in the Rn plane. And so I have to let t vary over all real numbers. So now the thing is one is interested in finding that value of t for which the this point this straight line hits the Rn plane. In other words, the n plus first coordinate of this should be 0. Find t such, so that the n plus first coordinate of sigma t is 0. So there will be unique t for which this happens.

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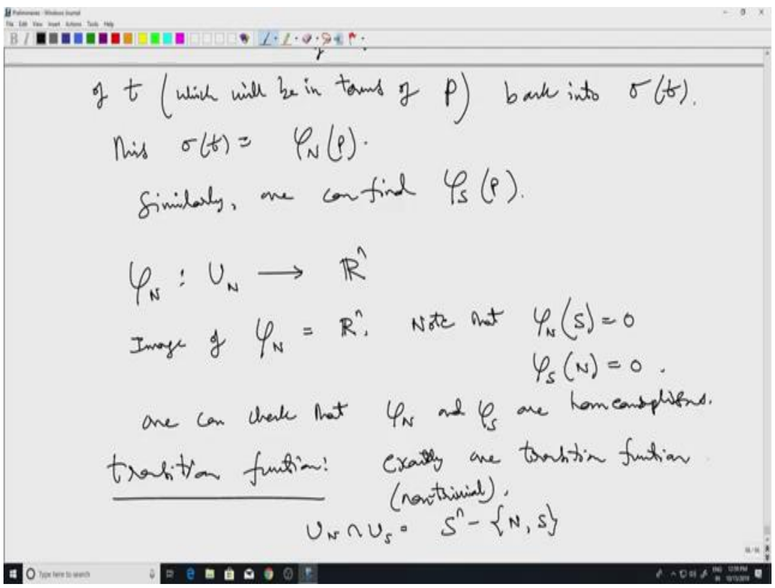

