An Introduction to Smooth Manifolds
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Lecture 01
Basically Linear Algebra

So, this class and a couple of more classes will be devoted to preliminaries for the subject. So, I will focus mainly on linear algebra in this class. And if I manage to finish the material, then I will move on to multivariable calculus. With these two in hand we should be able to start the study of manifolds.

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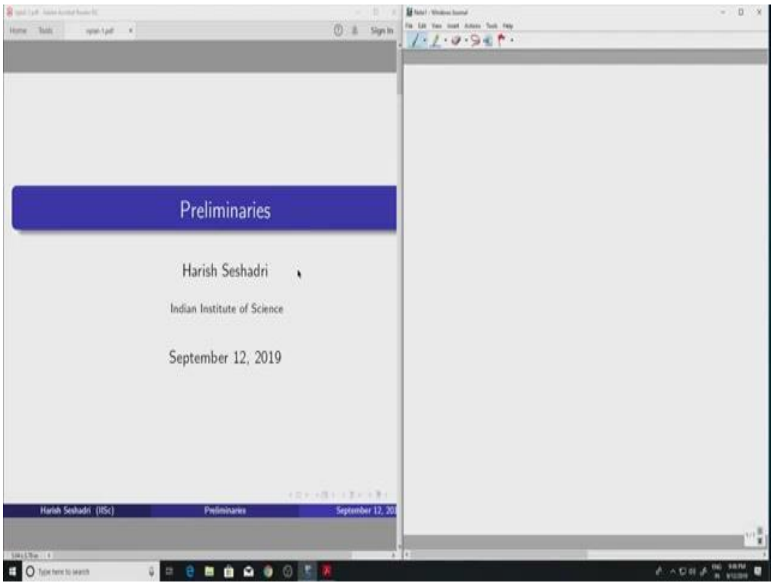

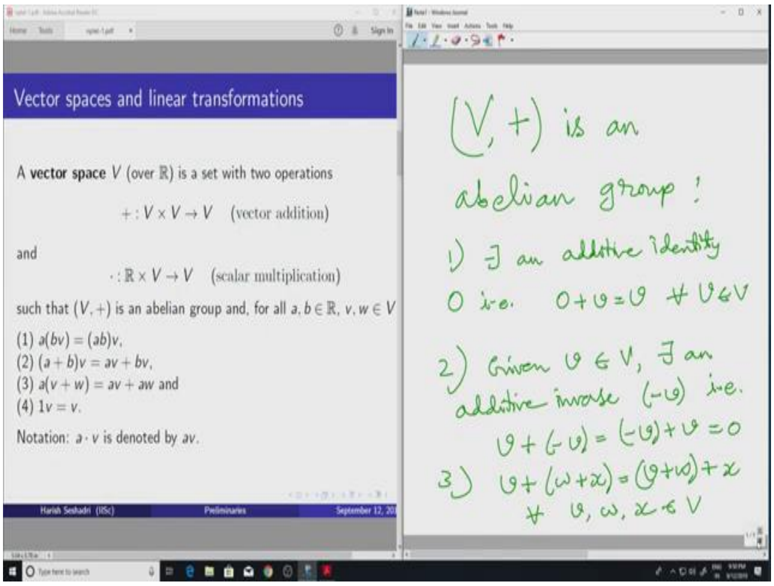

Well, okay, so let us recall some basics from linear algebra, which will be essential. And so I will start with the basic definitions. What is a vector space? Well, it is a set with two operations, in a addition operation, which is called vector addition and scalar multiplication. So formally speaking, these are functions from v cross v to v. And R cross V to V, here of course, I am assuming that every vector space is defined over R the set of real numbers as the field underlying field. Well, so we require that these two operations addition and scalar multiplication should be compatible in certain ways, which I list as properties one to four below. But before that, there is an axiom concerning just addition itself.

Namely, that with the additive operation v should be an abelian group. Well, this is just a shorthand way of saying that. So, what does it mean to say V with this plus is an abelian group, let me just, this just means that well, there is a first of all, there is an identity element inside the group. So there exists the unique there exists an additive identity, which we denote by zero IE.

The fact that the to say that it is an additive identity just means that zero plus v equal to V for all V in V. And the second thing is there is an additive inverse as well. Given V in V, there exists an additive inverse, which we denote by minus V. And the this additive inverse just means that v plus minus V equal to minus v plus v equal to the additive. Of course, even before these two properties, then one has something even more basic, which is that the additive operation, the fact that it is a group means that there is part of a definition that the additive operation is associated.

So, I want rather let me just v plus w plus x equal to v plus w plus x for all v w x in V.

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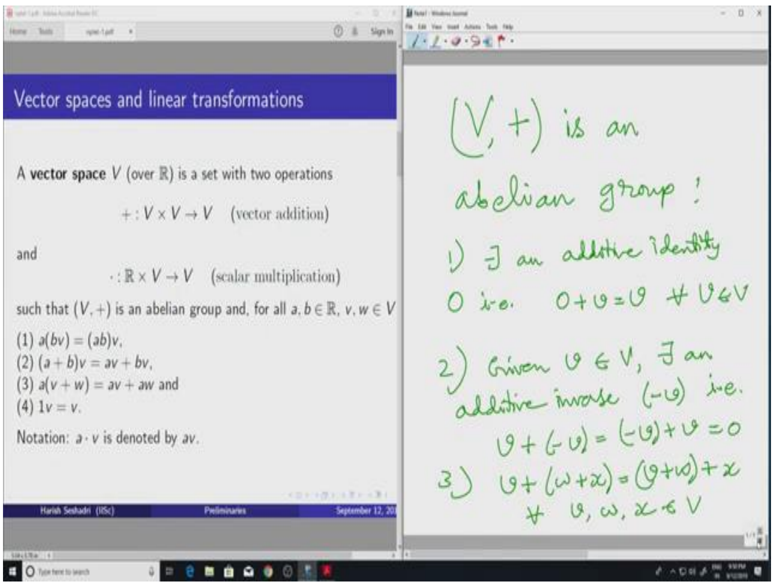

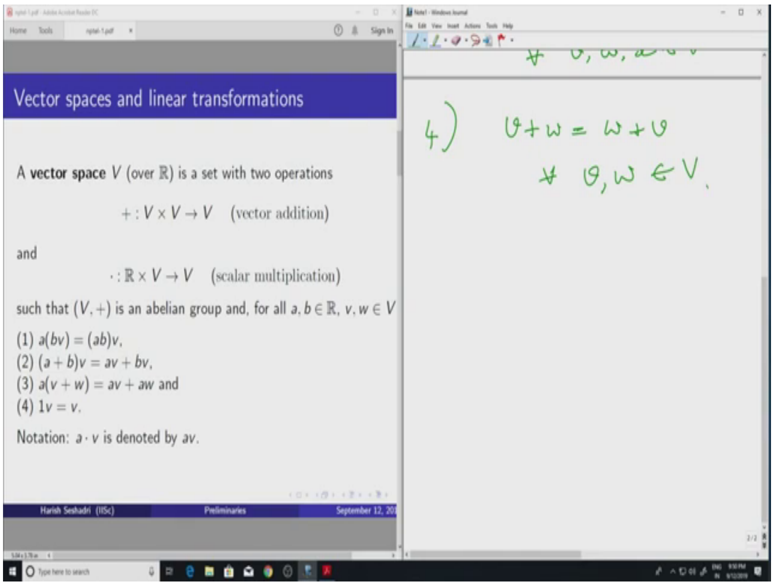

And, and, finally, the assumption that it is an abelian group just means that v plus w equal to w plus v for all v w in V. So, these four properties constitute what is called an abelian group. This has nothing to do with the other things scalar multiplication.

