Ordinary Differential Equations
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Lecture - 40
General Second Order Equations Continued

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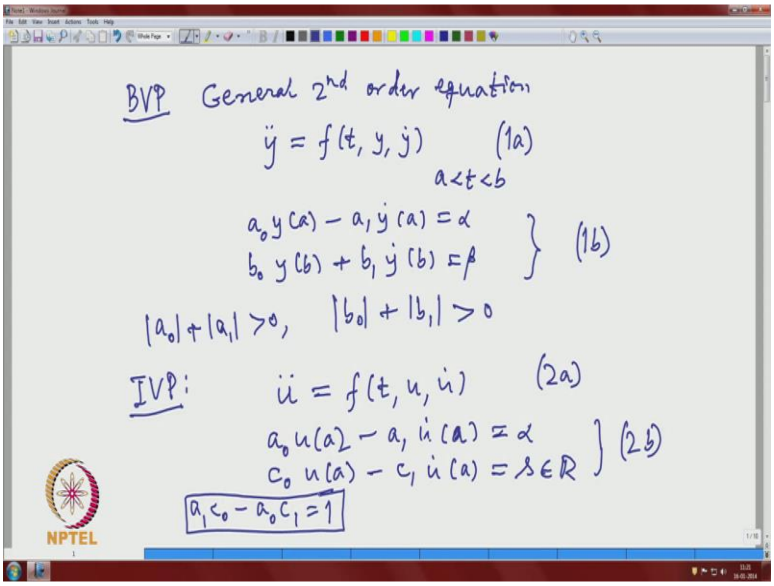

Welcome back. So, in the last class we were discussing boundary value problems for general second order equations. So, this is y double dot equal to f of t, y, y dot. So, this is the equation and the boundary conditions. So, this is in a less than t less than b and a 0 u a minus a 1 u dot a is equal to alpha b 0 u b plus b 1 u dot b equal to beta. Alpha, beta are given real numbers and so, this is 2 a. So, the boundary conditions, 1b.

So, the condition was this, a 0 and a 1 do not vanish simultaneously and similarly, b 0 and b 1 do not vanish simultaneously. A formal approach to a solution of this boundary value problem 1a and 1b, totally called 1, was through an initial value problem, discussed this last time, u dot equal to u double dot equal to f t, u, u dot, call it 2a. And now, we impose conditions on u and u dot only at t equal to a. So, this, the first one is, sorry for the thing, so this is not u, this is y, y b y dot u a minus a 1 u dot a equal to alpha. So, there is no change in the first condition.

And now, we want a, since it is second order equation, want to impose one more initial condition and this we want it to be independent of the first one. So, this is minus c 1 u dot a equal to s. s is a parameter at our choice and again a real number, and this is 2b. And by independence what I mean is, this a 1 c 0, we have fixed c 0 and c 1 such that this is 1.

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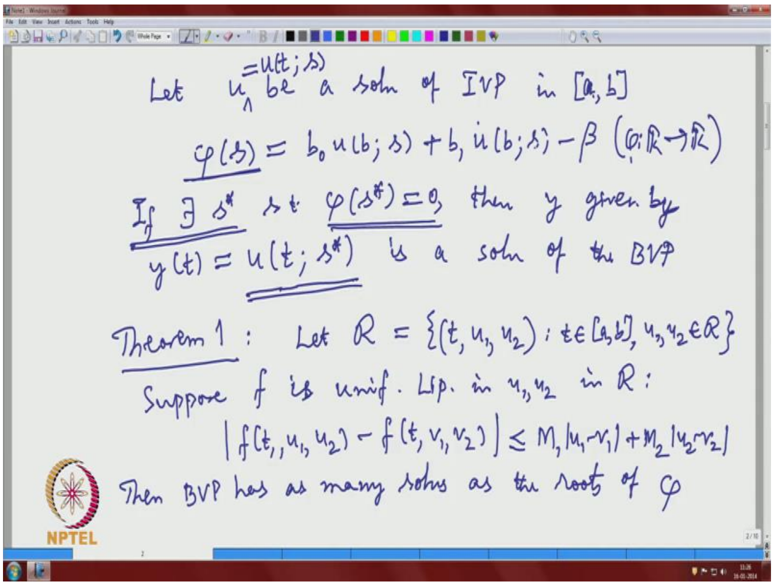

So, let u be a solution of IVP in the interval a b. So, last time we stated a theorem where some sufficient conditions were imposed on the right hand side function f, so that the solution of the initial value problem exists for all t in this interval. And with this u we form this real valued function phi of s. Remember, the second condition on u depends on s, so we form this b 0. So, u we emphasize the dependence on s. So, we say it u at t and emphasizing the dependence on s. So, this is u b s plus b 1 u dot b s minus beta.

