EXCELing with Mathematical Modeling

Prof. Sandip Banerjee

Department of Mathematics

Indian Institute of Technology Roorkee (IITR)

Week – 11

Lecture – 51 (Infection model)

Hello, welcome to the course EXCELing with Mathematical Modeling.

Today we will be discussing about an infection model, a discrete case.

So, let us consider I_n to be the number of infected population at any time n and r be the fraction of the infected population who has recovered in time $n+1$.

Now, the assumption is the number of newly infected population are directly proportional to the size of the infected population I_n and to the size of the susceptible population $(N - I_n)$, where N is the population size.

So, we have an infected population which we denote by I_n at any time n, r is the fraction of the population, that is, the infected population who has recovered in time $n+1$ and the assumption is that the number of newly infected population, they are directly proportional to I_n, that is, infected population and to the size of the susceptible population, that is, the population who are vulnerable to catch the disease, which is $(N - I_n)$, where N is the population size.

If you want to model this in discrete case, so I will write

$$I_{n+1} = I_n - rI_n + KI_n(N - I_n)$$

So once you get the model, you have to calculate the equilibrium solution.

So, you replace

$$I_{n+1} = I_n = I^*$$

So, this will give me

$$I^* = I^* - rI^* + KI^*(N - I^*) \Longrightarrow I^*(K(N - I^*) - r) = 0$$

$$\Longrightarrow I^* = 0, \quad \text{and} \quad K(N - I^*) = r$$

$$\Longrightarrow I^* = 0, \quad \text{and} \quad N - I^* = \frac{r}{K} \Rightarrow I^* = N - \frac{r}{K} > 0.$$

So, for the existence of this solution, we must have

$$N > \frac{r}{K}.$$

So, that is the existence condition for the equilibrium solution the non-zero equilibrium solution.

So, we have two equilibria

$$I^* = 0, \qquad \text{and} \qquad I^* = N - \frac{r}{K} > 0, \qquad \left(N > \frac{r}{K}\right).$$

Now, let us go for the stability analysis about the equilibrium solutions.

So for stability analysis, linear stability analysis.

So you have the model

$$I_{n+1} = I_n - rI_n + KI_n(N - I_n)$$

Let

$$f(I) = I - rI + KI(N - I) \implies f'(I) = 1 - r + KN - 2KI$$

At $I^* = 0,\ |f'(0)| = |1 - r + KN|$.

If this value is less than 1, then that the system is stable about the equilibrium point $I^* = 0$.

At $I^* = N - \frac{r}{K},\quad \left|f'\left(N - \frac{r}{K}\right)\right| = \left|1 - r + KN - 2K\left(N - \frac{r}{K}\right)\right|$

$$= |1 - r + KN - 2KN + 2r| = |1 + r - KN|$$

So, if this is less than 1, we say the system is stable about $I^* = N - \frac{r}{K}$.

Now, let us see the solution numerically. So, once the stability analysis is done, let us now find the value of k. So, you have the model

$$I_{n+1} = I_n - rI_n + KI_n(N - I_n).$$

So, I need to find this value of K and for this some initial conditions is needed.

So, let us take say $N = 10^6,\ r = 0.8, I_0 = 1000,\ I_1 = 1500$

For $n = 0$,

$$I_1 = I_0 - rI_0 + KI_0(N - I_0)$$

$$\implies 1500 = 1000 - 0.8 \times 1000 + K\ 1000(10^6 - 1000)$$

$$\implies K = \frac{1500 - 1000 + 800}{1000(10^6 - 1000)}$$

$$\implies K = \frac{1300}{1000 \times 999000} = 0.0000013013 = 1.3013 \times 10^{-6}.$$

So, that is the value of K. And, if you substitute, so you will get

$$I_{n+1} = I_n - 0.8\, I_n + 1.3\, I_n - 1.3 \times 10^{-6}\, I_n^2.$$

So, I just substitute the value of the value of k and I got the model in this form.

So, the value of

$$r = 0.8, \qquad N = 10^6, \qquad K = 1.3013 \times 10^{-6}.$$

Now, if you recall the stability condition, for $I^* = 0$,

$$|f'(0)| = |1 - r + KN| = |1 - 0.8 + 1.3 \times 10^{-6} \times 10^6|$$

$$= |2.3 - 0.8| = |1.5| > 1.$$

So, for this model it is not stable about the point $I^* = 0$.