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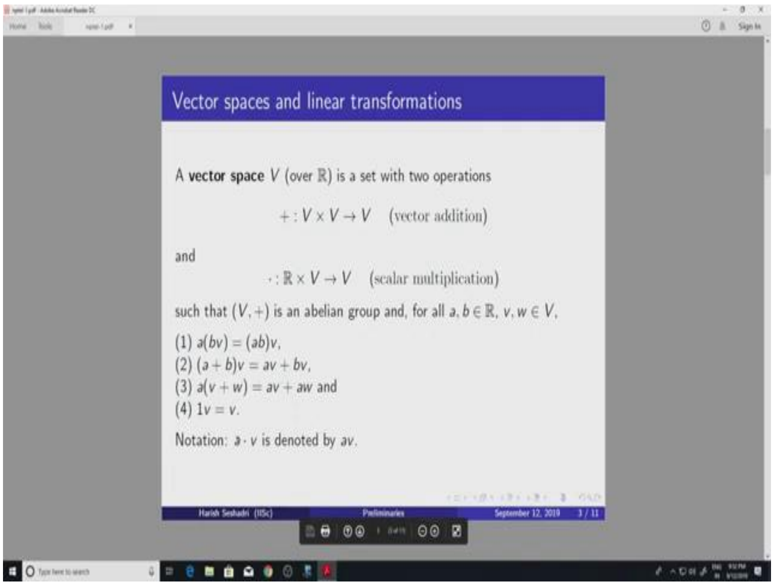

But for a vector space, these two operations, scalar multiplication and addition are required to be connected through this four properties that I have listed here. First one is, of course, this sort of, it does not matter whether you multiply scalars first and then multiply with a vector, or you just do scalar multiplication step by step, second one is distributivity of scalar addition and vector multiplication. And the third one is vector addition and scalar multiplication. Finally, the identity in the group are should act as identity by scalar multiplication as well.

So here of course, I have denoted scalar multiplication with a dot, but it is rather inconvenient to write it like this. So henceforth a dot v will just denote this by a v, where a belongs to R and v belongs to V.

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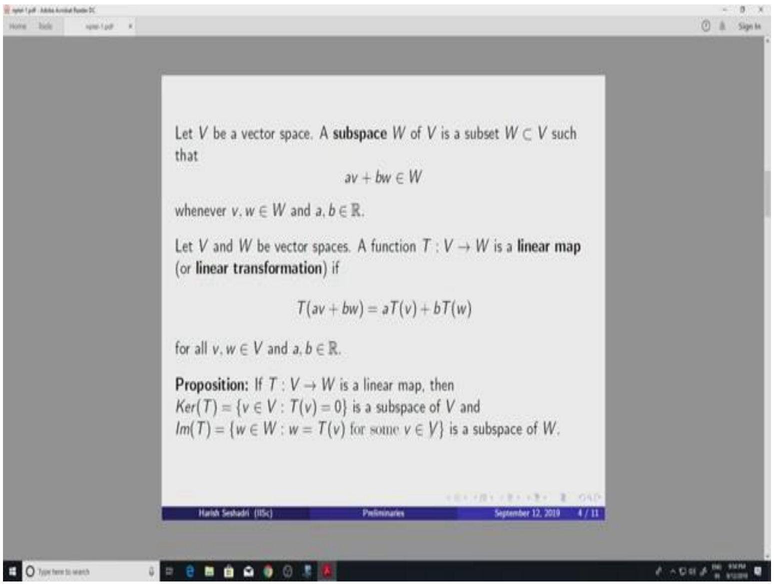

The next thing is a subspace. So, if you have a vector space is subset w in V set to be a subspace if whenever v and w belong to capital W. And if you take any two scalars a and b, you have a v plus bw is belongs to W. This of course, is equivalent to saying that W itself becomes a vector space when endowed with the same addition and scalar multiplication operation, which we has and the next thing a linear transformation.

So, let v and w be vector spaces a function from V to W is a linear map or linear transformation if essentially lose way of saying this is that it T respects the operations on V and W namely addition and scalar multiplication. Formerly T of av plus bw should be equal to a Tv plus bTw for all v w in V and ab in R. So, the moment you have a linear transformation, it automatically comes gives rise to two subspaces. One is if T from v to w is a linear map, then the kernel of T this is also known as the null space of T. This is the set of all V which map to the zero element by T. So, set of all V such that Tv equals zero is a subspace of V.

And this so this is one subset. The other one is the image. So these are all elements of W, which can be written as Tv for some V, in capital V. So this form a subspace of W, so you get two subspaces, one in V and one in W associated to any linear transformation.

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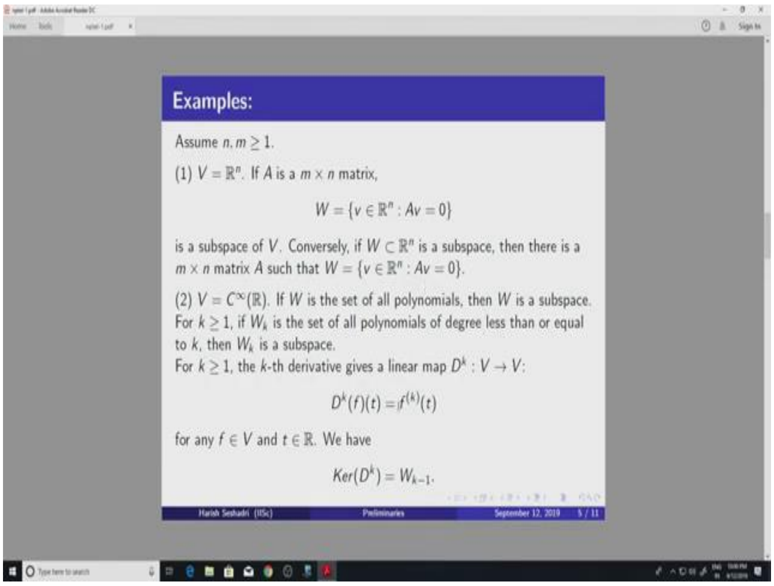

So, let us look at some examples. So, let us take two natural numbers at least greater than or equal to one. Of course, the standard the prototype for a vectors, finite dimensional vector spaces is R n. V equals Rn. If one can easily check that with the usual with the usual notion of vector addition so a point in Rn is just an (())(9:05)of real numbers. So when you have two points, you can add them component wise that gives vector addition in Rn and scalar multiplication would be just you multiply every component by a fixed scalar.

So let scalar multiplication. So you have which... And it is quite easy to describe all possible subspaces of V. Namely, if you start with any m cross n matrix and you look at the set W, which are all elements of Rn, all elements V in Rn and such that A v is zero. Note that this Av zero is just a compact way of writing system of linear equations. So essentially, what we are saying is W is the solution set of a certain set of linear equations. So if you look at such W, it is easy to check that this forms a subspace of Rn.