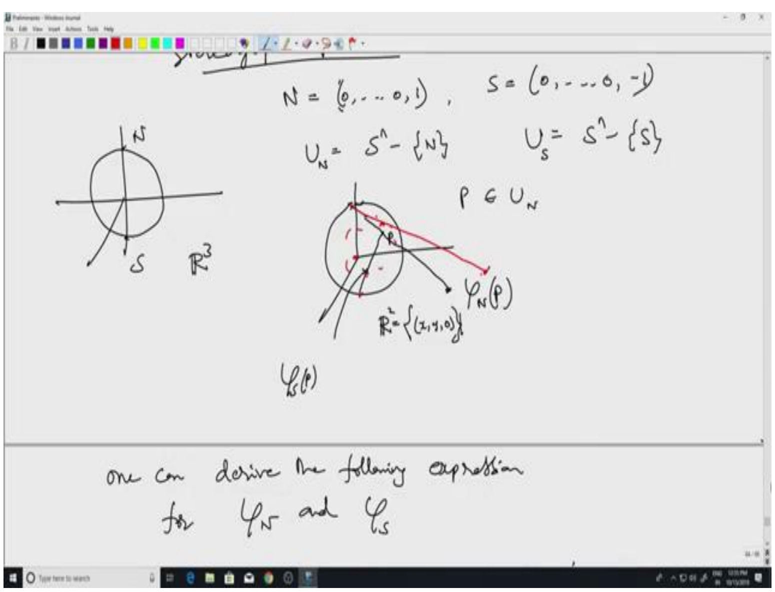

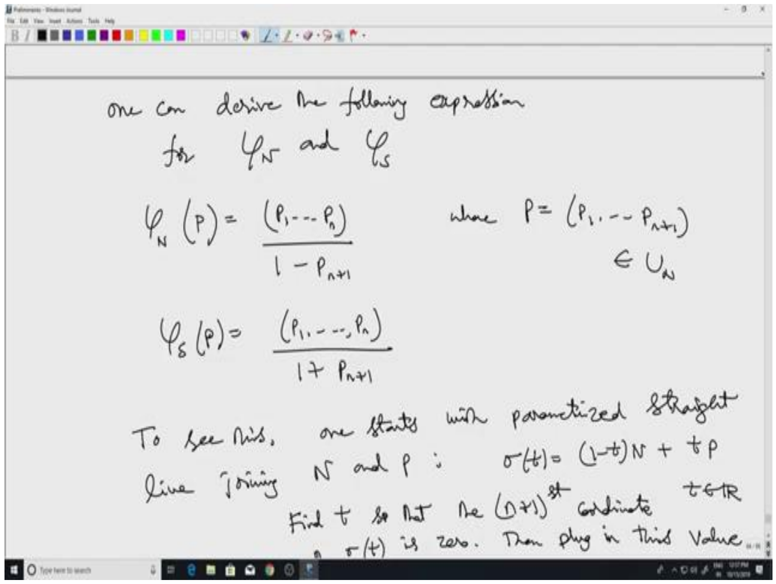

Once you do that, once then plug in this value of t which will be in terms of P back into sigma t. And that will give you this will be this sigma t equal to phi N of P. And a similar thing holds for, similarly one can find phi S P. Now so we have two charts. Well, let us just quickly see what the range of this is. So phi N is a map from Un to by definition Rn and image of phi, now it is, it is one can check easily by the that as P varies over all points on the sphere as this P varies over different points on the sphere I will get all possible values on the, for instance in this case in the picture that I have drawn in R2 every point in R2.

In fact, one can see this in a different way. So one starts with the point in R2 and I just connected to the north pole. It will have to hit the sphere somewhere. So in other words this point we started with is the image under the stereographic projection of this intersection point. Well the specific point is the origin, the origin comes from just by looking at the image of the south pole under phi N.

So I will just remark on this. For instance, rather than saying for instance let me just say that, note that phi N takes the south pole to the origin likewise phi S takes the north pole to the origin. So and one can check, one can check that both of these are homeomorphisms. In fact, one has a very explicit formula for phi N, phi N is obviously continuous and the formula it is very easy to write down the inverse of phi N as well. And one can see that, that is continuous as well.

So it is clear that phi N and phi S are homeomorphisms. Now the important thing is the transition. So if we just conclude it that these are homeomorphisms one would have that sphere would be is a topological manifold. Sn would be a topological manifold. But in fact in the what we need is that the transition functions are smooth and even that is rather simple transition. Notice that there is exactly one transition function in this, exactly one transition function non trivial.

Of course I have to add this non trivial because when I say transition function I can start with a chart and it intersects itself. So the transition function is the identity. So that is not interesting. If we have to get something non trivial we have to look at two different open sets, not the same open sets. And here exactly two open sets and their intersection is Un intersection Us. So remember that Un was the sphere minus north pole, this is the Us is the sphere minus south pole. So the intersection is sphere minus both the north and south poles.

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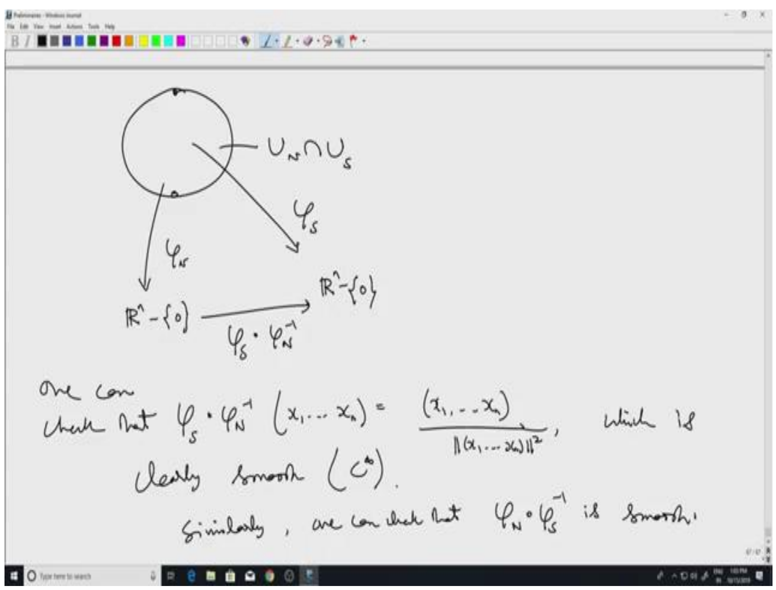

So let me draw the picture. So I have deleted this point. I have deleted this. So what I have here is Un intersection Us and I have this the homeomorphisms. So this is sigma N and on the same set I also have sigma S. Now where does sigma N map this Un intersection Us? As we observed last time the ray the image of sigma N is the whole of Rn and the point the south pole goes to the origin. But in Un intersection Us I have already removed the south pole so I am left with is Rn minus the origin.

Similarly, here it is Rn minus the origin, some slight notational mistake. It is not sigma, it is phi, phi N and phi S. So when I look at the transition map so first I have to go via phi N inverse. So phi S composed with phi N inverse would be this map from here to here. And one has to check that this is smooth and the other way around as well. So it turns out that, it is a formula for this. Again one can work out very easily.

