Ordinary Differential Equations
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Module - 5
Lecture - 26
2 by 2 systems and Phase Plane Analysis Continued

Morning, welcome back. In the last lecture, we have seen that every 2 by 2 system is linearly equivalent to one of the 3 types. That depends on the Eigen values of the matrix. If the system, matrix has two real and distinct Eigen values, then the given matrix can be diagonalized, and if the matrix has a real, but double Eigen value, then it goes to a form of the type lambda 1, 0, lambda, and the third type corresponds to the complex Eigen values. We also had the detailed analysis of the type 1, for the Eigen values are real and distinct, and the system corresponds to what is called as the equilibrium point; what is called a node; and if the Eigen values lambda mu, the product of the Eigen values is positive and the product of the Eigen values lambda mu, is negative; then it is called a saddle point equilibrium and that is a unstable equilibrium. When lambda mu greater than 0, and both the Eigen values are positive, then the equilibrium point is unstable and if lambda, both lambda and mu are negative, then the Eigen values, then the corresponding equilibrium point is stabled. So, that is a type 1 analysis, completely.

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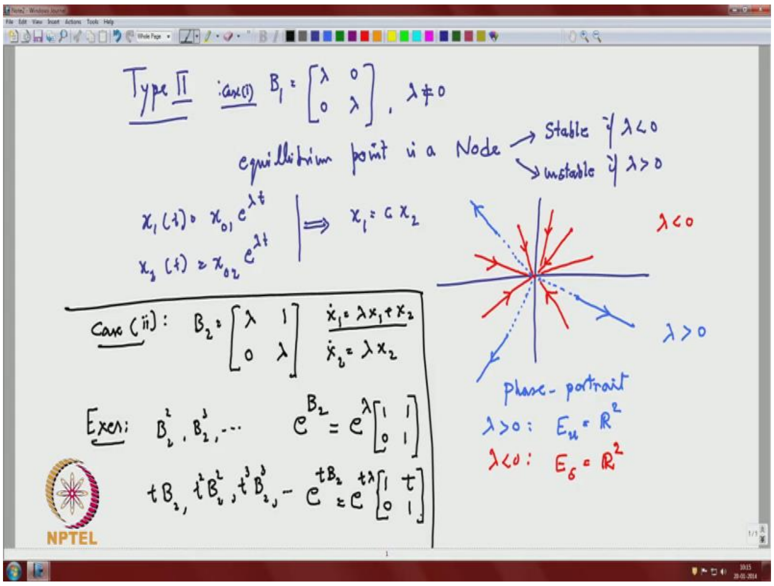

Now, we will quickly go to the type 2 and type 3. We will go to the type 2. In the type 2, there are two cases. When B 1 is of the form, B 1 is equal to lambda, 0, 0, lambda; this is the case in the type 2. There are 2 cases namely, case 1 here, where, the Eigen value is a double Eigen value, but it has two independent Eigen vectors. In this case, of course, lambda not equal to 0; this situation has absolutely, no difference; it is exactly like type 1 where, you can write down lambda equal to 1 and it is like a node. So, the equilibrium point is a node exactly, like in the previous class of the type 1 and absolutely, no different. It is stable, if lambda negative; it is unstable, if lambda positive. The solution can easily be written as in this case, x 1 t is equal to x 0 1, into e power lambda t, and x 2 t is equal to x 0 2, e power lambda t; same lambda. This, if you eliminate t, you will get x 1 is equal to a constant into x 2. So, it is nothing, but straight lines, all the time.

If you have the phase portrait here, you have the origin as an equilibrium point; x 1 equal to x 2 is nothing, but a straight line. If you have, this is the case. So, if you start any point here, and if this is the situation when lambda negative; it will be stable. So, it will move along that, and it will, this is the (( )). So, any point you start, it will be completely like this. Wherever you start, it has straight line. So, the entire trajectories are given like that. Like this, it will go. So, that is a phase portrait for lambda less than 0. If lambda greater than 0, I will draw it in the same thing. If lambda greater than 0, it is the same thing, if you start with, but then it will move away from that. So, this is for t negative part. From here, if you start; if you start anywhere, it will move along the straight line. This is the point, t negative part. So, it will move along this trail. So, trajectories are straight lines. So, if you start from here, again, if you start from, this is the starting point. From starting point, it will go along this straight line.

So, you will have unstability. This is the phase portrait; phase portrait of two systems; it is not just 1 phase portrait. For lambda less than 0, you have the complete. So, phase portraits for lambda greater than 0, you have your unstable thing; that is nothing, but completely r 2, and for lambda less than 0, it is a completely stable. There is no unstability or instability. So, you have the complete e v is equal to the stable subspace and unstable subspace are given like that. So, there is nothing. Now, we will go to the case 2. In the case 2, if you do, again, it behaves exactly like the node case to that, but not real difference between type 2 and type 1, when the real distinct Eigen values. Only thing in the type 1, when lambda mu less than 0; it is a saddle point equilibrium. That is the only difference; otherwise, it is a node. So, for this case, the B 2 is of this form; lambda, 1, 0, lambda. So, the corresponding system is x 1 dot is equal to lambda x 1, plus x 2, and the x 2 dot is equal to lambda x 2. So, the lambda for x 2; it is an independent thing. So, you can solve this system separately, if you prefer, because if x 2 can be solved immediately, because it is a decoupled part. Once you find the x 2, you can put it here for x 1, and that is a non homogeneous system for x 1; otherwise, you can immediately compute.