And a little bit of work shows that the converse statement is also true. In other words, if you start with any subspace of Rn, then one can find an m cross and matrix A such that that subspace is given in this form, namely the solution set of a Av equals to zero. So in short, any subspace of Rn is a solution of a certain number of linear equations.

Well, the next example is qualitatively quite different from this. So here I am looking at C infinity R, so this is notation for all functions on R, in other words, all real valued functions of one real variable which are infinitely differentiable. That is what this infinity stands for its differentiability of the function. So the V is the set of infinitely differentiable real valued functions on R so it is of course, it can add two functions and you can multiply a function by a scalar.

So, this in a very trivial way, it becomes a vector space. Of course, the point is that when you add to C infinity functions, you again get a C infinity function. And if you start with a C infinity function and multiplied by scalar, the result is also C infinity. And this has lots of interesting subspaces. So here, the First example is let W be the set of all polynomials. So here I do not specify the degree of the polynomial. So just look at all possible polynomials. Then of course, if you have sum of two polynomials, you get a polynomial and a scalar multiple of a polynomial is another polynomial.

So W is a subspace. And the in fact, the, you can specify the degree as well. So let us take k greater than or equal to one. And let Wk be the set of all polynomials of degree less than or equal to k. Then W k is a subspace as well. It is a W k is a subspace of W, which is in turn a subspace of V.

And, well, yeah, so here, why is Wk subspace? It is just that if your polynomials if your two polynomials of degree less than or equal to k, then there is sum as the degree of the sum is less than or equal to k as well. Same thing for scalar multiple.

And one has some interesting linear maps on the space as well. So let us say I want for k greater than or equal to one, the kth taking the kth derivative gives a linear map from which I denote D superscript k Dk from V to V. So in other words, if I start with an infinitely differentiable function, and I take it's kth derivative, the first thing to observe is of course, since it is infinitely differentiable, even after I take the kth derivative, the resulting thing itself is infinitely differentiable.

So I do get a map from V to V that written it as as follows Dk of f. So the input is a function f, it is value at a point t is given by just taking the kth derivative of f and evaluating at t. Of course, this is trivially it is a linear map. After all we know that taking the derivatives as linear operations, differentiating is a linear operation.

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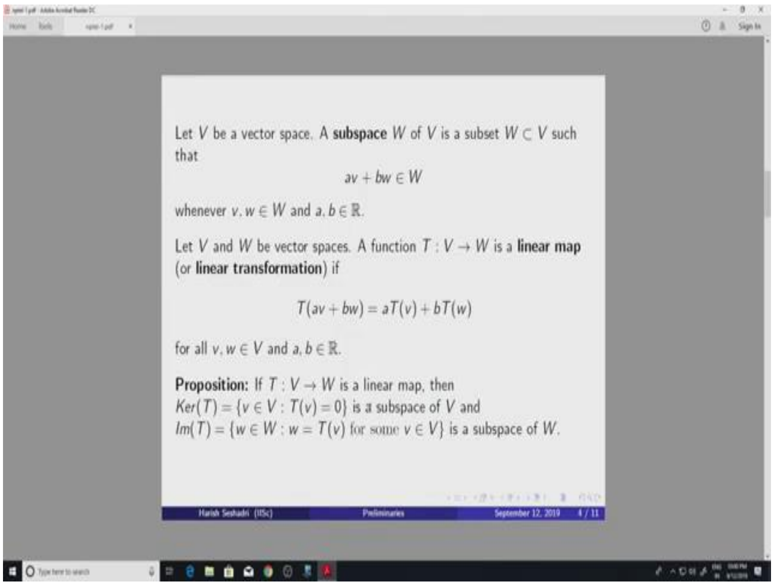

And what about its yeah, in the in the previous slide, I had remarked that whenever you have a leanear a map your two subspaces.

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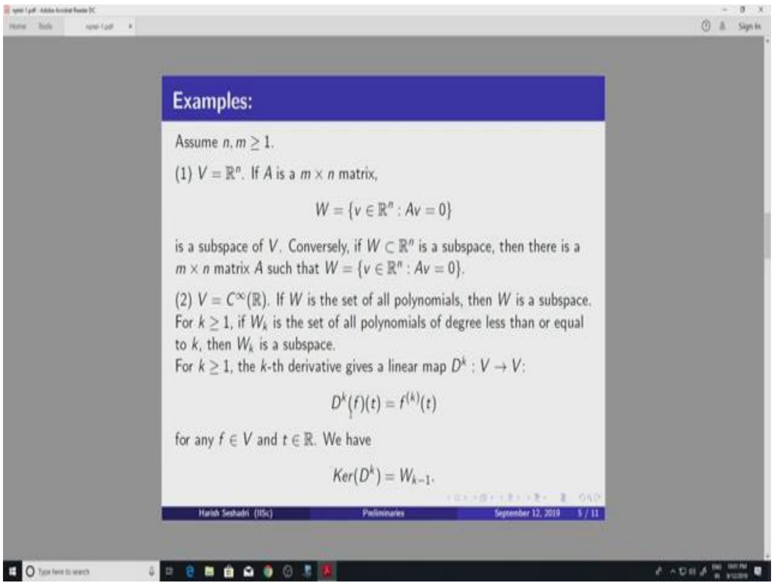

Well, in this case, what about the kernel of this kth derivative map? So to say that something belongs to the kernel of decay just amounts to saying that the kth derivative is zero and from that, it is quite easy to see that if the kth derivative of function is zero, then you just integrate k times and you will see that f is necessarily a polynomial of degree k minus one and the converse is also true, you start with a polynomial of degree k minus one, its kth derivative is zero.

So, the kernel of this map is Dk, is precisely this subspace W k minus one that I defined earlier.

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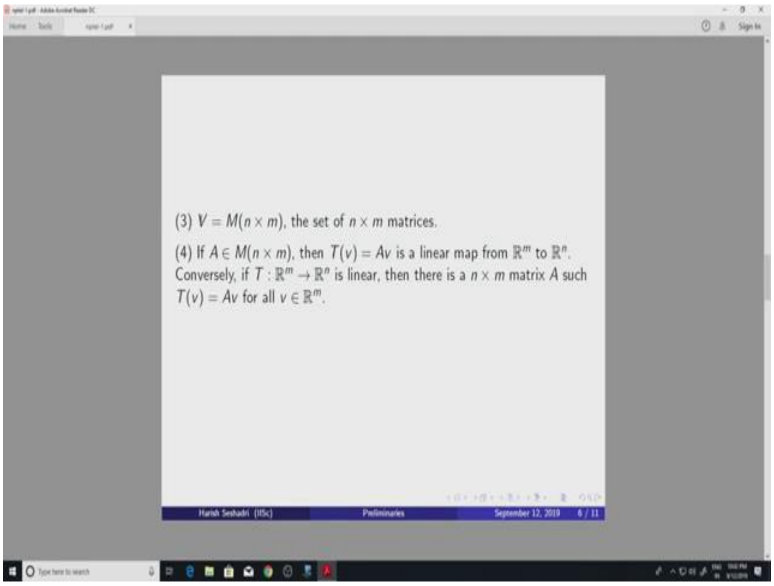