So, just notice, that this phi f s is nothing but the second boundary condition. So, if there exists, so this is the conclusion we do last time. So, if there exists s dot such that phi of s star is 0, then y given by y of t is equal to u of t s star is a solution of the BVP, and this is called shooting method. So, so we in this approach we attack this boundary value problem though an initial value problem and this is a big if. So, so they, this in general, for every s star satisfying this phi star equal to 0, that means the roots of the function phi, we get a solution of boundary value problem, ok.

So, last time I stated a uniqueness result. So, let me just again recall, so theorem 1. So, let R be the region. This t u 1 u 2 t is in the interval a b and u 1, u 2 in R. This condition looks bit stronger, but that is sufficient for the solution of initial value problem to exist the entire interval a b. So, suppose f is uniformly Lipchitz in u 1 u 2 variables in R, so that last time I stated that, let me just state one more time. u 1 u 2 minus f of t, v 1, v 2 is less than or equal to some M 1 u 1 minus v 1 plus M 2, u 2 minus v 2. Then, BVP has, has many solutions as the roots of phi. So, for every root of this function, real valued function, so remember this phi is from R to R.

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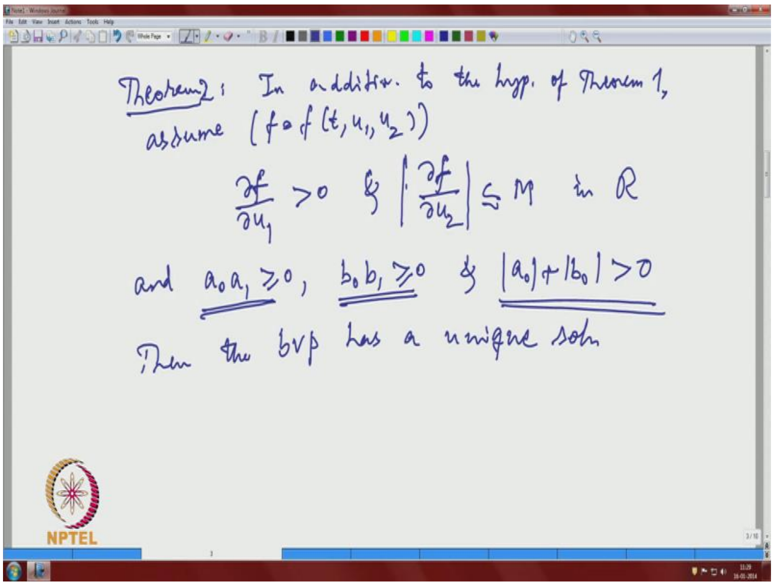

So, so now let us go to the second theorem, some uniqueness result, theorem 2 in addition to the hypothesis of theorem 1, assume. So, now some smoothness condition on f. Remember f is, this f of t, u 1, u 2, its function of three variables, t and e 1. So, this del f by del u 1 is positive and del f by del u 2 is bounded. It is not norm, just absolute value in R, that is the region stated in theorem 1.

And so, a 0 b 0 a 0 a 1 have same sign and b 0 b 1 have same sign and a 0 and b 0 do not vanish simultaneously. So, you already assume, that a 0 and a 1 do not vanish simultaneously and b 0 b 1 do not vanish simultaneously. Those are standing hypothesis on the boundary conditions. And in any addition to that now we are assuming, that a 0 and a 1 have both, both have the same sign, b 0 b 1 have the same sign and a 0 b 0 do not vanish simultaneously. Then the BVP has a unique solution.

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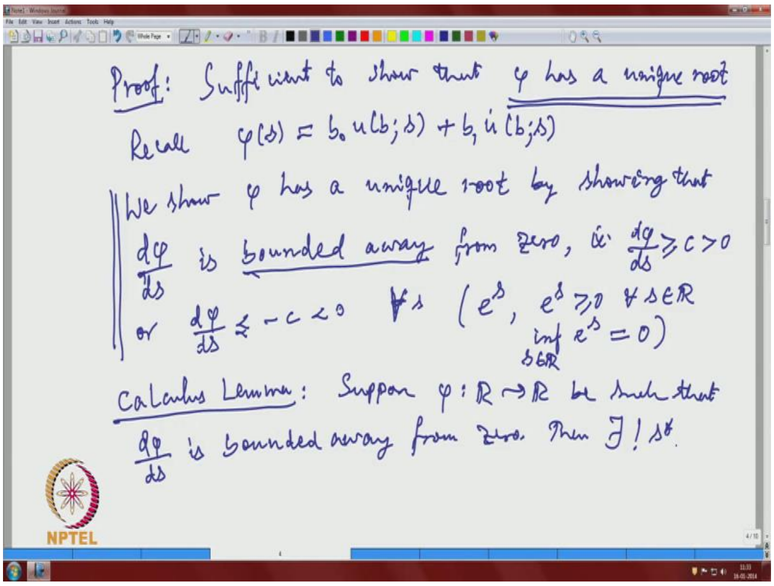