So, here is an exercise for you to compute now. This is an exercise 1, should do it immediately. You, for this matrix, for diagonal matrix, you already know, if it is a diagonal entries; lambda 1, etc. lambda n; e power the whole thing is the computation of exponential. So, it is a diagonal matrix. The diagonal entries will be e power lambda, but how does it look like if for a B 2? So, that is where, the exercise. So, you compute your B 2 square, B 2 cube, etc. and then you compute e power B 2. This computation is easy. I told you we are doing this, because the computation is not easy for us and also, for general matrix; it does not reveal the trajectory. So, this, you can compute to see that e power B 2 is e power lambda, 1, 1, 0, 1 and also, compute, because we are interested in writing the solution. You also compute t B 2, t square B 2 square, t cube B 2 cube, etc. and write down e power t B 2. This exercise is not hard; you just have to do it familiar. You get e power t lambda into; the matrix will be 1 here; t will be here; 0 will be, 1 will be here. So, you can write down now, the solution immediately.

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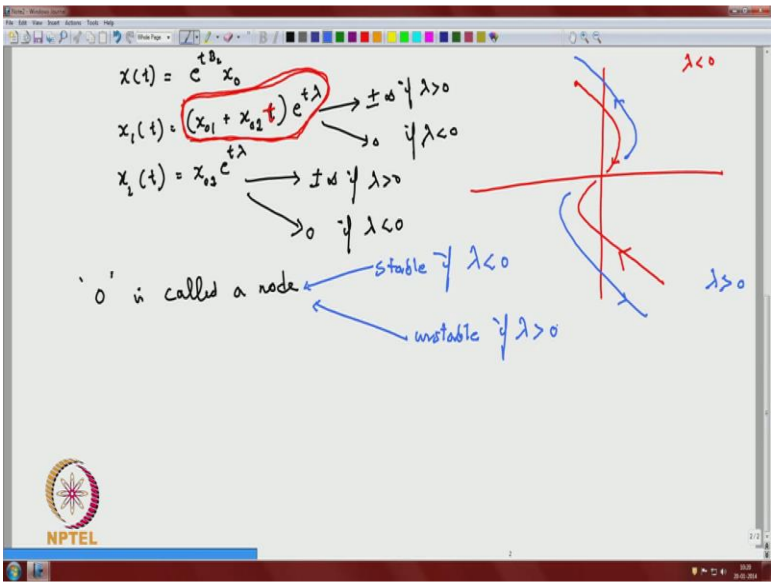

So, your solution will be, you can do it either, computing e power b 2 or you can directly, solve for this system. So, your solution x t will be for this system; e power t b 2 into x naught. If you decouple this again, you will get x 1 t is equal to immediately, you can decouple it; x naught x 0 1. So, you will write down your solution, x 0 1; you can do the computation; x 0 2, the initial values of t, e power t lambda; x 2 t is equal to x 0, because it is independent of that.

So, x 1 would not come there; x 0 2 into e power t lambda, you see. So, this will go to plus or minus infinity, if lambda positive, as t tends to infinity. This will go to 0, if lambda less than 0. This is the same case for this also. This will go to plus or minus infinity, if lambda positive. This will go to 0. So, you will have stability, if lambda (( )). So, it is again, the 0 is called a node; it is exactly like that, called a node. The only difference is that because of the appearance of this term, when you plot the trajectory in the neighborhood; it may have a different behavior, but then there is a t here. This will go to infinity. The effect of x 0 1 will be wiped out, you will see. So, it is eventually, it will behave like this term because of the t.

So, if you plot the solution here, in a typical case, if you phase portrait, if you do; if you start here, in the neighborhood, there will be a twist, because of a turn like this, I told you. So, the exercise again, is to plot these curves in, properly. You may, if you start with the trajectory for lambda less than 0, may come, if you start with. Initially, there will be some twist; it may behave like that. It will go to the origin, if you start here. So, from here, if you start it may. So, it will go to 0, if for the lambda case. So, it will be something like that. There may be some trajectory difference of this thing, because of the neighborhood of the origin, there will be, but eventually, it will go to the origin. On the other hand, for lambda positive, if you want to see the lambda positive case, the trajectory will be the same, but then it may, if you start from here, the trajectory may go like this; the same behavior will be there.

If you start with only thing, it will go to infinity. It may, it depends on the initial values. In the neighborhood, there may be a motion difference, but eventually, either, it will go to 0, along a curve, given by this x 1 t and x 2 t. It is given by the parametric representation, x 1 t x 2 t; it will go to 0, as lambda less than 0, and it will go to infinity, plus infinity or minus infinity from which, quadrant you are starting with and the behavior of that one. So, basically, it depends on this here; the sign of that. So, it is a case of node. So, the type 2 is exactly, one case of type 1 in the node situation. So, again, the equilibrium point in all this is called a node, and there, a node is stable, if lambda negative, and it is unstable situation; unstable, if lambda positive. So, that completes the situation of type 2.