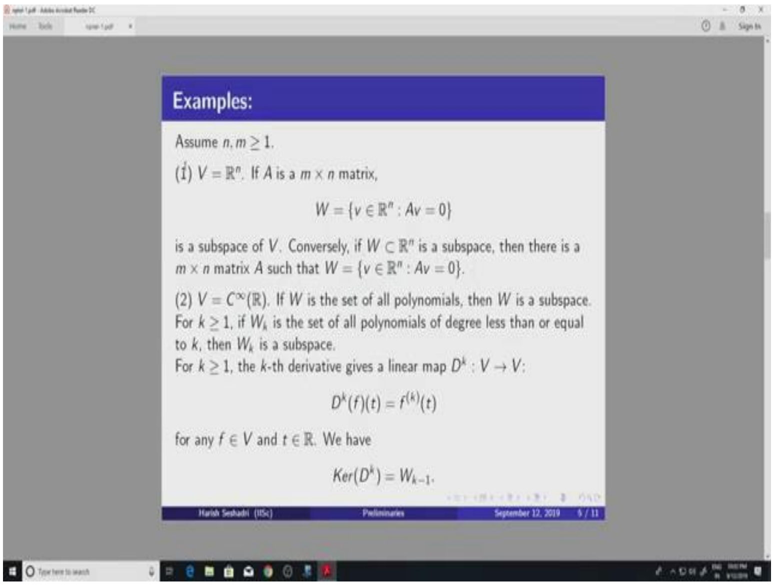

And, of course, one can also ask, What is its image which I will not discuss, right now. The next example is the vector space consists of all n cross m matrices, all n cross m real matrices. So one can add two matrices in the usual way and then multiply matrix by a scalar. So it does indeed become a vector space. And in this case, m, n equals m, namely square matrices is particularly interesting because it has some nice sub spaces.

But even if n is not equal to m, this is indeed a vector space. So what about now going back to the first example Rn, if what are all possible linear maps from Rm to Rn, if you have a n cross m matrix, then you can define a linear map just by multiplying take v in Rm and multiply with A. So Av will be an element of Rn. And it is a standard fact that any linear map from Rm to Rn can be described like this. In other words, if t from Rm to Rn is linear, then you can find a matrix A such that T v equal to A v for all v in Rn.

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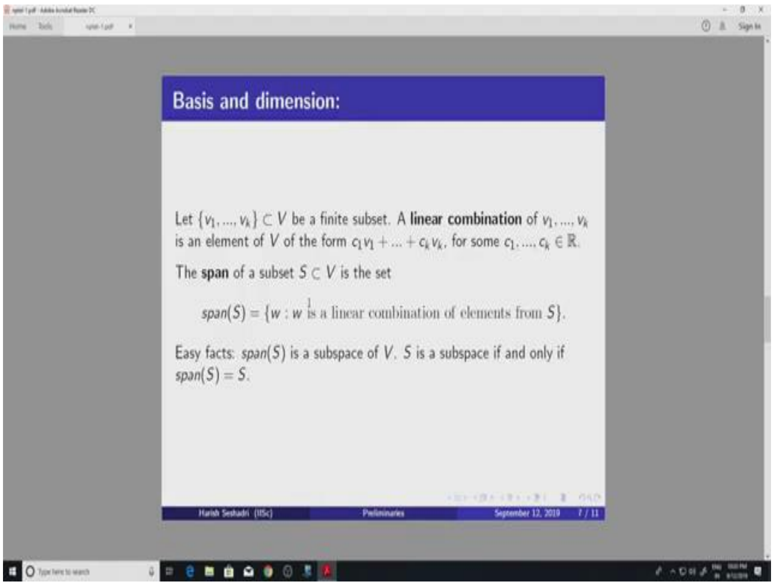

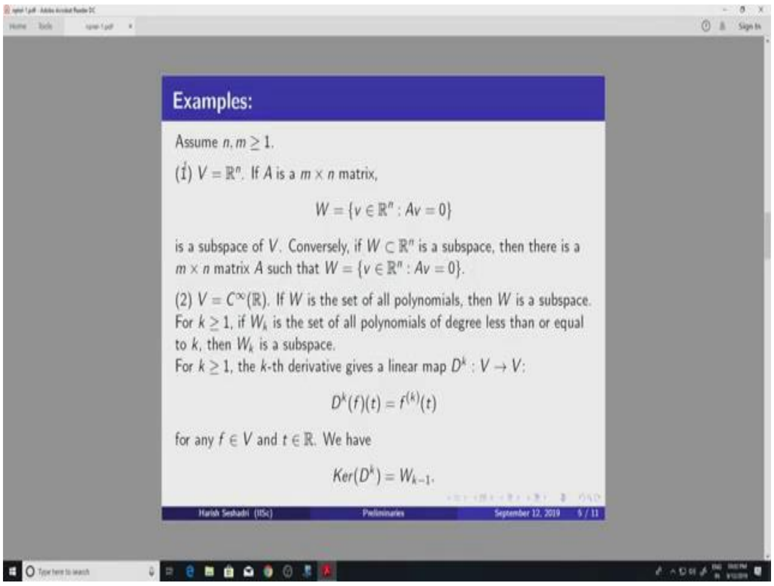

Now, we come to the important concept of dimension of a vector space. So, this is something very fundamental this will for instance, it will help us differentiate this, the first example and the second example, fundamentally different and the difference arises from this concept of dimension. First, let us recall that if you have a finite subset of V, finite set of vectors, a linear combination of v one v two v k is an element of V, which is of the form c one v one plus c two v two etcetera plus ck vk where c one c two and ck are all scalars. So, you just multiply each of these with some scalars and add them up.

And the span of a subset S is the set where is the set consisting of all linear combinations of elements from S. So I just take finite number of vectors in S, take all possible linear combinations, and then put them all together that is the span of S. So it is a simple fact to check that span whatever set S you start with the span S is a subspace of V. And, in fact, S itself is a subspace if and only if span is just itself.

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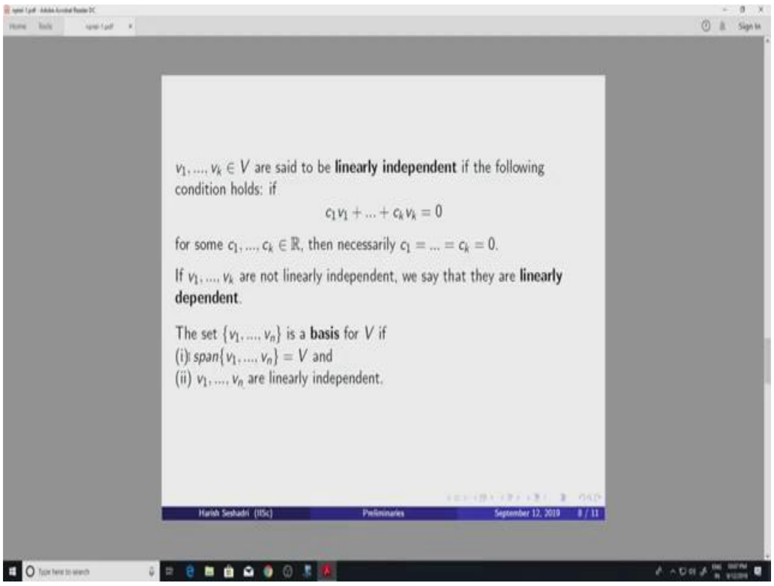

So, this is yeah. The other concept which we need is that of linear independence of vectors. So again you start with a finite set of vectors, the set to be linearly independent if the following condition holds, if, c one v1 etc, plus ck vk zero. So in other words, if any linear combination of them is zero, then necessarily all these coefficients have to be zero. And if v one v two v k are not linearly independent we say that they are linearly dependent.