So, let me sketch a proof of this, it is really simple and just uses one variable calculus, but little technical in detail. So, sufficient to show, sufficient to show, that phi has a unique root. So, I we just recall the conclusion of theorem 1.

The BVP will have as many solutions as the roots of phi. So, if we show, that phi has a unique root, then BVP will have unique root. So, again let me recall. So, recall this phi of s is b 0 u b s plus b 1 u dot b s and u is the solution of the initial value problem, ok. And since phi is a real valued function defined on real line, so this is, this is accomplished. So, we show phi has a unique root by showing, that its derivative d phi by ds is bounded away and d phi by ds is bounded away from 0, that is, d phi by ds is bigger than equal to c positive or d phi by ds less than or equal to minus c strictly less than 0 or all s. So, either it stays all the time all, for all s it stays positive or it stays negative.

So, just as an example, ok, if you take e s, its derivative is e to the s, the derivative e to the s is bigger than or equal to 0 for all s in R, but it is not bounded away from 0. So, infimum e to the s, s R is 0. So, e to the s is not an example. So, once this is satisfied, so you can show that. So, this is a simple calculus lemma. So, suppose phi from R to R is be such that d phi by ds is bounded away from 0.

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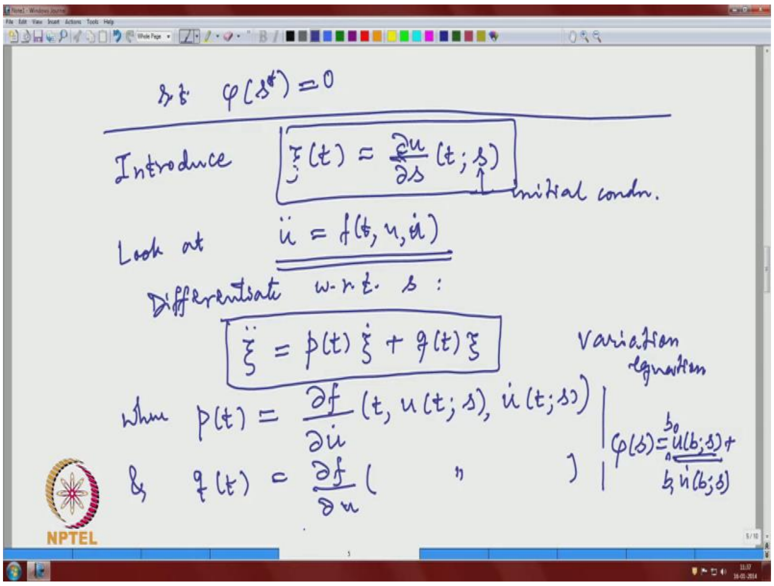

So, then there exists unique s star such that, such that phi of s star is 0. So, it is very simple thing. So, you just, it is not hard to see that, ok. Now, coming back to the, so proof. So, we have to establish, that d phi by ds is bounded away from 0. So, for this we introduce something. So, introduce psi t is du by ds t s. So, I am taking partial differential derivative with respect to s. Remember, s appears in the initial condition, so now, I am varying that s. So, I am considering u as a function of s initial condition, it appears in the initial conditions, and now look at the equation again for the u. So, look at, recall that. So, u double dot equal to f of t u u dot, ok. So, u now implicitly depends on this s. So, we will differentiate this equation with respect to s; so differentiate with respect to s.

So, this is, this is what we will learn in general theory, the continuous dependence on the, of the solution of the initial valued problem with respect to initial data. So, if you do that thing, then we get psi double dot equal to, so let me just write it p t psi dot plus q t z. So, remember this is psi, the derivative of t solution of the initial value problem with respect to s.

So, what is p? So, where p t, it is coming from this, you are differentiating the right hand side with respect to s and so, you are using just the implicit differentiation. So, this is what you get. So, df by du dot, so now everything you let me write once. So, then it is understood and q t is just, t does not depend on a. So, you do not have to worry about that and same thing. So, this is called variation equation. So, this plays an important role now, variation equation.

