EXCELing with Mathematical Modeling

Prof. Sandip Banerjee
Department of Mathematics
Indian Institute of Technology Roorkee (IITR)

Week – 11

Lecture – 55 (Kidney Function Model)

Hello welcome to the course EXCELing with Mathematical Modelling.

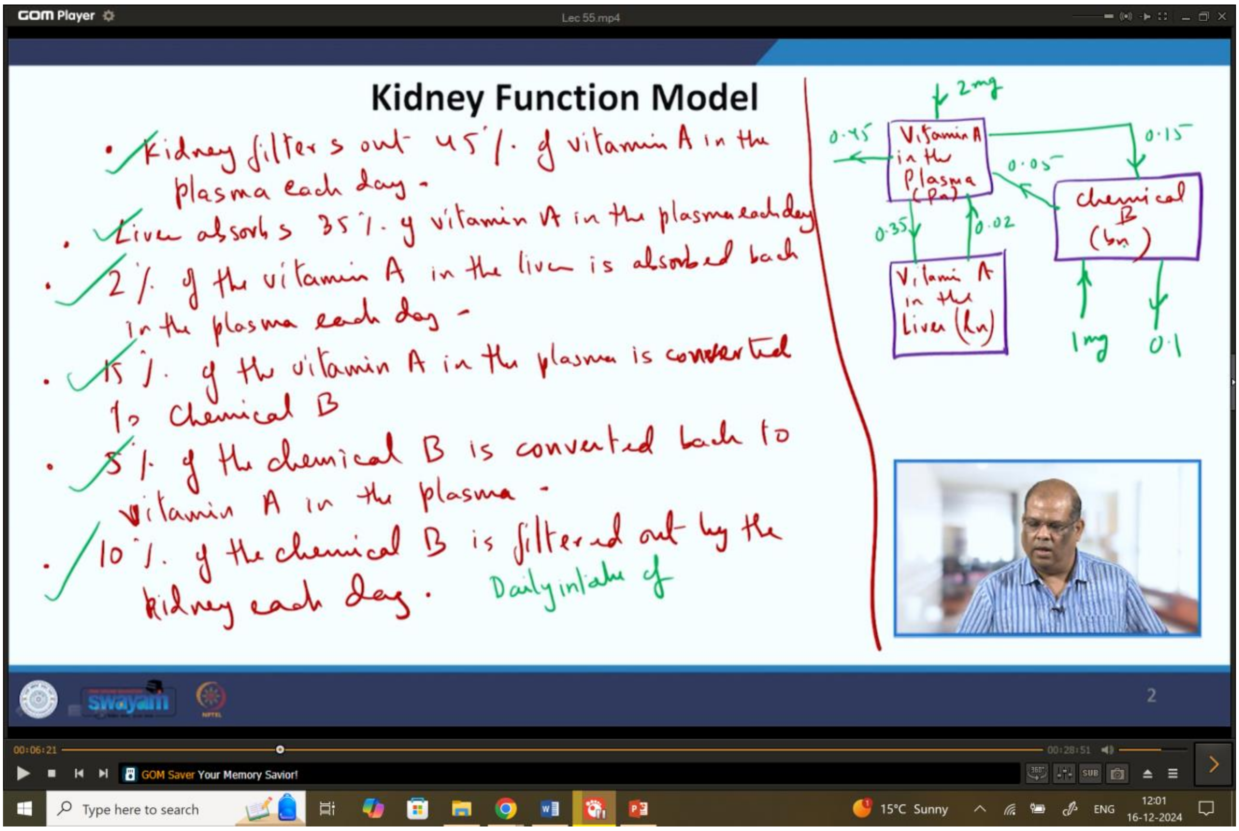
Today we will be talking about kidney function model.

Now, it is not a general kidney function model, but we have taken a typical problem that is related to this kidney function. So, let us move on to the problem.

So, let me write it point wise:

- The first point is, say, kidney filters out 45 %t of vitamin A in the plasma each day.
- The liver absorbs 35% of vitamin A in the plasma each day.
- 2% of the vitamin A in the liver is absorbed back in the plasma each day.
- Next one, 15% of the vitamin A in the plasma is converted to some chemical B.
- 5% of the chemical B is converted back to vitamin A in the plasma.
- 10% of the chemical B is filtered out by the kidney each day.

Now, with so many information if you want to formulate a model, what you have to do is you have to rely on schematic diagram first. So, it is also better that if you visualize this whole information in a chart.



So, here is vitamin A in the plasma, this is vitamin A in the liver and this is the chemical B. To make it interesting change the colour, so this is vitamin A in the plasma, this is vitamin A in the liver.