So, the now we combine these two properties the spanning property and rather the linear combination property and this linear independence property. So, we say that a finite set of vectors is a basis for V if span of v one v two v n is equal to V.

In other words, so, this is another way of saying that every element of the vector space can be written as a linear combination of v one v two v n. And the other thing is that these vectors are linearly independent. So, these two properties as such have nothing to do with each other, but we demand that both of them hold for the set to be for it to be called a basis. Now, there are other ways of saying that v one v two these are linearly independent.

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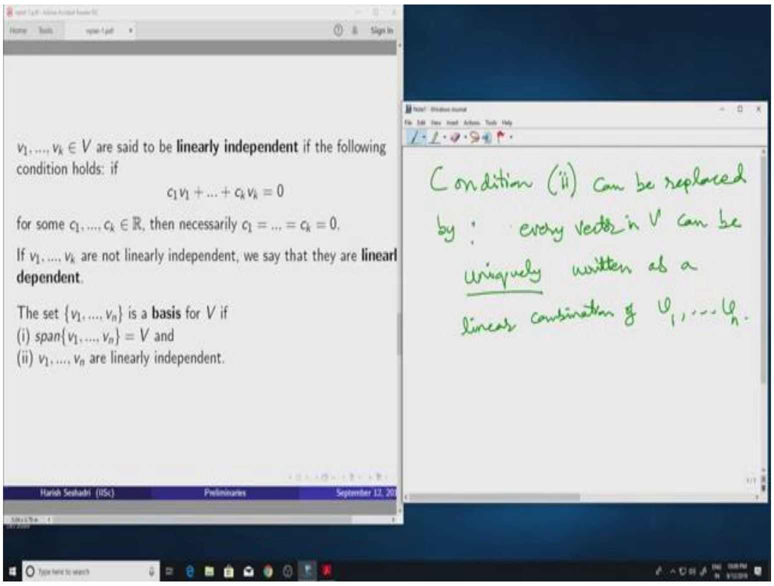

So, let me just write one other way of so, the first condition says that, every vector in v can be written as a linear combination of this v one v two vn and instead of saying that there linearly independent one can just say they can be uniquely written as a linear combination of v one v two vn. So, in other words, both of these can be replaced you can remove both one and two and put this condition here I just said condition two can be replaced.

But the here the emphasis in what I wrote is the uniquely part, uniquely written as a linear combination of v one v two vn.

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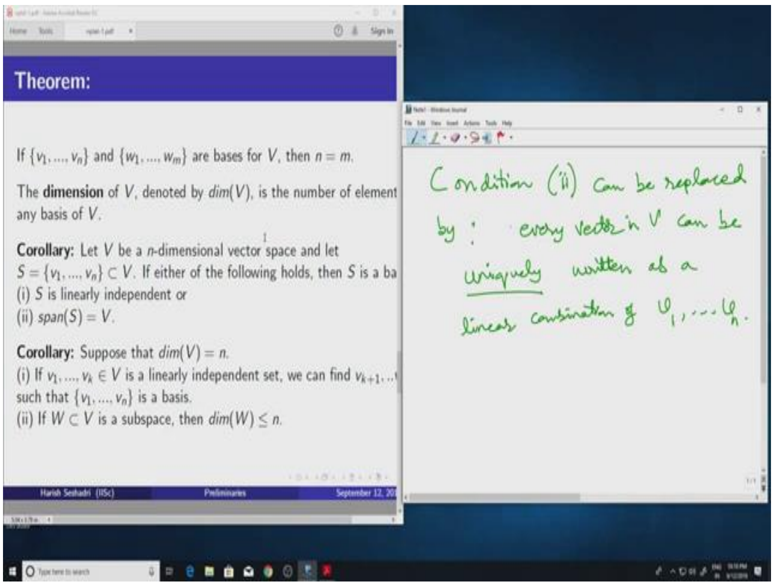

Now we come to the one of the Basic theorems and linear algebra, which is that if you have two different basis for V. Let us keep in mind that given a vector space, there is no guarantee that there is any basis at all. Basis in the sense of finitely many vectors satisfying these two conditions.

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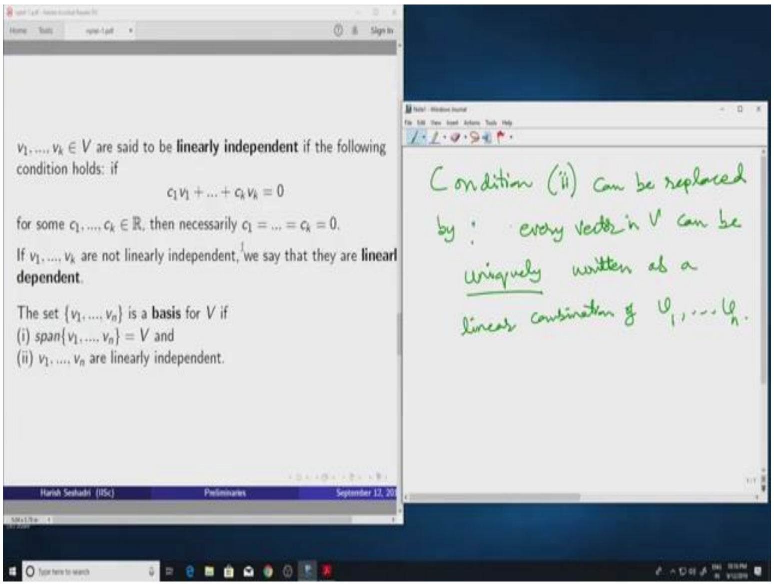

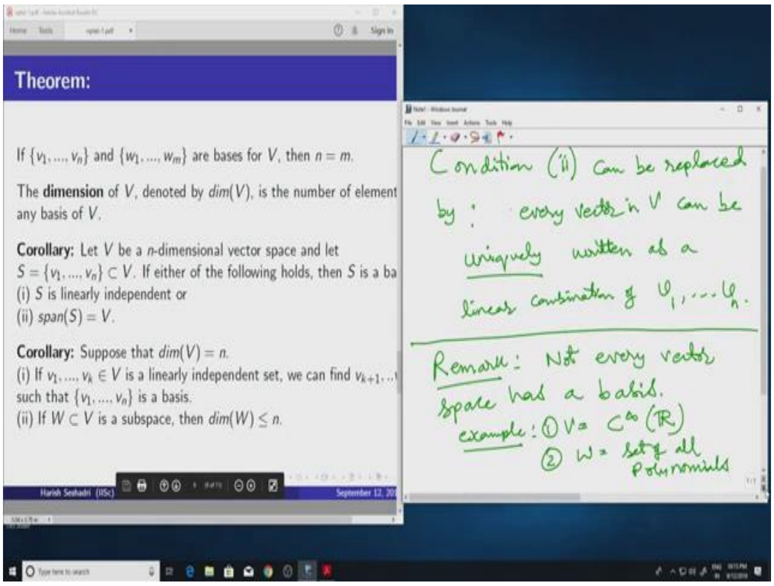

So, given any way, an arbitrary vector space, there is no guarantee that there is any basis. But what this says is that, if we do have two basis then the number of elements have to be the same in each. Of course, every vector space can have, if it has one basis it can have, it is easy to see that the it has infinitely many basis but whatever what the theorem says is whatever basis you take it has the number of elements has to be the same.