And remember why this psi is important. If you look at the expression for phi, phi we are analyzing this phi. Phi is nothing, but u b s b 0 plus b 1 u dot b s and we are trying to show the derivative of phi is bounded away from 0. So, what is the derivative of phi? Phi is nothing but this derivative of u with respect to s and that is precisely psi. So, that is why this role of psi is important, ok.

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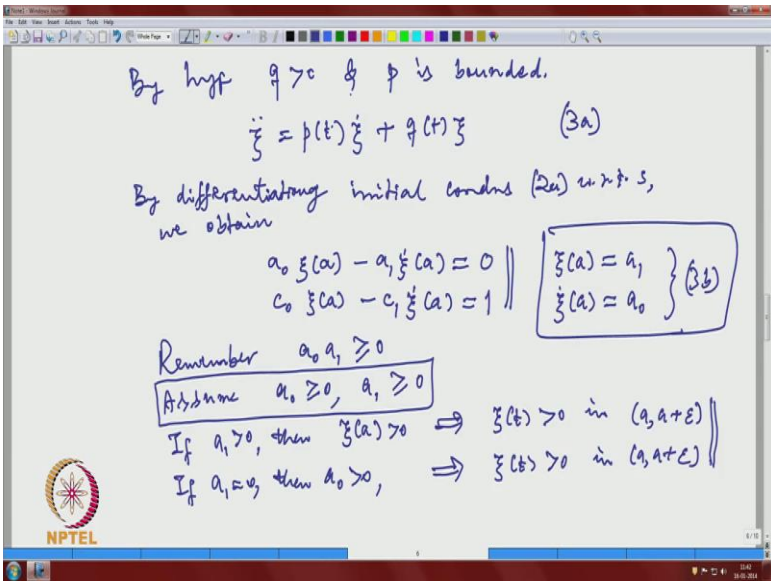

So, now you see, that from hypothesis, this q is positive. Let me say, by hypothesis q is positive and p is bounded, ok. We are going to make use of this. So, let me again rewrite here. So, psi double dot equal to p t psi dot. So, you keep on recalling this. So, this let me call it 3a.

And similarly, if you now differentiate the initial conditions to a, what you get is psi of a is equal to, you do that thing, so let, let me just write that. What you get is by differentiating initial conditions 2a with respect to s, ok. So, everything with respect to s we obtain, a 0 psi a minus a 1 psi dot a. This is 0 because this is alpha there. So, alpha does not depend on s, so that is we get that one. So, c 0 psi a minus c 1 psi dot a and there is an s here. So, when I differentiate with respect to s we will get 1 and by your choice of c 0 you immediately see, that psi a, if you solve these two equations, you get a 1 and psi dot a is equal to a 0. So, let me call this as 3b, ok.

Now, just concentrate on this condition and remember a 0 a 1. They have the same sign and they do not vanish simultaneously. So, let us fix. So, assume, so the other case is similar, assume a 0 is nonnegative and a 1 is nonnegative and look at the conditions 3b. So, if a 1 is positive, if a 1 is positive, then psi a is positive and this implies psi t is positive in a small neighbourhood in this interval a to a epsilon. So, there is no problem there. And if a 1 is 0, then a 0 is positive by our choice.

And look at the second condition, psi dot a equal to a 0, a 0 is positive, so psi is again increasing. So, again that implies, so psi t is in some a to a epsilon. This epsilon may be different. What we are trying to say here is, psi remains positive, with our this assumption important, if we assume both are negative, then we will have negative thing here, but for definiteness let us fix the signs of a 0 and a 1. So, that is that is what we get, ok.

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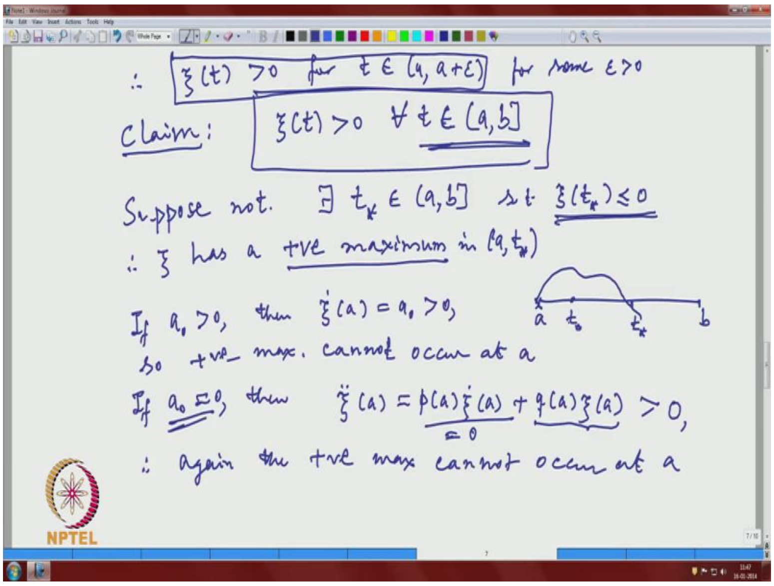