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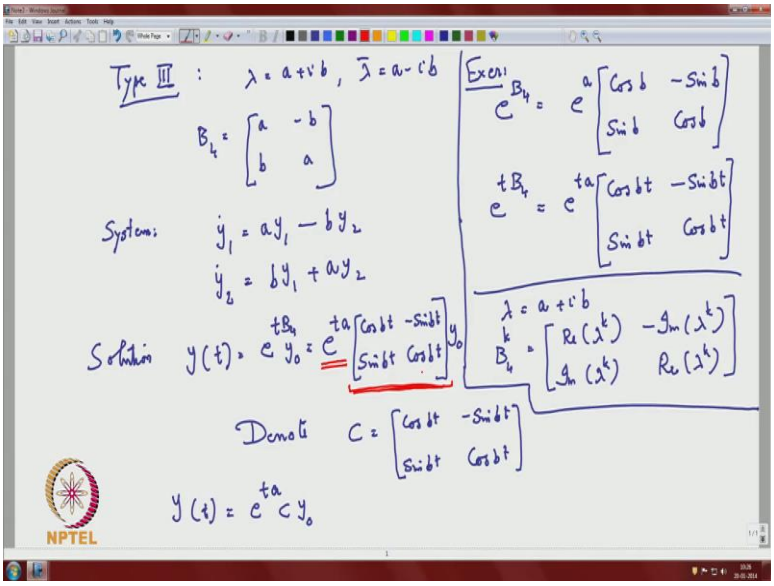

So, you have your type 3. You have your Eigen values, lambda is equal to a plus I b; lambda bar is equal to a minus I b; and your matrix B 4 is equal to a minus b, b a. This is the case. So, your exercise will be to compute, for this special type. For the general matrix, it is difficult. The matrices we came upon, is easy to compute. So, I want you to compute as an exercise, e power B 4; that is what our always, the job will be; e power a. Now, computation will be cos of b, minus sin of b; it will be sine of b, cos of b; this is your e power b, but we are interested in e power t b 4. So, you have to compute this one. If you compute this one, you will get e power t a, and matrix will be cos power b t, minus sine b t, sine b t and cos b t. You can directly compute B 4, B 4 square, B 4 cube, b, etc. or t B 4, etcetera.

Another way to compute probably, this is then may be this easy to compute. You can just try, if lambda is equal to a plus I b, you can actually, see that a power k; you have to compute; not a naught a plus I. You will have B 4 power k will be; please, check this; the validity of this one, it will be real part of lambda power k, minus imaginary part of lambda power k, real part of lambda power k. That is why you get it, cos b. If you add it

B 4 power of that factorial, you exactly, get cos b t, sine b t, etc. If you can do it, that one B 4 all the thing, you have to work with t B 4. So, correspondingly, there will be t here. So, you can do this thing, exactly. With that, you can write down your solution, now. So, what is the system, corresponding to this one? Corresponding system in y system is y 1 dot is equal to a y 1 minus b y 2, and y 2 dot is equal to b y 1 plus a y 2. If you write your solution for this system, solution y t is equal to e power t B 4 y naught. So, we have already computed e power t B 4. So, it will be e power t a, into cos b t, minus sine b t, sine b t cos b t, acting upon some y naught. So, you have a representation of the solution. We want to see the phase portrait. The aim is to understand the phase portrait.

So, let us denote for the time being, denote c is this matrix, cos b t minus sine b t, sine b t cos b t. So, this is the exercise part. Therefore, your solution y t is in a short form here; e power t a, c y naught where, c is; this is a temporary notation. So, what is here, to understand the phase portrait; that is what you have to understand, now. As t tends to infinity, look at these terms here; depending on the sign of a, e power t a will go to plus infinity or minus infinity. If a is negative, t power t a will go to 0. If a is positive, t power a will go infinity, if it is. On the other hand, what will be this is doing? These terms, cos and sine, are periodic with period 2 pi; cos t and sine t. So, this shows the second terms here, when act on y naught, shows some rotation, because as t tends to infinity, these terms, this action of this c, c y naught; if you start with a point y naught, it will rotate around the origin. That is what c will do. So, c on the action of c on y naught, will try to rotate around the origin. On the other hand, the term t power t a, if a not equal to 0, and that will either, that is typically, the amplitude.

If a is positive, e power t a will go to infinity. So, it will start rotating, but rotation, the radius of the rotation, will start increasing if a is positive, because that is amplitude. If a is negative, that will go to 0, the amplitude. So, the e power t a will give you the amplitude of the rotation. This will give you the periodic rotations around that point. So, we have to distinguish the two cases, when a not equal to 0, and a equal to 0, because when a equal to 0, e power t a will vanish, and there is only a pure rotation. There is no amplitude change leading to what is called the periodic solution. We would be studying this thing in the non-linear analysis module, looking for periodic solutions of the trajectory.

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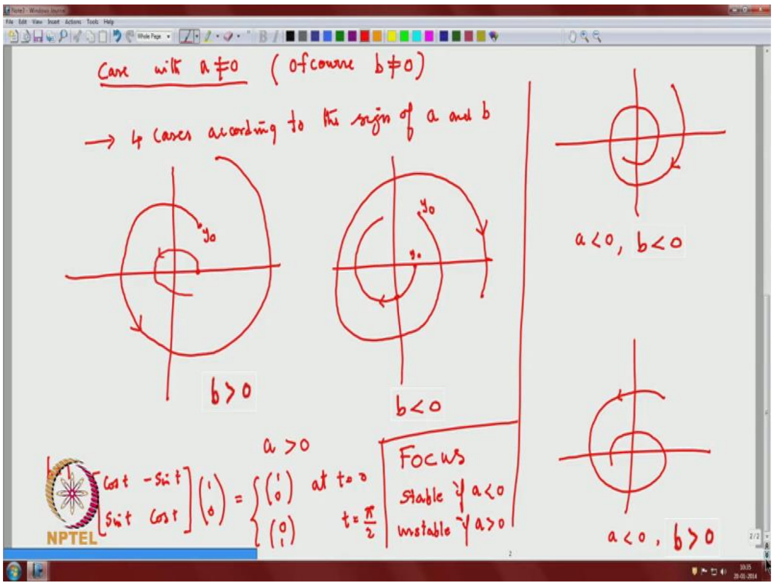