So, basis for this and then the dimension of V. So, once we know that it is the, this n equals m, then one can define the dimension of V, which we denote by dim V is the number of elements in any basis of V the whole point of this theorem is. This concept of dimension is a well defined number. Since we already know it does not matter which basis we take. So, we have a unique number associated to a vector space. Of course, this dimension makes sense only assuming that there is a basis to begin with. So, in fact, I should remark here. Well, not every vector space has a basis. As an example, we will just go back to the example of C infinity functions. So, V equal to C infinity R. This is one example. The second example is inside this W is the set of all polynomials.

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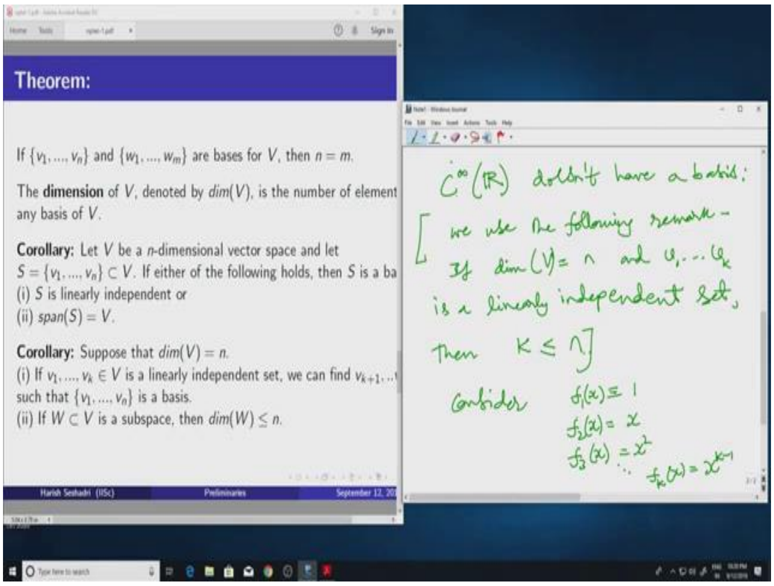

Neither of these, has a basis. Well, how do we see, for example, that C infinity R is does not have a basis. To see this in fact, what we will do is, we will go a bit further. And let us read some corollaries of, let us go through some corollaries of this statement that any two basis have the same number of elements.

First corollary is that start with an n dimensional vectors space. So here, of course, it means that V has a basis consisting of n elements. And I start with a subset S consisting of n vectors. So this of course, to begin with need not be a basis, it is just a set of n vectors. Now, what one says is that if either of the following two conditions holds, then S has to be a basis, the first condition is S is linearly independent that the set of n vectors of course, it is the important thing here is that this n and this n are the same.

In other words, the dimension and the number of elements in this set S have to be the same for this corollary to hold. So if you have n vectors, which is a linearly independent set, then we are assured that it is a basis or if we know that if its spans V, then also we know that it is a basis. So, what in effect this is saying is that normal definition of basis is that both of these conditions should be satisfied one and two.

Here what we are saying is if one of the conditions is satisfied, the other one is automatically satisfied. All this under the assumption that dimension is n and the set n S has n element that is one thing. Second thing is that we know suppose that dimension of V equal to n, again n dimensional vector space. If you have any… v one v two with any set of linearly independent vectors. We can find v k plus one etc. such that this is v one v two v n is a basis.

Here, there is already something implicit in this statement. Use the following remark namely that if dimension v equal to n and v one v two v k is a linearly independent set. Then the scale should necessarily be less than or equal to n. So that this statement actually should have come here. So, if v one v two vk is a linearly independent set, then k is necessarily less than or equal to n. And if k strictly less than n then we can find v k plus one etc. all the way up to vn so that v one up to vn is a basis of V.

So this is the part which I will use for to say that C infinity R does not have a basis. Next thing is that if W contained in V is a subspace, so then dimension of W is less than or equal to n in particular W itself has a basis. If I know that V has a basis, then it says that W has a basis and number of elements in the base in any basis of W is less than or equal to n.

So, but now, let me go back to this example that I was talking about that C infinity with this in hand, let us prove that C infinity are does not have a basis. Now, C infinity R so I use this remark that if suppose C infinity R has a basis, and suppose its dimension is n to have a basis means that I can talk about dimension I call the dimension n, then we know that there is an upper bound on the number of linearly independent vectors. Namely, there cannot be more than n linearly independent vectors elements of this space.

But we know that consider I look at the following elements the constant function one and so on. So, if kx is x to the power k minus one power functions, these are all of course, infinitely differentiable functions and they belong to C infinity R.

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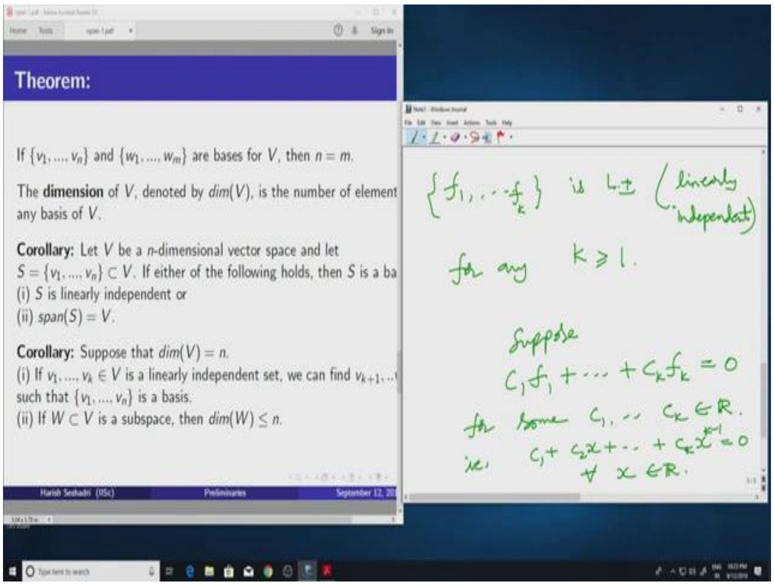