For $I^* = N - \frac{r}{K}$,

$$\left|f'\left(N - \frac{r}{K}\right)\right| = \left|1 - r + KN - 2K\left(N - \frac{r}{K}\right)\right|$$

$$= |1 - 0.8 - 1.3 \times 10^{-6} \times 10^6|$$

$$= |1.8 - 1.3| = |0.5| < 1.$$

So, the system is stable about this equilibrium point $I^* = N - \frac{r}{K}$.

Let us now check this numerically.

So, here I have already done, but let me generate them again.

So I have n, I have I_{n+1}. It starts with 0.

This will be 0 plus 1 and I drag it to 30 values.

Now to calculate it I will follow this law though I have written two more but I will explain later.

So the value of r is 0.8, the value of K is 1.3013 into 10 to the power minus 6, n is 10 to the power 6,

We will talk about this K later.

So, it is i n plus 1 is i equal to i n minus r times i n plus K times i n multiplied by n minus i n. So, we will use this to calculate.

So, this was 1000.

This is equal to I(n) minus r times again I(n), r is a constant, so I put dollar sign plus K, this is the value of K, which is again a constant, so I put dollar sign K times I(n) which is this value multiplied by N, which is the constant minus I(n).

So you can see that this value came to be 1500 and you can check from the initial condition for this particular value of K it has to come 1500 because we have used this value to calculate the value of K. Next we just generate the value till 30.

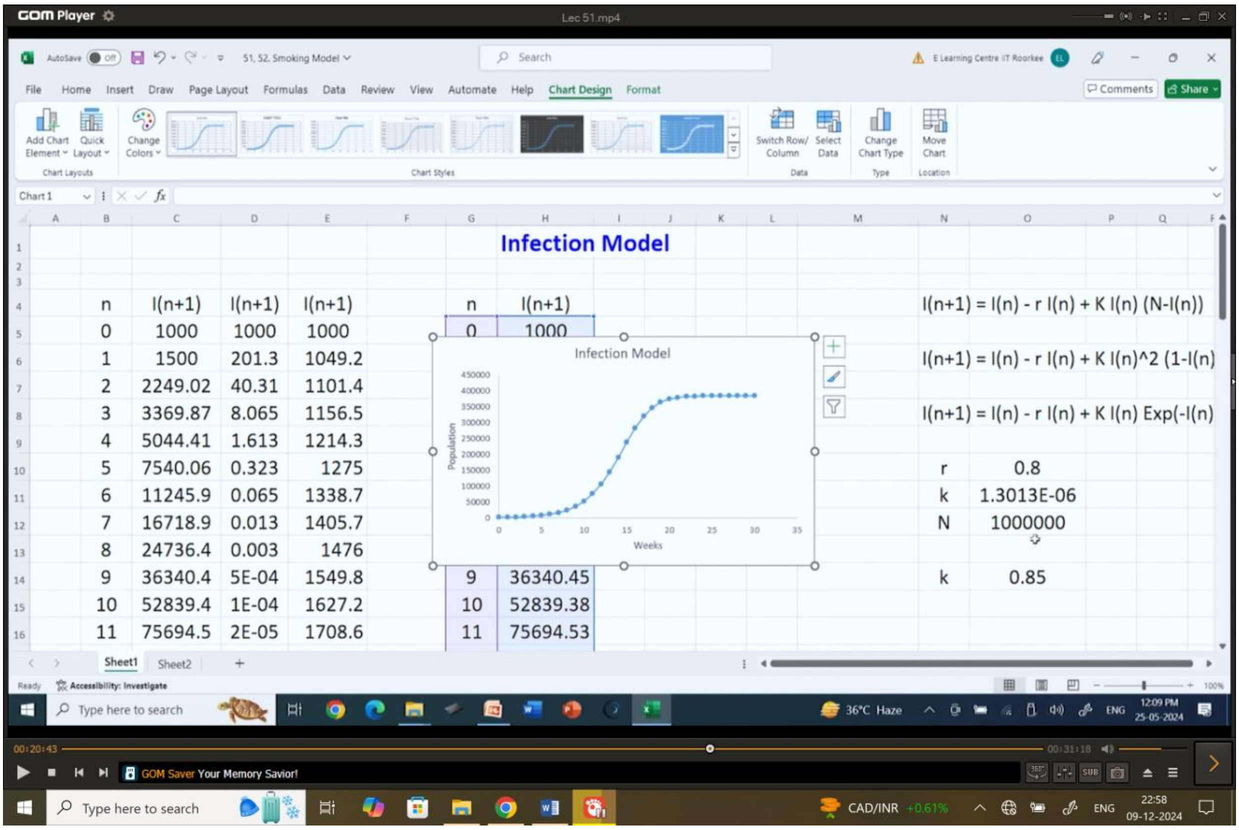
So if you get this kind of symbol it means that there are more digits all you have to do is come here and double click and you will get the number of digits.