So, therefore, psi t is positive for t in a a epsilon for some epsilon positive. So, claim, so this is again very simple calculus argument, psi t is positive for all t in (a, b). So, up to b, the psi remains positive. So, only at a, it may be 0 or it may be positive. So, that is the claim. So, once we have this thing it is the proof follows very easily and this is also very simple argument.

So, suppose, not a calculus argument, suppose not. So, what does that mean? So, there exists some t star in (a, b) such that psi t star is less than or equal to 0. So, may be better draw some diagram. So, there is a, here there is t star there and there is b here. So, we already know, that psi is positive in some small neighbourhood. That is important, it is positive there. So, it may be 0 here and then, it comes again down. So, looking at the picture, so therefore psi has a positive maximum, it could be just relative maximum, no problem in a t star, and we want to rule out this a. t star is already ruled out because psi's t star is less than or equal to 0, so it cannot be a positive maximum. So, we want to rule out this positive maximum at a.

So, how to do that thing? So, if a 0 is positive, then psi dot a is a 0 positive. So, positive maximum cannot occur, cannot occur at a. And if a 0 is 0, then look at psi, psi double dot a from the equation. So, this is go back to the equation. So, psi double dot a is equal to p t psi dot a plus q t psi a. And now, we are assuming a 0 is 0. So, that means, this is 0 and this is q t psi a and psi a is equal to a 1. So, this is strictly positive. So, and at a maximum, the second derivative cannot be positive. It can only be less than or equal to 0. So, therefore again the positive maximum cannot occur at a. So, that means, what the positive maximum occurs at an interior point t 0. So, let us go back, ok.

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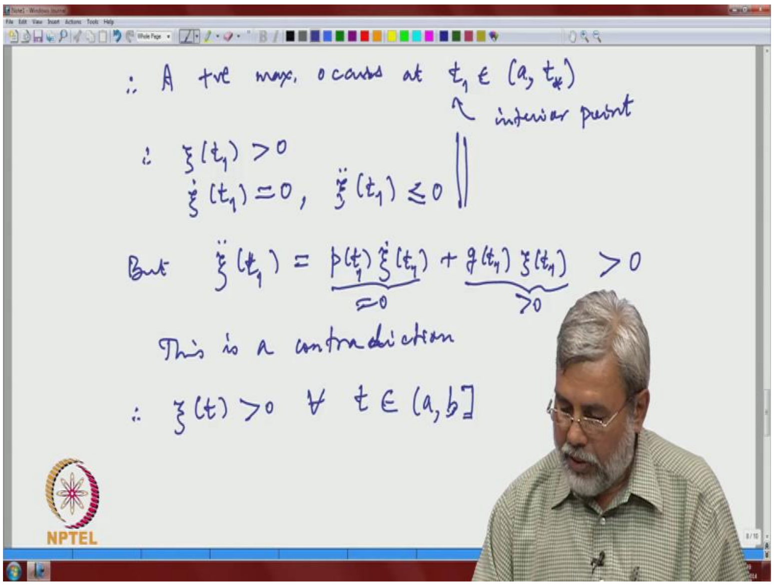

So, therefore, a positive maximum occurs at t, call it t 0 or t 1, t 1 in a t star. So, any at an interior point, so this is an interior point, it is not either a or t star. So, then you know, that when a maximum occurs at an interior point, so therefore, we have this psi of t 1 is strictly positive, psi dot t 1 is 0 because it is a maximum and one more condition, this psi double dot t 1 is less than or equal to 0. This is from calculus.

But again, you go back to the equation, variation equation. So, psi dot t 1 is equal to p t psi dot p t 1 psi dot t 1 plus q t 1 psi t 1 and again this is 0 and this is positive. So, whole thing is positive. But at a maximum, psi dot psi double dot t 1 is negative. So, this is a contradiction; so, this is a contradiction. So, therefore, our claim is true. So, therefore, what is our claim? Psi t is positive for all t in (a, b) up to b. So, that is important, ok. So, this is the first step and the remaining steps are very straight forward.