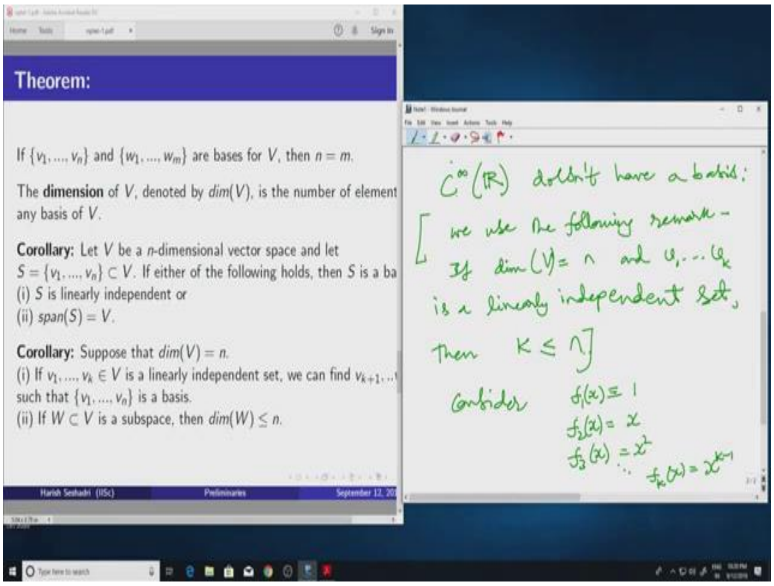

Well and there are infinite number of them. Then I also claimed that f one etcetera f 2 this set is linearly independent. Let me stop at some k does not matter is linearly independent. So this is shorthand notation I just wanted to write. Let me write clearly. So right. So I would like to claim that this is this set is linearly independent LI. For any k greater than or equal to one.

Well, this would immediately give a contradiction, because we know that if C infinity R had a basis and the basis had n elements, then we know that there cannot be more than n linearly independent vectors. Here we are saying that there is no restriction on k. You take any number of power functions and then you will get a linearly independent set. To see that this is linearly independent, what it just amounts to saying that if I take any linear combination of these, suppose this is true.

Well, this is a to say that this is zero. Let us keep in mind that this is actually supposed to be a function. So this function is the zero function. In other words, if I evaluate it on any x, it should be zero. So what we are saying is that c one, f one x is just the way we defined it, f one was a constant, C two F two was x, x to the power k minus one, zero. And this is supposed to hold for all x in R. So what we have is? we have a polynomial which is zero for all x.

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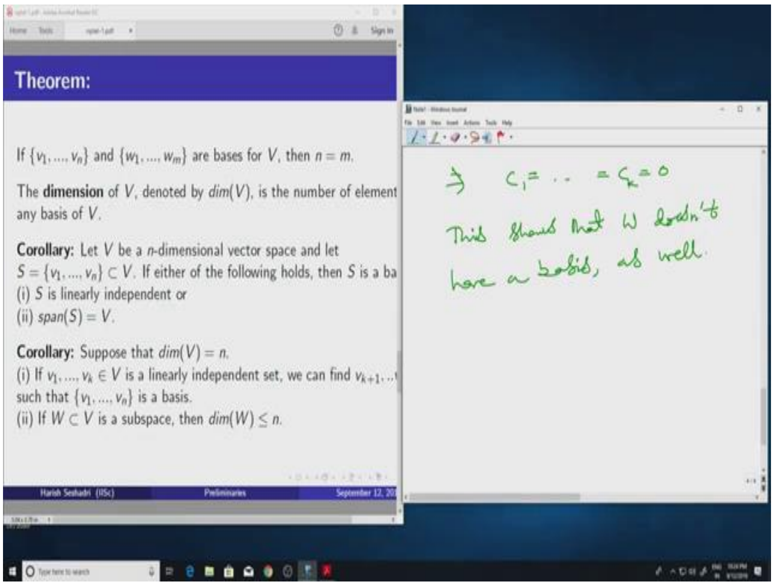

Of course, we know that this implies that if it is going to be zero for all x, the only way that can happen is all the coefficients are zero. So that shows that they are linearly independent. In fact, the same proofs shows that the set of all polynomials is also, does not have a basis as well. This shows W does not have a basis, the reason being that this power functions belong to W. So again, if it had a basis, the same argument will work.

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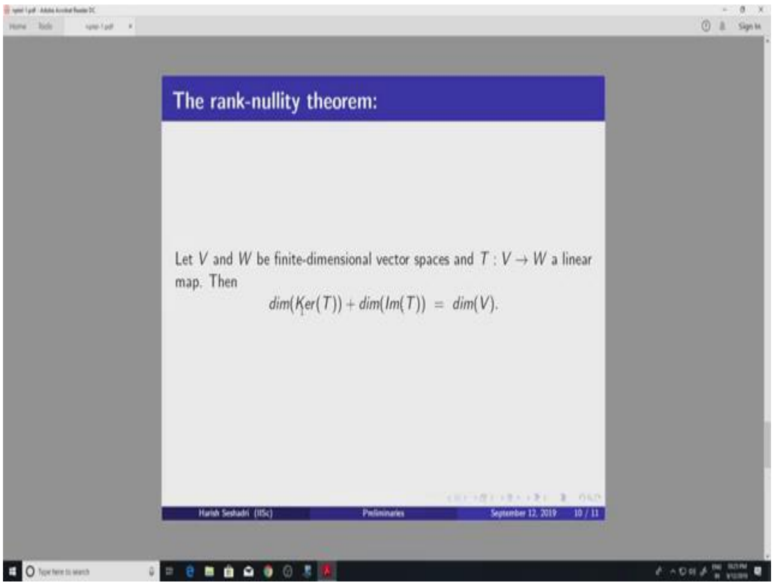

So let us recall one another important thing about which relates linear transformations and these subspaces associated to that and the concept of dimension. So whenever you have a linear transformation, so the assumption is V and W be finite dimensional vector spaces. This just is another way of saying that both V and W have basis in the sense that we have defined a basis. So finite dimensional vector spaces, and T from V to W a linear map, then the statement is that the kernel, which is a subspace of this, an image which is a subspace of this, the dimension of the kernel plus dimension of image should add up to that dimension of this space V, the domain space.

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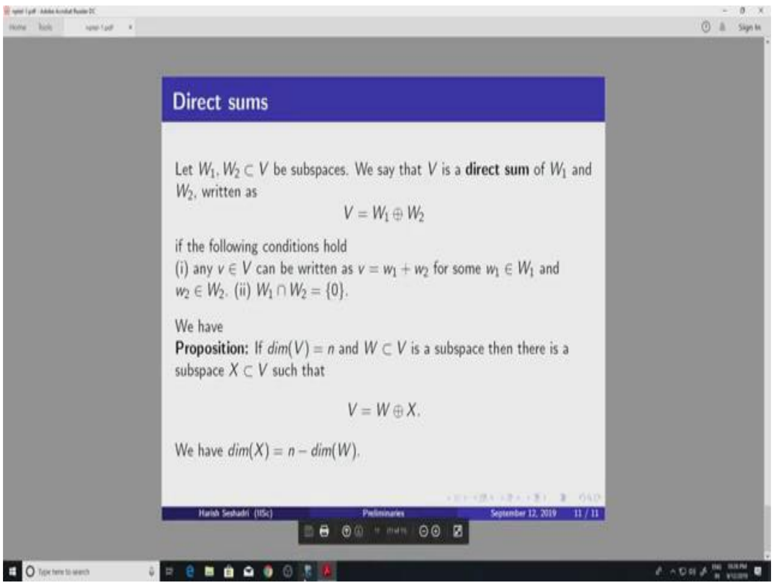

The last topic that I will deal with in this lecture will be the concept of a direct sum. So, if you have two subspaces, W one and W two, we say that V is a direct sum of W one and W two and we write it like this, V equal to W one directs on W two with the following conditions hold, first thing is that any v can be written as small w one plus small w two for some w one in this space and some w two in this space. The second condition is that w one intersection and W two should be this the zero element.