So, we will consider in these two cases, case with b, with a not equal to 0. Of course, recall that; of course, b is always not equal to 0 in type 3, because if b equal to 0, then there is no complex part. There is no imaginary part; the Eigen values are just real, and the real case, we have already studied. So, b is always no equal to 0 in type 3, because we are in the situation of complex Eigen value. So, we will start with the case, a not equal to 0. This leads to four cases, according to the sign of cases, according to the sign of a and b.

So, what are the four possibilities coming into picture? Let us try to understand one by one; the case 1. We will have two cases. First, we just plot the case when a positive. When a is positive, the amplitudes are increasing. If you are starting from any point here, it will rotate, but then rotation can take place; either, it will rotate in such a way that the amplitude should become bigger and bigger, in both cases, but the only thing is that whether, it will rotate clockwise or anticlockwise. So, there are two options for a positive; the rotation can take place in the clockwise direction, and the rotation can take place in the anticlockwise direction. What is the exercise I am going to suggest here; you try to state, if you want to say whether, clockwise or anticlockwise, versus, sign you have to choose; you start with some particular values of b and also, particular values of a. So, consider two specific situations; choose some b say, equal to 1, and choose a equal to 1. Then, you take b equal to minus 1 a equal to 1, and all that possibilities; you try to see that rotation, how it take place. To see that one specific case, if you want to see, let us take the case for, yes; I want to know that the rotation amplitude is going to infinity.

So, what will happen is that if you think that one go, one possibility is that it will take an anticlockwise direction and the amplitude will go; that is the case. So, your arrow will be something like that. The other situation; if you take, the second amplitude is increasing, the clockwise, it can go, you see; it is going to infinity. So, this is your situation with y naught. This is the y naught here. The only thing is that one will be for b negative; one will be for b positive. The best way to check is that you consider a specific case for example, with b positive; what will happen to that? If you want to check it here, say, choose b equal to 1. So, your matrix is cos t, minus sine t, sine t cos t; this is the way you have to look at it. Look at the vector, which is starting from 1, 0. So, I am choosing a vector from here, or you can see that vector from there; it is choosing. What will happen? This one, this will be; suppose, I take t equal to 0; that is the situation when, t equal to initial point is at the origin.

When t equal to 0, cos 0 is equal to 1; sine 0 is equal to 0, 0. So, it will be 1, 0, 0, 1. So, it will go to 1, 0 only at t equal to 0. Where it will happen? Suppose, I am taking a rotating pi by 2; I am taking t equal to pi by 2 at a later time. When t equal to pi by 2, cos pi by 2 is equal to 0, now; sine pi by 2 is equal to minus 1, and sine pi by 2 is equal to minus 1, which I want you to see that; that is equal to yes, and this is 1, 0; it will move to 0, 1; yes, if you compute. So, what will happen is that if I start from here, I will have my situations here. This is with, if I rotate here, next point will be; it will take this point, y naught. If I start y naught from here, it will go along this one; it increases. So, this is the situation with b positive clockwise, and this is a situation b negative. Now, this is the case with a positive. Now, you can verify, when two more cases will come, what are the two cases? Both these cases are with a negative. In this case again, the same thing, you will get a clockwise rotation. You will get a clockwise rotation, if you start, but then the amplitudes are reducing. So, it should go towards the origin. So, it will come here, like this. So, this is the situation with b positive.

Then, if you start from here, see, it will go to the origin, spirally. So, this is a negative with b negative. This is a positive with a negative with b positive. To see, of course, we are taking a minus b, b a; that is a matrix we consider. Certain places, we me take; this also, may be written in the form of a b minus b a, but you have to check the sign, properly; look for thing. So, the best exercise is that take some particular values of a and b with different, all the four combinations and try to focus here. This situation is called focus. So, we will have, this is called the focus. It is a stable focus in all these situations, all these four situations; we call the equilibrium point, a focus, and stable, if a negative. So, it is depending on the real part of the Eigen value. The stability always, depends on the real part of the Eigen value; not about the complex part. The sign of complex part determine the orientation of the trajectory. So, you have a focus, this unstable in the first situation, first two figures, and it is a stable in the next two figures. So, we have left with one more case with a equal to 0.

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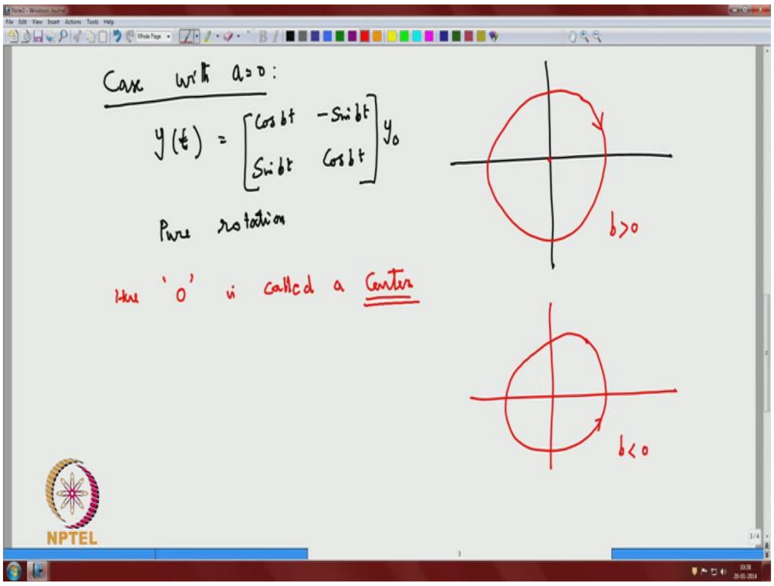