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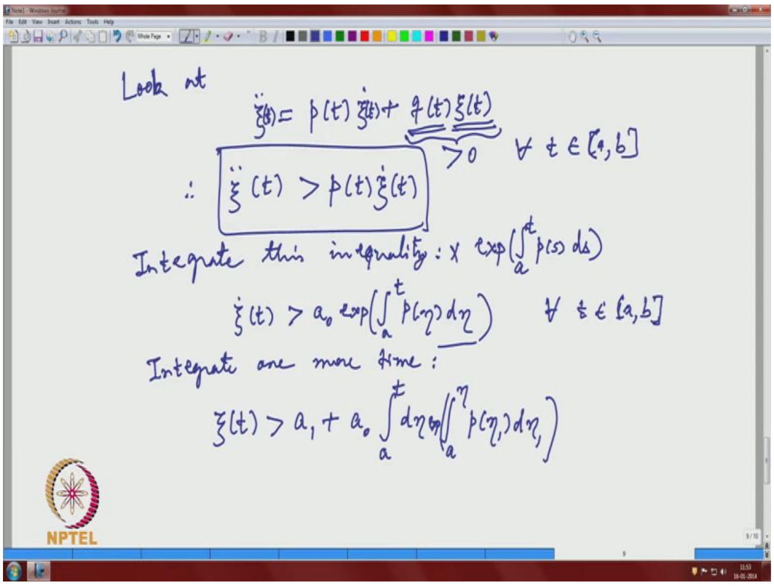

So, now look at, again go back to the, look at the variation equation, psi double dot equal to p t psi dot plus q t psi. So, everything at t. So, let me just write that. And just now we have proved, that this psi t is positive up to b and q is positive by assumption. So, this one is positive for all t in (a, b). So, therefore we have this inequality p. You can put greater than or just, and fortunately, we can integrate an inequality, we cannot differentiate an inequality, but we can certainly integrate an inequality. So, integrate this.

And preserving the sine of inequality integrate this, you will not do directly, so we have to multiply by exponential 0 to t. Let me write p s multiplied by this and then, this is the integrating factor. So, multiply and integrate. So, what you obtain is, so psi dot t is bigger than a 0. This is coming from the condition psi dot a equal to, not 0, here a. So, our point is a. So, this is exponential a to t p s t s, not t s, so let me use some other variable, may be eta, eta d. So, this is true for all t in a up to b, no problem.

And now this, integrate one more time. So, integrate one more time to obtain. So, finally, we have psi t, it is bigger than a 1 plus a 0. So, there is a double integral. So, let me write it first and then. So, a to t, let me write d eta and then exponential, as it is a to eta p eta 1 d eta 1. So, there is a double integral here, ok.

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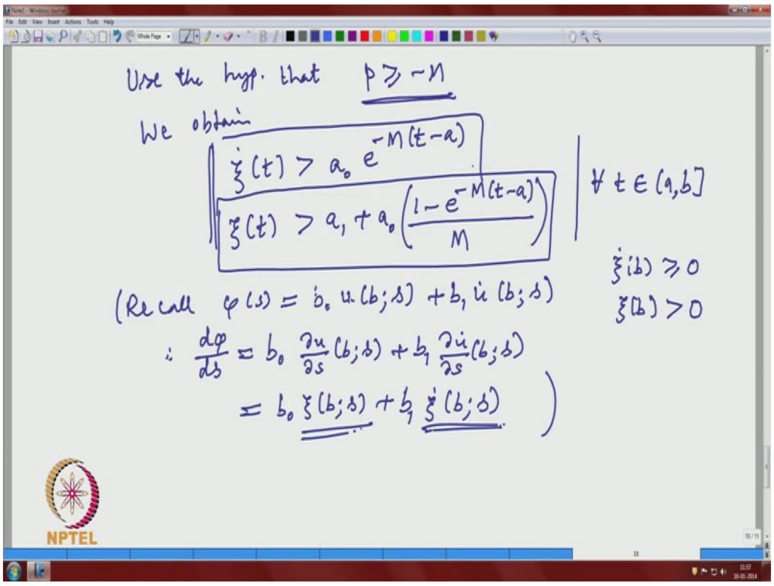

Now, you use the hypothesis. So, use hypothesis, that mod p is less than or equal to M. So, we are just using one-sided. So, p is bigger than or equal to minus M; M is some positive constant. And then, if you plug in this hypothesis on p, so we obtain psi dot t is bigger than a 0 e to the minus M t minus a and psi t bigger than a 1 plus a 0, that is, so we have to do some integration there. Let me just write it, e to the power M t minus a divided by M.