So now I have to plot I just highlight these up to 30 values go to insert take this value and get the value.

I can just write here infection model if I want axis title I click this here it is weeks and here it is population.

If you want the grid lines to be removed go here and just click this grid lines and it will go.

So, you can see that for this value of r, k and n you get the model to be stable at $I^* = N - \frac{r}{K}$.



Now what will happen if I take a different law?

So you can change the laws and you can play with the model.

Here, the second law says that new population will vary directly to the product of $I_n^2 \left(1 - \frac{I_n}{N}\right)^2$.

So, if I calculate this, I will get this is again a new I_{n+1}.

I take this to be 1000.

So, under this new law, let us calculate these values.

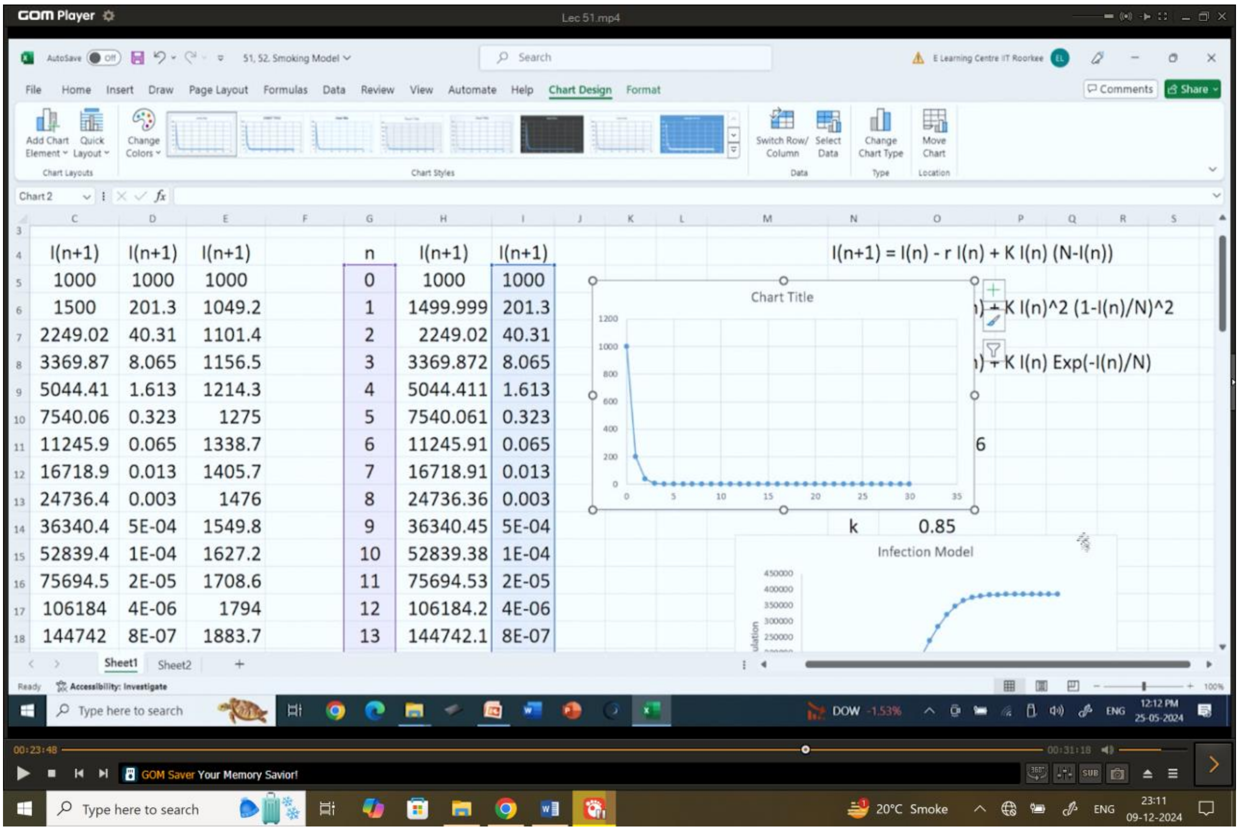
So, this is equal to I(n) minus r, which is a constant, multiplied by I(n) plus K, again a constant, multiplied by $(I(n))^2$ $(1- I(n)/N)^2$.