Now, the last case which we want is case with a equal to 0. In this case, the Eigen values are purely, imaginary and your solution, y t is nothing, but you see, cos b t minus sine b t, sine b t cos b t; it is a pure rotation. It will rotate; pure rotation, see again. There, it is always compact, if you have the trajectory. So, if you are trying to do a phase portrait here, and you have a equilibrium point, only thing is that it will rotate like this. The other situation which, you get is that it will rotate like this. That is the only two possibilities. Again, you have to see that which direction it will see. This is the clockwise direction with earlier case, we have already seen that. So, you have this situation. So, you see, this is the b case, positive case. You have a clockwise and b negative, you have anticlockwise. So, you have, this is the case with b positive, and this is the case, b negative. Such a point here, 0 is called a center. So, with a nonzero, it is a focus in the complex situation, when a equal to 0; it is a center. The equilibrium point is called a center, completely. So, that gives in a 2 by 2 systems, we have a complete analysis of this one.

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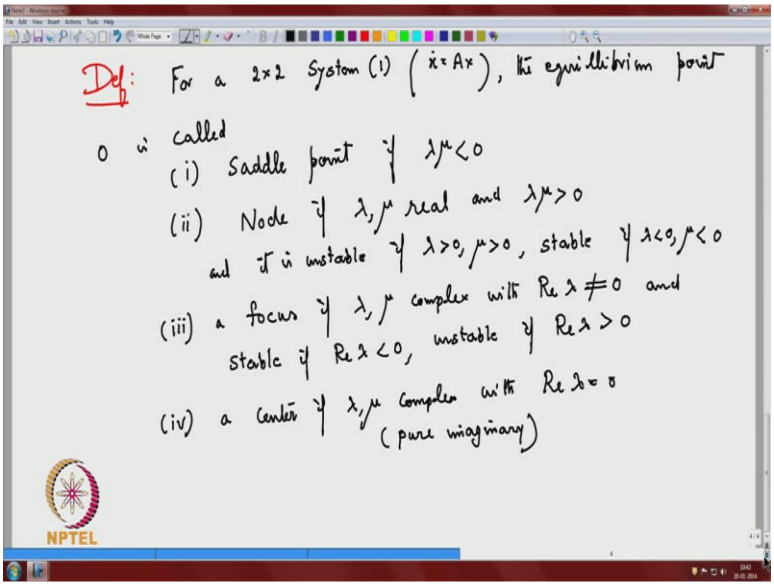

Let me give a fine definition finally. Consolidating all the results, we will write the definition. For a 2 by 2 system, any system 1; what is the 1 system? Let me recall; x dot is equal to a x. The equilibrium point 0 is called one; saddle point, if lambda mu negative. Note here that the moment lambda mu is negative; it cannot be complex Eigen values. So, it automatically, tells you that it is Eigen value. Two; a node, if lambda mu real; here, you have to say it is real, and lambda mu positive; lambda mu positive, because if lambda mu even, in the complex case, will be real. Even, if lambda and mu are complex, because lambda will be a plus I b; mu will be a minus I b, and hence, lambda mu will be a square plus b square, here. So, it is always positive. It is unstable, if lambda positive, mu positive; stable, if lambda negative, both the Eigen values are negative.

Third situation; the equilibrium point is a focus, called a focus, if lambda mu complex with real part of lambda; both real part of lambda is not equal to 0, because real part of lambda is not equal to 0, and that is also, a condition; stable, if real part of lambda is negative; unstable, if real part of lambda is positive. The last case 4; a center, if the same thing, lambda mu complex with real part of lambda equal to 0, you see. So, you have, in other words, purely imaginary; that is what pure imaginary. So, we have a complete analysis of the 2 by 2 system. There is an interesting bifurcation diagram. We will end this 2 by 2 part, before going to high dimension.

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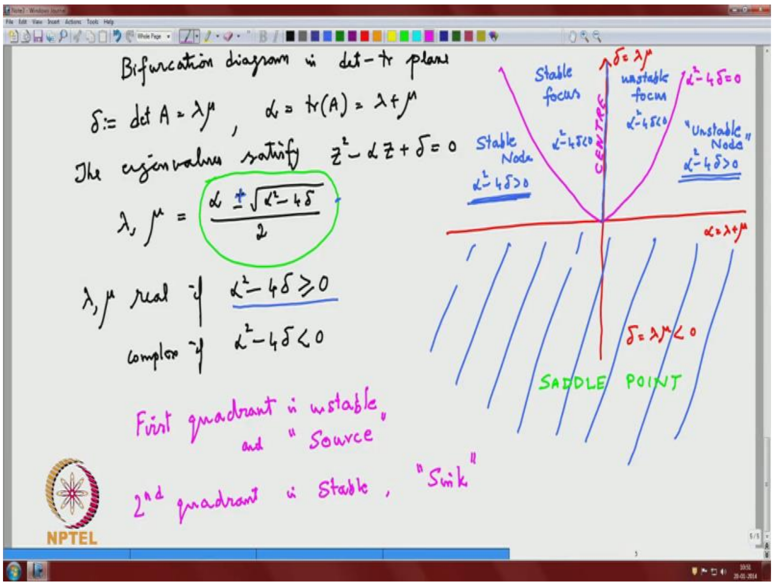