Let us put it as p_n, this is l_n and this is chemical b_n, and then let us see various relations.

So, kidney filters out 45% of vitamin A in the plasma each day.

So, 45% goes out which is 0.45.

Then this is done.

Liver absorbs 35% of vitamin A in the plasma each day.

So, the liver absorbs 0.35 of vitamin A in the plasma. This is done.

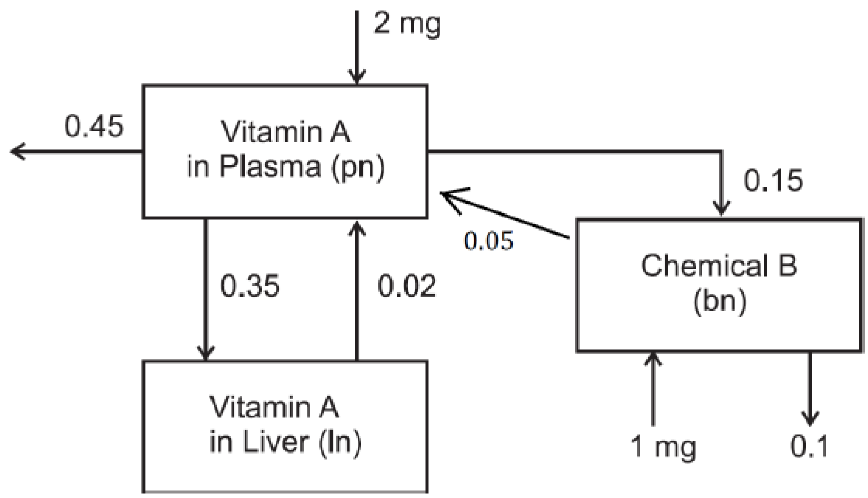
2% of vitamin A in the liver is absorbed back in the plasma each day. So, 2% goes back 0.02.

15% of vitamin A in the plasma is converted to chemical B. So, 15% goes to chemical B 0.15.

5% of chemical B is converted back to vitamin A in the plasma.

So, from here we have 5 percent, 0.05 and 10 percent of chemical B is filtered out by the kidney each day. So, you have 0.1 which is filtered out.

Suppose in addition to that I have two more information which says that daily intake of vitamin A in the plasma is 2 mg and daily intake of chemical B is 1 milligram 1 mg.



So, now we have this complete model and the variables are p_n, l_n and b_n.

Let us now form the equations. Okay, so I already have the diagram.

So, what you have to look into that from this, what is going out and what is coming in.

$$p_n = p_{n-1} - 0.45p_{n-1} - 0.35p_{n-1} + 2 + 0.02l_{n-1} - 0.15p_{n-1} + 0.05b_{n-1}$$

Now, let us look into the liver. So,

$$l_n = l_{n-1} + 0.35p_{n-1} - 0.02l_{n-1}$$

Next, come to this the chemical B

$$b_n = b_{n-1} - 0.1b_{n-1} + 0.15p_{n-1} - 0.05b_{n-1} + 1$$

So, this becomes the whole model.

So, if I just go through the model one more time, it looks a bit complicated though if you remember it step wise that this is the amount of the vitamin in the plasma at time n, this is at n minus 1 and then you see what is going out and what is coming in.

So, let us first see what is going out.

So, one is this 0.45, so 0.45 is going out, then this 0.35, so 0.35 is going out, so that is why both are negative then a constant input 2 mg, so that has been added. Now, another 0.15 going out, so this is also 0.15 going out.

Now, let us come to the liver, here 0.02 is going out, so minus 0.02 and it is going to the vitamin A in the plasma, so here is the term and 0.35 is going out and coming to the vitamin A in the liver. So, from here it has come here and then go to the chemical B in the chemical B 0.1 is filtered by the kidney. So, here it is 0.1 then from vitamin A in plasma it is adding to chemical B. So, plus this is this minus and it came here.

Then from the chemical B 0.05 is going to vitamin A in plasma. So this is going and it is being added here and plus 1 is the constant input.

So once you get this, then we have to look into the equilibrium solution and the stability analysis. So if you go for the equilibrium solution. Then I have to replace

$$p_n = p_{n-1} = p^*$$

$$l_n = l_{n-1} = l^*$$

and

$$b_n = b_{n-1} = b^*$$

If you do that, you will get

$$p^* = p^* - 0.45p^* - 0.35p^* + 2 + 0.02l^* - 0.15p^* + 0.05b^*$$

$$\Rightarrow 0.02l^* + 0.05b^* - 0.95p^* = -2$$

So, this is one equation.