So, you get some value you just drag them.

So, if I now plot this and this insert the charts.

So, I can see this particular law gives the model going to zero.

So, this model is stable about $I^* = 0$.



Now, if you change anything this will have effect on the model.

For example, say I want to change the value of K. So, let us see how it affect the model.

Suppose I put this as 0.85, let us see what happens.

Okay, something weird happens, so 0.6, nothing much, 0.06, 0.00006.

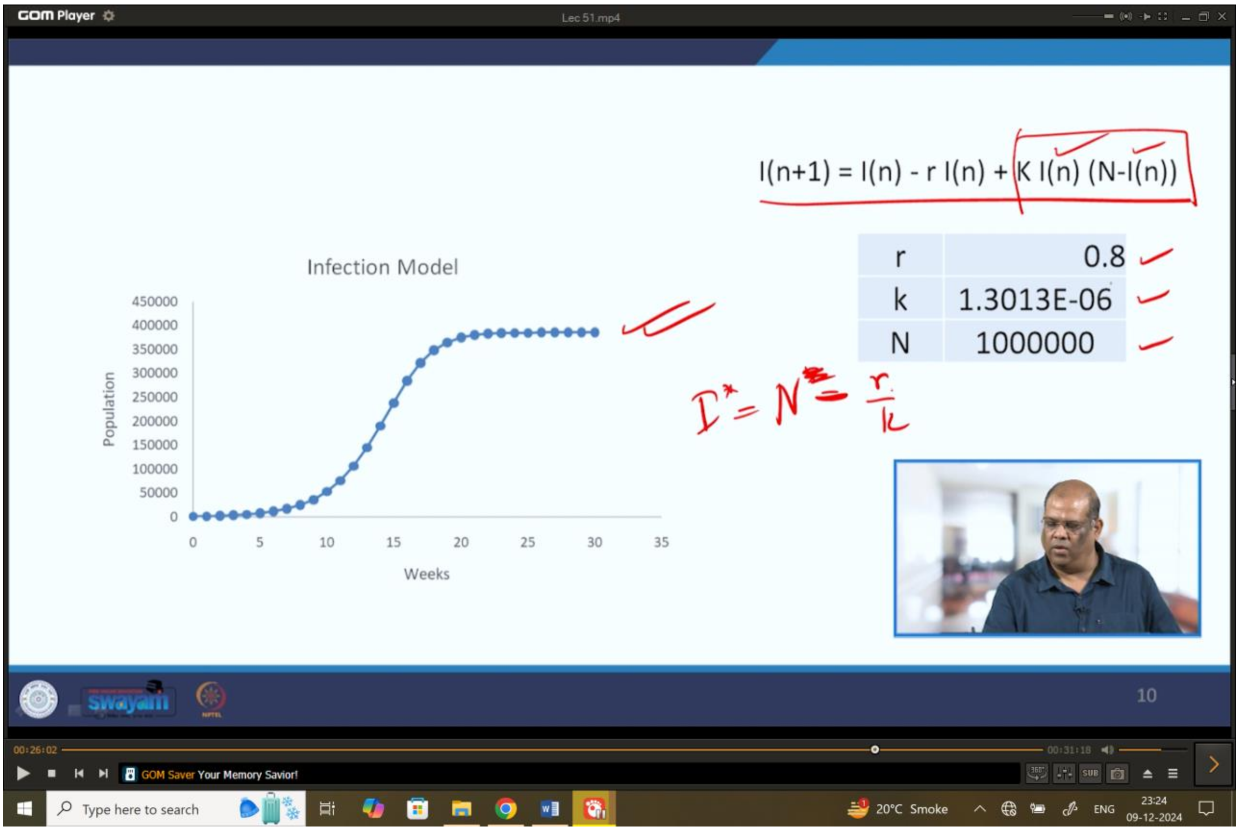
So, it gives the same kind of dynamics even if I change certain values of K. So, like that you can play with the other laws like here it is given that it varies directly to I(n) × Exp(- I(n)/N). Let us go back to the slides.