There is a bifurcation diagram, nice bifurcation diagram. You can get a picture. This bifurcation diagram is in the determinant trace plane. What do I mean by that determinant trace plane? Given a thing, given a matrix, you can write determinant of a is nothing, but the product of Eigen value. Let me call it, this is delta by definition and then you call it alpha. The trace of a, that is nothing, but the sum of Eigen values. Then, you know that clearly, because this is the product and this is the sum, and it solves a second order algebraic equation. So, this will be in the coefficient of the power term. The Eigen values satisfy the quadratic equation, right; quadratic equation, z square minus alpha z; that is the sum of the Eigen values, because it is a root. So, it is a coefficient of that, plus delta equal to 0. So, you can write your lambda mu, as solve that equation; lambda mu. The Eigen values can be written in terms of the determinant and the trace; that is equal to alpha, plus or minus square root of alpha square, minus 4 delta by 2, you see. So, you can write your alpha is 1, is the alpha plus square root of minus 4 delta by 2, and the other one is alpha minus square root of alpha square minus.

I am going to draw the diagram in the plane, determining by this is alpha; that is nothing, but the trace, and this is delta. Delta is equal to lambda mu and this is equal to lambda plus point. This is nothing, but alpha is equal to lambda plus mu. What is this? Let us consider all the cases. Below this domain, below this alpha axis, delta; this is the region delta less than 0; delta is equal to lambda mu less than 0. So, what is this region? This region is this region; the complete region, complete negative region. On that, you have all four points. For lambda mu, satisfy, it belongs to this region; that means delta less than 0. So, you have your saddle point here. So, you have your saddle point. This is the saddle point, this region. Now, let us look at the situation here, when this Eigen values are real Eigen values. It is real so, you have lambda mu real, if alpha square minus 4 delta, greater than or equal to 0. So, you have to consider the equation, and complex, if alpha square minus 4 delta, less than 0. So, in the alpha delta plane, alpha square minus 4 delta equal to 0, determines a parabola.

You have to draw the parabola here now. You have to draw the parabola here. So, you have your parabola here, and this is nothing, but the region, alpha square minus 4 delta, equal to 0, and what it shows that, you have a different thing. This is the region; alpha square minus 4 delta positive. This is also, the region; alpha square minus 4 delta positive. Here is alpha square minus 4 delta negative, and here, alpha square minus 4 delta negative. So, that gives; we will consider this case again. On that case, when alpha square minus 4 delta, greater than or equal to 0, one Eigen value here, with plus sign here, will remain to be positive. So, one of the Eigen values minus that will be negative, because alpha square.

So, if you look at here, alpha square, this region, if you look at all this region, corresponding to this one; alpha square minus 4 delta positive; you will have the stable focus, unstable. In this case, it will be unstable. Here is the region with node. So, you have unstable node, because it is a situation where, you have real Eigen value. So, you have only node. Here also, is a situation; it is a node thing, but then you have the other thing, and you have your stable Eigen value here. So, you will have this stable here, stable node, you see. So, this is a situation; alpha square minus both are nodes, but here, you look at it. Here, alpha is equal with alpha positive, and this is the region, with this part is the region, with alpha negative. So, you will get your Eigen values in this region. Both Eigen values will be negative in this region, if you look at it.

Similarly, if you look at here, you will get unstable focus, and here, you will get a stable focus. So, you have to understand that this is the region; this entire region is where, delta is equal to lambda mu positive. So, you will have a region here, delta lambda mu positive with lambda plus mu positive; that gives you both lambda and mu are positive, and here, you will get the stability. What about this here, this region, this line? On that line, you will see that the real part is 0. So, that is a region where, you have centre, you see. So, this is the region. So, this gives you in the delta alpha plane, a complete analysis of the, you have your complete analysis of the thing.

So, this portion, the first quadrant is the unstable situation. Of course, the third and fourth are already unstable, because of the saddle point equilibrium. The first quadrant is unstable, and this is called the source. Second quadrant is stable quadrant part; this is called the sink. So, ever thing will sink there. So, this gives you a complete; you do not have to think in a determinant trace plane, you can completely, understand where your stability and instabilities happening; where it is saddle point; where it is node; where it is focus; where it is these. So, that gives you the complete picture. Of course, we are in an easy situation of 2 by 2, and hence, you have the complete picture. Such a picture in general, is much more harder, when you go to the higher dimensional situation. So, we will now, try to give some interesting one or two examples, before going to higher dimensional situation.