The second one which is

$$l^* = l^* + 0.35p^* - 0.02l^*$$

$$\Rightarrow l^* = \frac{0.35}{0.02}p^* = 17.5p^*$$

The third equation

$$b_n = b_{n-1} - 0.1b_{n-1} + 0.15p_{n-1} - 0.05b_{n-1} + 1$$

$$b^* = b^* - 0.1b^* + 0.15p^* - 0.05b^* + 1$$

$$\Rightarrow b^* = \frac{1 + 0.15p^*}{0.15} = 6.667 + p^*$$

So, I got l^* in terms of p^*, I got b^* in terms of p^* and I will substitute it here to calculate p^*. So, if I do that, I have the equation

$$0.02l^* + 0.05b^* - 0.95p^* = -2$$

So, I substitute

$$0.02(17.5p^*) + 0.05(6.667 + p^*) - 0.95p^* = -2$$

Simple linear equation, if you solve, you will get the value of

$$p^* = 4.24$$

And similarly, your

$$l^* = 17.5 \times p^* = 17.5 \times 4.24 = 74.24$$

and your

$$b^* = p^* + 6.667 = 4.24 + 6.667 = 10.91$$

So, I have the equilibrium points $(p^*, l^*, b^*) = (4.24, 74.24, 10.91)$

Let us now look into the stability analysis.

So, if we rewrite the model after simplification, you will get

$$p_n = 0.05p_{n-1} + 0.02l_{n-1} + 0.05b_{n-1} + 2$$

$$l_n = 0.35p_{n-1} + 0.98l_{n-1}$$

$$b_n = 0.15p_{n-1} + 0.85b_{n-1}$$

So, the coefficient matrix $A = \begin{pmatrix} 0.05 & 0.02 & 0.05 \\ 0.35 & 0.98 & 0 \\ 0.15 & 0 & 0.85 \end{pmatrix}$

Now, once you get the coefficient matrix, you have to calculate the eigenvalue, that is,

$$|A - \lambda I| = 0$$

and this will imply

$$\begin{vmatrix} 0.05 - \lambda & 0.02 & 0.05 \\ 0.35 & 0.98 - \lambda & 0 \\ 0.15 & 0 & 0.85 - \lambda \end{vmatrix} = 0$$

So, this is a bit complicated though you can do it by hand and you will be able to find the values of lambdas to be 0.99, 0.86 and 0.033.

So, the modulus of all $\lambda_i' s < 1$ and the system is stable.

Now, let us use Microsoft Excel first to find the eigenvalues and then to visualize this graph.

So, I already have this one open and the coefficient matrix is here 0.05, 0.02, 0.05, 0.35, 0.980, 0.150, 0.85 and here I have written the identity matrix.

So, to calculate the eigenvalue we have to find the determinant, $|A - \lambda I|$. So, I put arbitrary value of λ and I first calculate the matrix $(A - \lambda I)$. For that you, since this is a 3x3 matrix.

So, first you select 3x3 cells and then put an equal to then select this A 3x3 minus lambda multiplied by I. And to get the values for all these 9 cells you have to press control and shift at the same time and then press enter and it will come.

So, if you see that, because lambda is 0, obviously it is going to return A and you got this as A. If I change this value to 1 and you can see this has also changed. So that is how it works.

The next thing is so I make this as 1 now.

So the next thing is that this I have to calculate determinant.

So this is equal to the formula is MDETERM and you will get this which calculates the matrix determinant.

So double click this and choose the matrix that is A minus lambda I, that this much, close the bracket and enter.

So, you have a matrix A, identity matrix I, you have calculated A minus lambda I which is this matrix and you have calculated this.

Now, go to this data and click this what if analysis and then click Goal Seek.

So, this kind of window will open.

So, what is the goal?

The goal is to make the determinant A minus lambda I 0. So, in place of set cell you choose this one.

So, this determinant will go to 0 by changing which cell? By changing this value of lambda. So, this cell and you click okay. 0.99 is one of the eigenvalue.

So, I can take another one I just put it say 0.5. So, you have to play since there is 3x3 you know there will be 3 eigenvalues.

But one thing you note if it is an imaginary eigenvalue, this will not give imaginary eigenvalues. So, it has to be real. So, this has its limitations.