So phi S composed with phi N inverse, phi S composed with phi N inverse of x1 up to xn turns out to be just x1, xn divided by norm of the same point square. One can check that, so this and this map is clearly smooth since the denominator never become 0. The domain is Rn minus 0. So this map is which is clearly smooth, C infinity.

Similarly, one can check that, one can check that the other way around is also smooth. In fact, the, this map phi N composed with phi S inverse is just the inverse of this (map) these two are inverse of each other. And it is easy to check that the inverse of the transition map is itself. So inverse is itself and therefore it is a, one can see that this is smooth. So that is a first, that gives us ray infinite family of manifolds and notice that all of these are compact.

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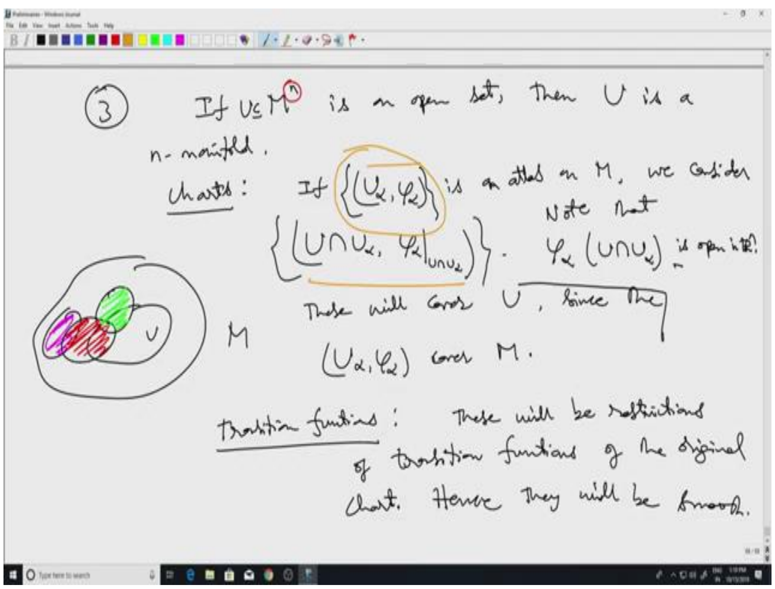

Now here is another trivial class of, here is a trivial class of manifold. So if Mn if U contained in Mn is an open set, then U is a n-manifold. So as usual when I put the super script and here it just indicates that the dimension of the manifold is n. So if I start with a n manifold and take an open set then the open set itself it becomes an n-manifold. So what would be the charts on this open set b.

The, the topological conditions namely that this is Hausdorff and second countable etcetera those are inherited from that bigger space. So I will not mention that but rather my interest is the charts and the transition maps. Now so I have a big manifold M here and I have an open set U. M is covered by some charts, etcetera. So these are all charts, this is one chart. So charts on M in order to get charts on U all I do is I just intersect these charts with U.

U alpha, phi alpha is a chart on M, we consider U intersection U alpha and then the same transition, same homeomorphism phi alpha except that I now restrict it to U intersection U alpha. And so this since this U alpha covers the whole of M, when I do this intersection, U intersection U alpha it will cover all of U. These will cover U since the U alpha, phi alpha cover M. So actually to be better way of putting it as rather than starting with a single chart, we will start with an atlas.

So I will just put curly brackets is an atlas on M. We consider the system of charts. These will cover U since the U alpha cover M. And again since the chart maps the homeomorphisms are just the restrictions of the original one, these transition functions so these will be restrictions of transition functions of the original chart. So in other words the original chart namely this, this would give rise to transition functions which are by definition smooth maps of certain open subsets of Rn.

Now when I consider this one here I will be getting the same maps, same maps but except that the domains of this maps in Rn will be open subsets of the original domains. But smoothness is not affected at all. Since it is, they are just the restriction of the original map to open subsets, so hence they will be smooth. So this is right. Oh, here at this actually at this early stage I should remark that after all the definition of a manifold is that it should map an open subset of the space into an open subset of Rn.

So here one should just note that phi alpha of U intersection U alpha is open in Rn. And the reason is clear because U alpha here by definition phi alpha of U alpha would be open, phi alpha of U alpha would be open and therefore since this is an open subset of U alpha its image would also be open, it just follows from the definition of homeomorphism. So I will stop here and then in my next lecture I will again continue with some more examples of manifolds. Then we will talk about smooth maps and tangent spaces. Thank you.