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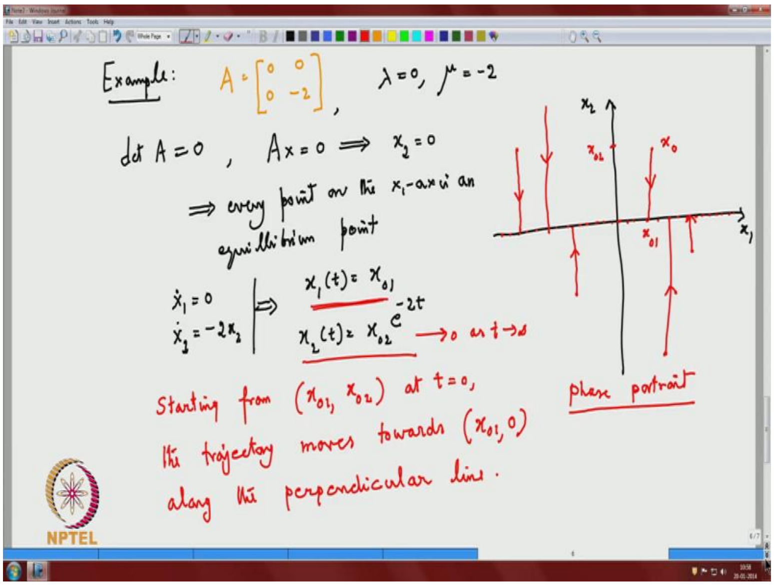

We will at least, give two examples. Here is an example 1. We will have more examples later. One example is a situation where, your Eigen values, yes, previous thing. There is one more thing; where is the picture? Here, yes, this is the situation, this case. Let me have a different color. This is the case where, you have the degeneracy. You see, this is the situation where, determining the 0, which is a kind of degenerate situation, is a degenerate. You will have the degenerate case. You see, that is a region; that is what we skipped. So, we will see an example of the next one. Consider an example with a matrix a is equal to 0, 0, 0, minus 2. What are the Eigen values? Eigen values are lambda equal to 0, and mu equal to minus 2. So, we want to determine, of course, x equal to 0 for any linear system, the origin is in equilibrium point, but if a is invertible, origin is the only equilibrium point, because the equilibrium points are given by a x equal to 0; set of all x, such that a x equal to 0. So, if invertible, only x equal to 0 is the solution to that one, but here is a case, determinant of a not equal to 0. This is a case, sorry, determinant of a equal to 0.

So, you have to look for what are all a x equal to 0. You want to find all the equilibrium points; a x equal to 0, immediately, will give you x, the second component; this is x 1 x 2; the second component is 0; that is all it will give. First component, it does not give you anything, because the equation say, a x equal to 0 is equivalent to 0 x 1, plus 0 x 2 is equal to 0; second one is 0 x 1 minus 2 x 2, equal to 0. So, that shows x 2 equal to 0. That means, every point on the x axis, the x 1 axis, every point on the x 1 axis is an equilibrium point. It does not put x equal to 0, does not put any conditions on the first component of x. That implies, any point, every point on the x 1 axis, is an equilibrium point, you see.

Now, I want to solve this equation. What are the equations? You do not have the equation. The first equation, the e is equal to x 1 dot is equal to 0, because a x equal to 0 implies, x e is equal to 0; x 2 dot is equal to minus 2 x 2. So, if you solve this equation, you will get x 1 dot, is x 1 t is a constant; that is nothing, but a constant is the initial point, and x 2 t is equal to x 0 2, into e power minus 2 t, you see. So, what does it show? Now, if you are trying to find; this is nothing, but an equation of a line, but it is an equation of a perpendicular line. That is why, it cannot be written as a function of x 1 variable, anyway, x 1; this is x 2. So, what I am trying to say is that every point on this is an equilibrium point. All the points are equilibrium points; not just the origin. All points are equilibrium points.

What does it say? That is how, if I start a point from here x 0, what does it say that if the first component; this x 1 trajectory says that the first component will not change. So, this will be your x 0 1; this will be your x 0 2. If you start from here, the first component will not change, and the second component says that it will go to the same x 0. It will go to the 0; this goes to 0, as t tends to infinity. So, the second component goes to 0, and x component will not change; it will remain. If it does not change, it has to come along this one. So, it will move along this one. Whenever you start here, the trajectory will move here, as t tends to infinity. If you start from here, it will move to that. Anywhere you start, it will go towards that, because of x 2 component. So, all your trajectories are given like this. Whatever you trajectory you start, it will move towards that. So, this is the complete phase portrait of this system. In other words, starting from x 0 1, x 0 2 at time t equal to 0; the trajectory moves towards only, at the infinity; moves towards x 0 1 origin, along the perpendicular line, you see, the behavior difference in that one.

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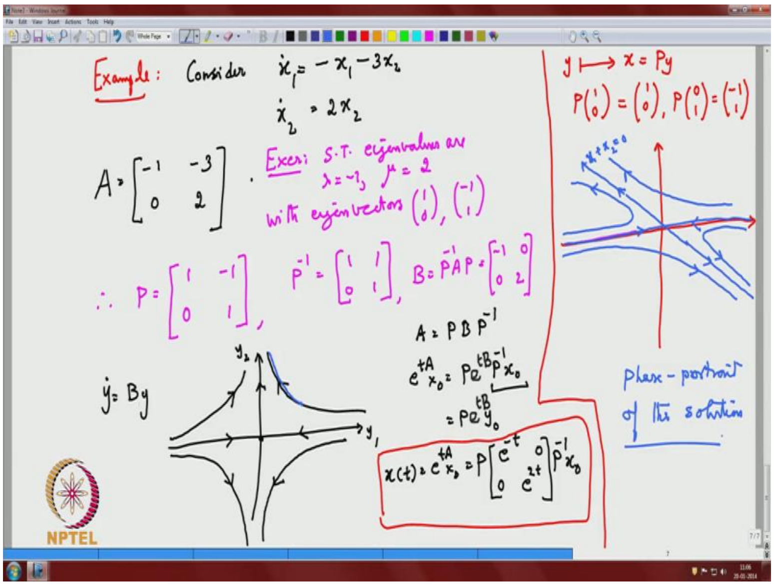