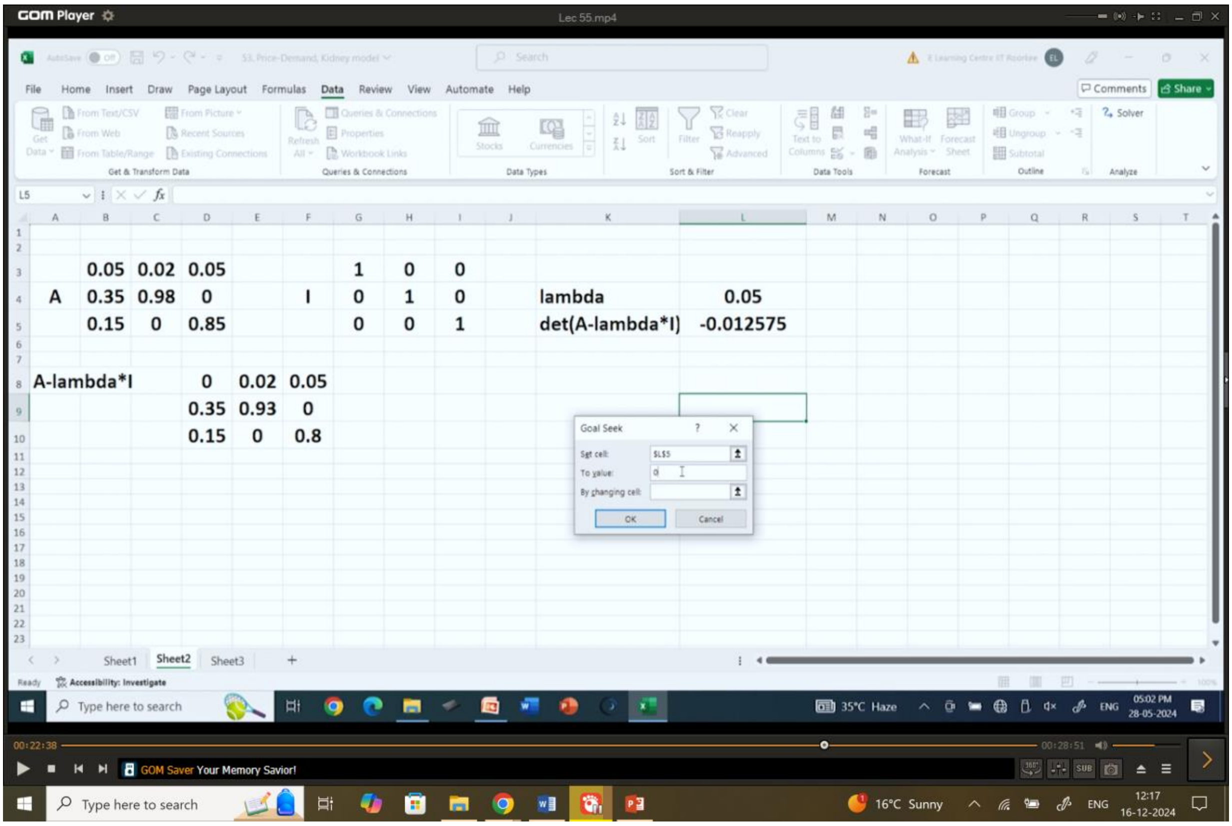
So, I change this to lambda equal to 0.5, I again go to what if analysis, I go for goal seek, I choose determinant A minus lambda I which I want to be 0 by changing this initial value of lambda and I got that to be 0.86. So, one value is 0.99, another value is 0.86.

I make it to 0.05. Again, I click what if analysis, go to goal sec, choose the cell, determinant of A minus lambda I which should be equal to 0 by changing this value of lambda and I get 0.034. So I got three eigenvalues.

Now here is the you know the problem or the catch is that okay you have to play with this a little so that you can get to the eigenvalues.

The initial condition of the lambda is very important here. If you fail in this, you know, getting the approximate value of eigenvalue close to the actual eigenvalue, this may not work or this may repeat the same eigenvalue again and again.

But in that case, you just have to, you know, take the help of your manual calculation. But otherwise, it is sort of a good way of calculating the eigenvalues using this Microsoft Excel spreadsheet.



Next we move to the numerical solution of the model. So I already have the equations here.

$$p_n = 0.05p_{n-1} + 0.02l_{n-1} + 0.05b_{n-1} + 2$$

$$l_n = 0.35p_{n-1} + 0.98l_{n-1}$$

$$b_n = 0.15p_{n-1} + 0.85b_{n-1}$$

I have the n. Let's me take the initial values. So I put it 10, 20, and 5.