In the second inequality, there are double integral. So, if you do that double integral, you get this thing and this is valid for all t in (a, b). So, remember again, so you have to just remember that things. So, recall phi s is b 0 u b s plus b 1 u dot b s. So, therefore, d phi by ds is just b 0 du by ds at b s plus b 1 du dot by ds at b s. And by our notation, this is nothing but psi of b s plus b 1 psi dot at b s, ok. And our aim is to show, that d phi by ds is bounded away from 0. That you remember.

So, now we have got the expressions for psi and psi dot for all t, so in particular, for t equal to b. And let us see, whether they are bounded away from 0. So, that is our next task, just. So, if a 0, so look again, let me go back there, just look at this, look at here. So, both psi dot b is greater than equal to 0 and psi b since there is a 1 there and so, and a 0 and a 1 are not simultaneously 0, psi b is always bounded away from 0. There is no problem about psi. This is always bounded away from 0.

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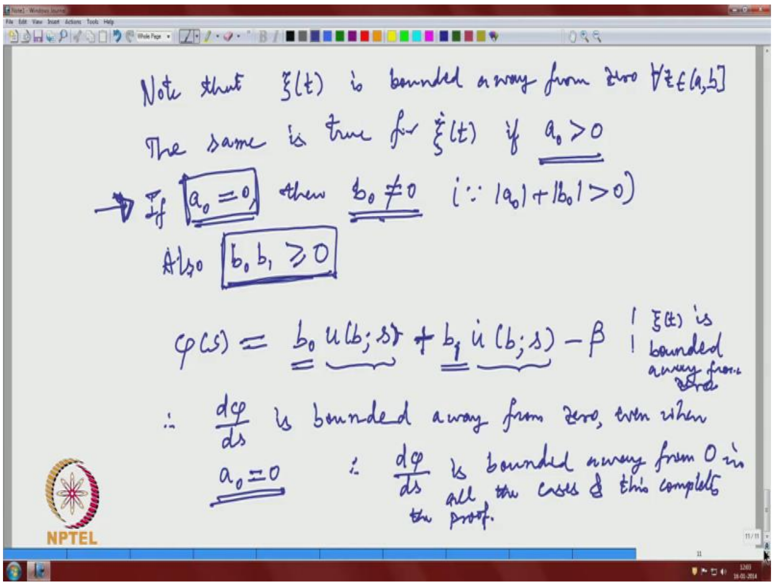

And this, in general, not that is just I want to, so note, that psi b is, now let me write b, let me write t, is bounded away from 0 for all t in (a, b). Of course, we are just interested in the final point p. The same is true for psi dot t if a 0 is strictly positive. So, the trouble occurs when a 0 is 0. And now, again look at the conditions. So, if a 0 is 0, then b 0 is not 0. So, this is one of the hypothesis, ok.

In the second theorem we want this a 0 b 0 should not vanish simultaneously and also, b 0, also b 0 b 1 have the same sign. If a 0 is 0, b 0 is not 0 and b 0 and b 1 have the same sign. And again, now look at phi s. Recall this b 0 u b s plus b 1 u dot b s. So, our only concern now is what happens if a 0 is 0. So, if a 0 is 0, then b 0 is not 0 and b 0 and b 1, they have same sign. So, if b 1 is 0 that is no problem. If b 1 is not 0, b 1 and b 0 have same sign. And we have already seen, that this derivative of u with respect to s is bounded away from 0 and this is just, this infimum could be 0. But in this case, we have b 0 not 0. So, therefore, again d phi by ds, d phi by ds is bounded away from 0 even when a 0 is 0.

So, let me again recall this. So, if a 0 is 0 by our hypothesis, b 0 is not 0 and b 0 and b 1 have the same sign and that is important, because the b 0 and b 1, they let me, there is a beta here, that does not matter. So, there is a beta there, b 0. And when I take derivative with respect to s, that beta anyhow goes away. So, important, that b 0 and b 1 have the same sign and b 0 is not 0. So, when I take the derivative, it again I already know that, I already know that psi, psi is bounded away from 0. I already know that; I already know that.

The only problem with respect to psi dot, psi dot and this is now taken care of by this a, additional hypothesis. So, remember this is additional hypothesis. So, when a 0 is 0, we required, that b 0 is different from 0 and we also required, that b 0 b 1 have the same sign. And that assures, that d phi by d s is again bounded away from 0 even when a 0 is 0. So, therefore, d phi by d s is bounded away from 0 in all the cases and this completes the problem.

So, this very simple calculus, one-dimensional calculus tools, very nice proof. Of course, one can always question this whether they are necessary or they can be weakened and that is, we can always look into the literature and see if a weaker hypothesis are possible.