So, we will go to another example; how the linear transformation, linear equivalence will change? We will start with a system, which is not in its normal form. So, you consider a system; let me write down x 1 dot is equal to; part of it is an exercise also; equal to minus x 1, minus 3 x 2 and x 2 dot is equal to your 2 x 2. So, your matrix a is nothing, but minus 1, minus 3, 0, 2; this is you’re a. Here is an exercise for you. So, let me put it in a different color; exercise. Show that; the Eigen values are clear, Eigen values immediately, now, Eigen values are lambda equal to minus 1; mu is equal to 2 with

Eigen vectors; find the Eigen vectors; with eigenvectors 1, 0, minus 1, 1. Therefore, the matrix p, which converse to a linear equivalent matrix, because of that two distinct Eigen values; so, immediately get as it is 1, 0, minus 1, 1. Compute p inverse; p inverse is nothing, but 1, 1, 0, 1, and that will give you your B equal to p inverse of a p, that is of the form minus 1, 0, 0, 2, you see.

That is a diagonal system as expected, because you have two distinct Eigen values and hence, you will have thing. What is a y system? Therefore, your y system is y dot is equal to a y; that is already, we have studied. It is a standard problem, which a saddle point equilibrium. Now, it is clear, because you have two real Eigen values with opposite signs and hence, it is a saddle point equilibrium. So, you can write down, which we have done already, the solutions. If you plot this curve here, immediately, I do not want to go further, here; you can do that one.

So, if you plot your y, this is for the y axis; this is your y 1; this is your y 2, and this also says that the first Eigen value, corresponding to the first Eigen value, minus, and you have the first component stability, the second component stability, totally, instability, but then it is a saddle point equilibrium. You will have this one. So, this is your phase portrait, and y 2 go to infinity, and y 1 go to 0. So, y 1 go to 0 means, your graph will be like this. So, y 1 will go to 0, you see it will go to 0; this will go to 0. So, this is your phase portrait on that, and it is an invertible matrix. We want to know the solution, corresponding to x 1. You can, of course, do the analysis, because you can write down from here, e power a t; you can write down, a is nothing but p b p inverse, you can write down the solution.

Then your solution is, you can write down as e power t a is equal to e power t a of x naught, is nothing but p b p inverse of x naught. So, you choose this p inverse of x naught as y naught. This is starting with that p b of y naught, and p b; you can write down the solution completely, immediately. So, the solution is x t is equal to e power t a of x naught, is nothing, but p you can completely, compute; b is the diagonal matrix. So, it will be e power minus t, 0, 0, e power 2 t of p inverse, we have already computed. So, this is your solution. So, you have your solution, completely. You can plot it, separately. But then just try to understand the transformation in this slide itself, so that, we do not need another slide. What is your transformation going? So, your transformation y going to x equal to p y or p, yes, x equal to p y; this, we have already discussed. So, you compute that one. If you do that transformation, how does may axis? I told you in the beginning, this kind of transformation, linear equivalence is the coordinate change. So, when I want to know that how does my y 1 coordinate changes. So, I want to first compute p of 1, 0. If you compute p of 1, 0, you just take p is equal to 1, 0, is nothing, but 1, 0 itself.

So, the y 1 coordinate axis goes to x 1 axis itself, but what about my second coordinate; axis p of 0, 1. The y 2 axis, you can see that it is minus 1, 1; that is how it will go to 0. So, if you plot, if you know, try to sketch in the x 1 x 2 plane; if I sketch the x 1, the y 1 axis go to your x 1 axis. So, I will plot here. Let me choose a proper color. So, you see, it will go to the same color, but then the x 2 axis goes to, p 0, 1 goes to minus 1 to 1. So, it will go to this axis; that is what the y 1 minus y 2 in; it will go to this plane. So, this is the new coordinates now. This is this is the coordinate, y 1 plus y 2 equal to 0 or x 1 plus x 2 equal to 0, you see; that is where it goes. So, the y 2 axis actually, transforms this 1. Now, the trajectories do not intersect. So, that trajectory, corresponding to this one, if you take it; it will go like this. So, the trajectories will be like this, and the corresponding trajectories will be, even, the directions would not change it. So, the directions will be like this. See, the trajectory here; it will be only shrinking. So, the saddle point equilibrium, this will be the trajectories in the; so that is the trajectories here, and this is the trajectories here.

So, this will all be unstable; the second component is unstable; the first component is stable. Still, you get the stability here. This will be in this direction, you know, there is, it is not we are not talking about that. So, this will be here and this is direction. So, this is the phase portrait; phase portrait of the solution, you see. So, that gives you the axis part as in the beginning, I have told. Under linear equivalence, under this thing, the trajectory behavior, the equilibrium, the stability of the behavior; do not change. Whatever stability you have it, you will get the same thing.

So, the picture something like, will be changed to transformed into a new coordinate system, yes. So, with this we will stop the complete 2 by 2 part. Now, we will not be able to do such a detailed analysis for the higher dimensional system, but we will appeal to the Jordan decomposition; what are the best possible things we can do it in higher dimension. We will begin with one or two examples and possible ideas behind it, and then we will see what are the blocks, coming in higher dimensional system in the next lecture.

Thank you.