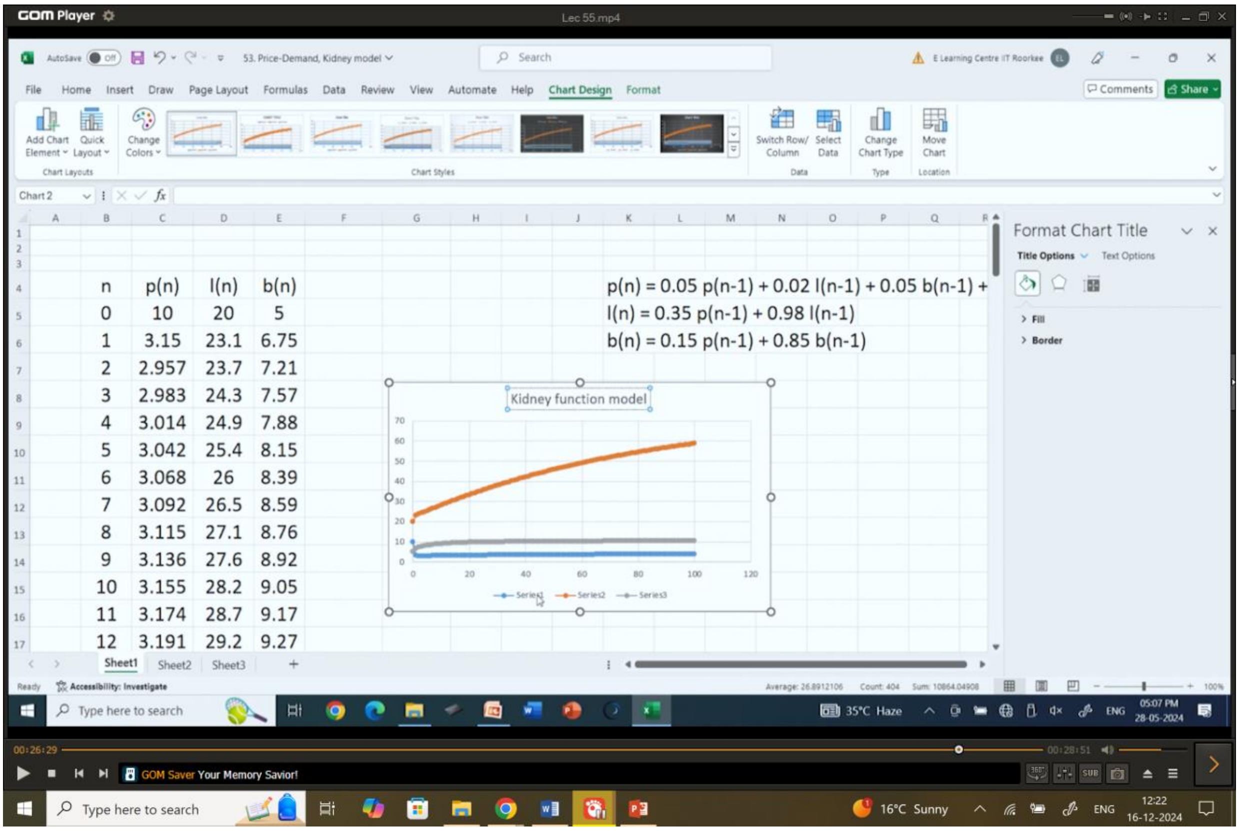
So this is equal to 0.05 multiplied by p plus 0.02 multiplied by the l value plus 0.05 multiplied by initial chemical value plus 2.

Similarly, this is equal to 0.35 multiplied by the p-value plus 0.98 multiplied by l and this is equal to 0.15 multiplied by the p-value plus 0.85 multiplied by the p-value plus 1.

So, let us drag them a little. Go to insert, go to the chart and get the chart.

So, as you can see that they are slowly approaching the equilibrium values.

If you want to have the whole set, you just go to some drag to some more values up to 100 of them. Let us draw this one. So highlighted 100 values and then go to insert, go to this chart and calculate this.



So as you can see that this is kidney function model, and this is go to select data, series 1, edit, V in the plasma, just make it short, V_plasma, V_liver, series 3 is chemical B. So, you got this visualization of the model and the model is stable. So, it is going to the equilibrium points.

If you want to copy them, just right click and go to copy, but here what I find is, if you want to paste it in this, they will say that you have to close the excel sheet first, so you can just close and then you can try paste here, so it will come, so as you can see that while solving the model, we got equilibrium point and all the eigenvalues are positive and their modulus is less than 1.

So, it is a stable equilibrium.

So, with this we come to an end of this kidney function model.

In my next class, we will be looking into some data handling or estimation of parameters.

Till then bye-bye.