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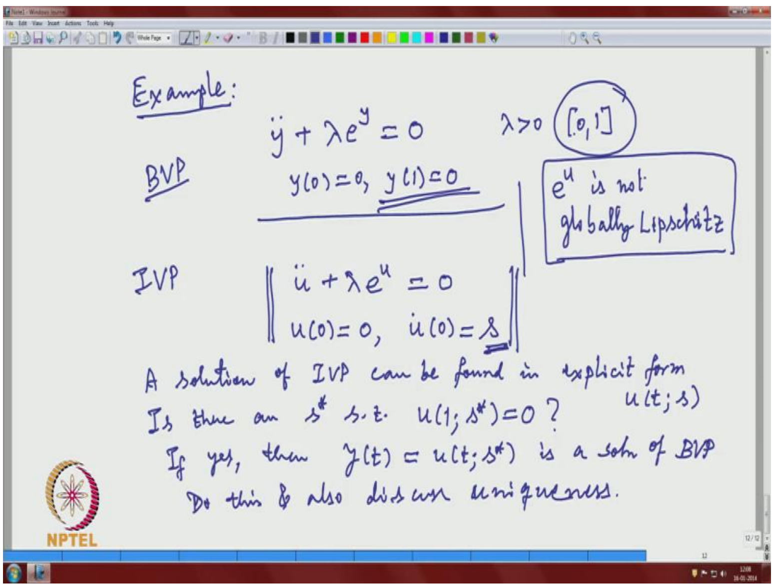

So, again let me recall that this is the one. So, again last time I stated this example, hope you have worked out. So, again let us recall that. So, this y double dot plus lambda e y equal to 0. So, y 0 equal to 0, y 1 equal to 0. So, this lambda is positive, ok. So, I approach this boundary value problem via this initial value. So, this is BVP; so, this is IVP. So, u dot u double dot plus lambda e u equal to 0. So, you put one this initial condition same thing and now, this is an independent initial condition. I put a parameter s. So, last time I mentioned, that this can be solved explicitly. A solution of IVP can be found in explicit form. So, do it. So, it is similar to the one we did in case of conservative equations. So, this is also a conservative equation, an explicit form, ok. So, using that explicit form, obviously it contains this parameter s.

So, next question is, is there an s star? So, you call this solution as u t s, is there an s star such that u at 1 s star is 0 because that is the condition we want at t equal to 1. So, that is the question. So, if yes, if yes, then y of t is equal to u of t s star is a solution of BVP. So, do this thing and also do this and also discuss uniqueness. So, why I am stating this example is, that this e to the u s in this e to the u is not globally Lipschitz, is not globally Lipschitz, it is locally Lipschitz, ok.

So, this condition is not really necessary. What we want is this solution to the initial value problem to exist for the interval in question. In this situation we have just 0 1. So, we are satisfied if a solution of the initial value problem exists for all t in this interval. We are not bothered whether it blows up outside that thing. So, there are, I mean the possibilities of weaker hypothesis. So, with that thing I come to an end of this simple discussion on boundary value problems.

So, we first discussed the linear boundary value problems in which we have the advantage of fundamental solutions from the linear system, but in the case of non-linear system our approach was through this IVP. And you can now refer literature and even see more general problems involving these boundary values. With that brief discussion on the boundary value problems for second order equations we come to the end of this course on ordinary differential equations. We hope that you the listeners have enjoyed the course.

The topics of the course were mainly based on the syllabus of ODE courses in most of our universities and other institutions. As you might have noticed, we have omitted some important topics, such as regular and singular Sturm Liouville theory and corresponding Eigen value problem or spectral problems and also systems with periodic coefficients and the corresponding flow k theory. At the same time we have added as a tradeoff some discussion on qualitative theory of autonomous systems and shooting method for the boundary value problem of a general second order equations.

The qualitative theory for autonomous systems is an important topic, not only in physical and engineering sciences, but also in mathematical biology, especially mathematical epidemiology where many problems have been modeled as first order autonomous ODEs. The subject field of ODE is quite vast and it is not at all possible to include all the topics in a single course to satisfy the various users of the subject.

We however hope, that those who have followed the course meticulously, will be able to read and understand the cited differences, further topics of their interest with our energy levels permitting and time permitting. We certainly hope to return with another course on ODE in future comprising of Sturm Liouville theory, flow k theory and some advanced optics in qualitative theory, and perhaps some topics in non-autonomous systems also. We would like to certainly hear from you regarding the contents and usefulness of this course. Your constructive criticisms are always welcome.

Thank you.