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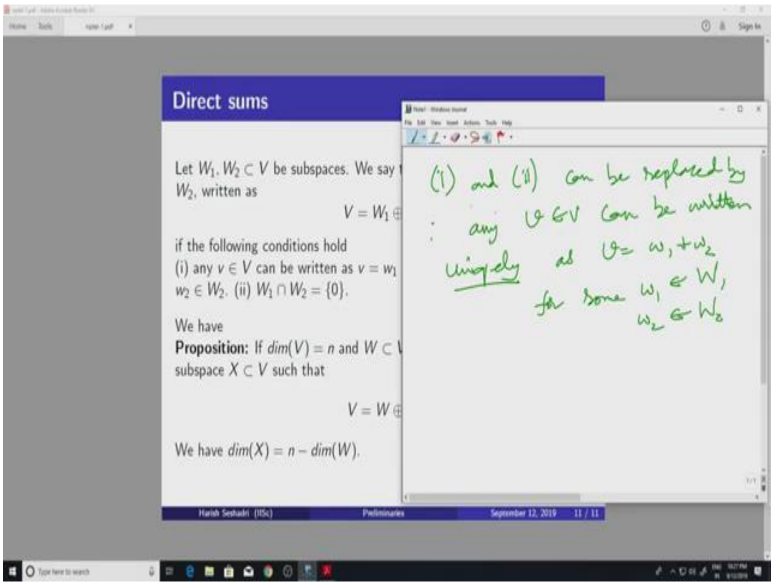

Again condition 2 in the definition of a direct sum I can replace, one and two can be replaced by a single statement. Any v in V can be written uniquely as v equal to w one plus w two for some w one capital W one capital W two. So, there are unique elements w one and small w one and W two given V, so that I can write it like this, this single statement is equivalent to both of these.

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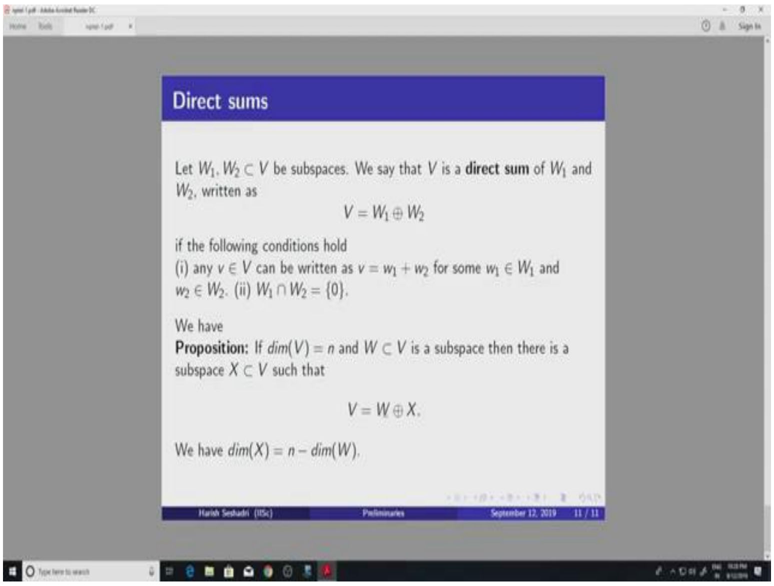

Then we have a in general if you start with any vector space and a subspace then there may not be any other subspace w two so, that v equal to written but if you assume that V is a finite dimensional vector space if dimension V is n and w is a subspace then one can check that there is a sub this statement there is a subspace x such that V can be written as W, the W we started with direct some x.

So, this is x is called… , x is not unique, there can be lots of subspaces which have the same property given W I can find many x usually, this is called any set subspace x is called a complimentary subspace of w. So, what we are saying is that in a finite dimensional vector space given any sub space I can find a complimentary sub space and the dimension of a complimentary subspaces just n minus dimension of this minus dimension of W.

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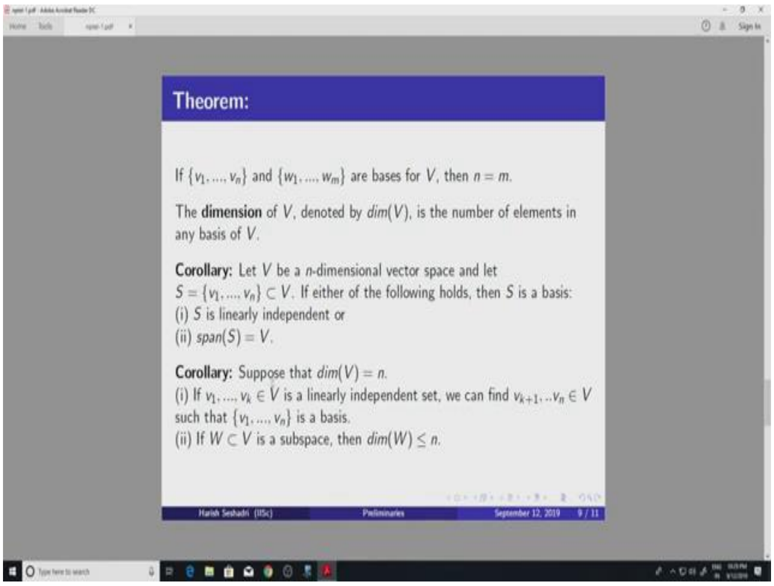

And the proof of this is just an immediate application of this corollary namely that if I given a subspace or other, this one that if I start with any k linearly independent vectors. I can expand that set so that I get a basis of the big space.

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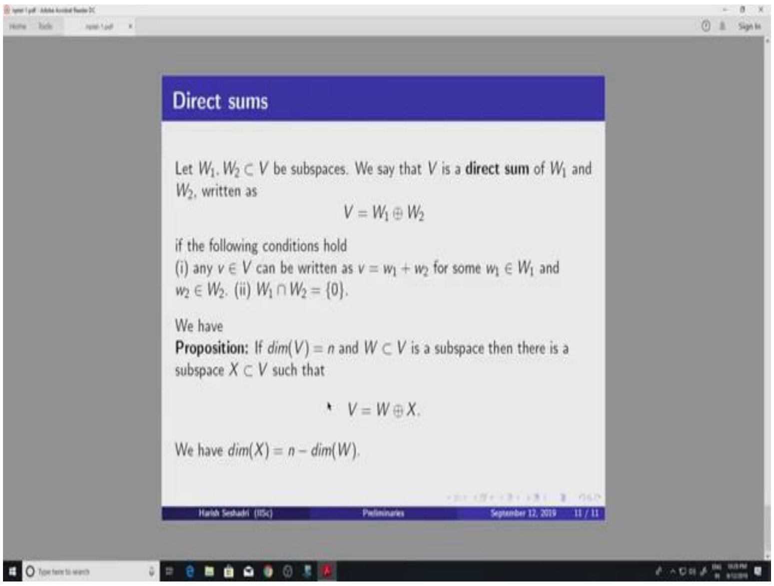

So using that, it is easy to check find a complimentary subspace. So with this, with this topic, so we will end this lecture and next time we will resume with multivariable calculus.