So, let us look in the numerical results one more time.

So, this is the one we already have seen that when your model is like this where the law says that it varies directly to $I_n(N - I_n)$ with r value 0.8.

This is 1.3×10^{-6} and you have seen that the model is stable about the equilibrium point $I^* = N - \frac{r}{K}$ and this is the value where it has reached.

So with this model the population will reach a steady value which is equivalent to $N - \frac{r}{K}$. So you can substitute these values and you can easily calculate.



Now this is the model where I have changed the law, it now directly the new infection directly depends on K $I_n^2\left(1 - \frac{I_n}{N}\right)^2$.

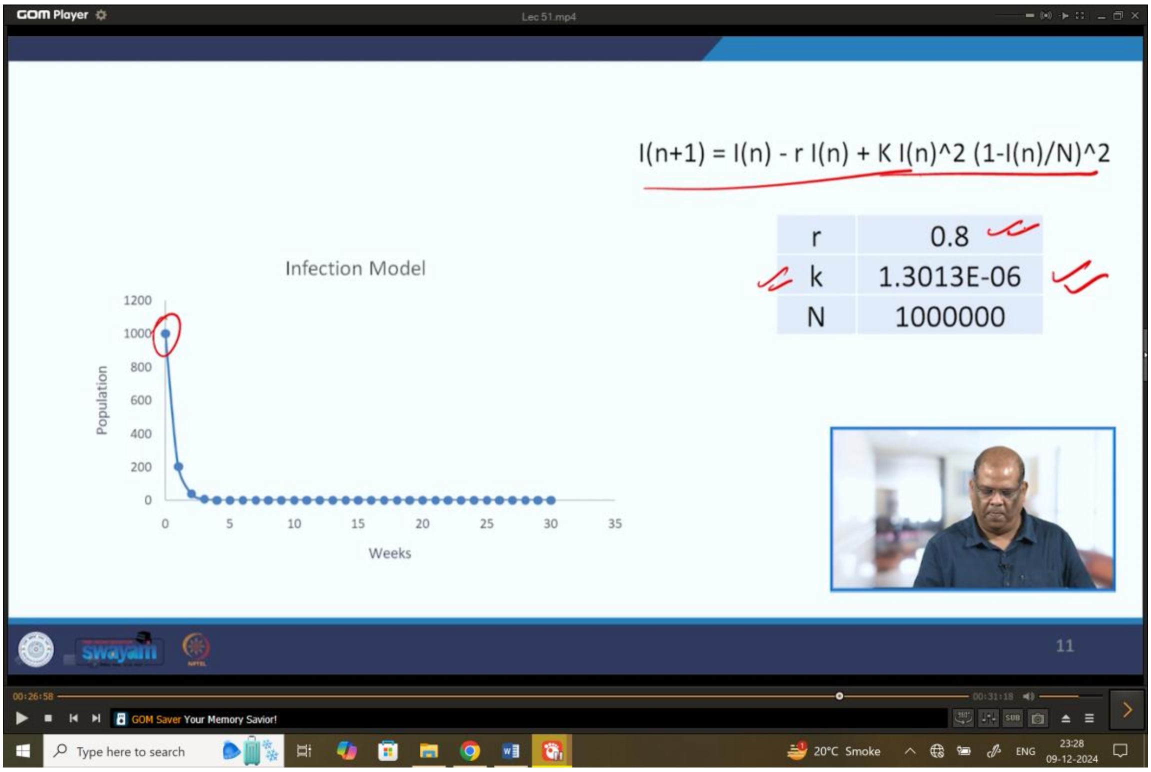
And we have seen that if we put that with the same initial condition 1000 and the same value of K, please note that this value of K has been taken arbitrarily.

If you use some initial condition this value of K may change.

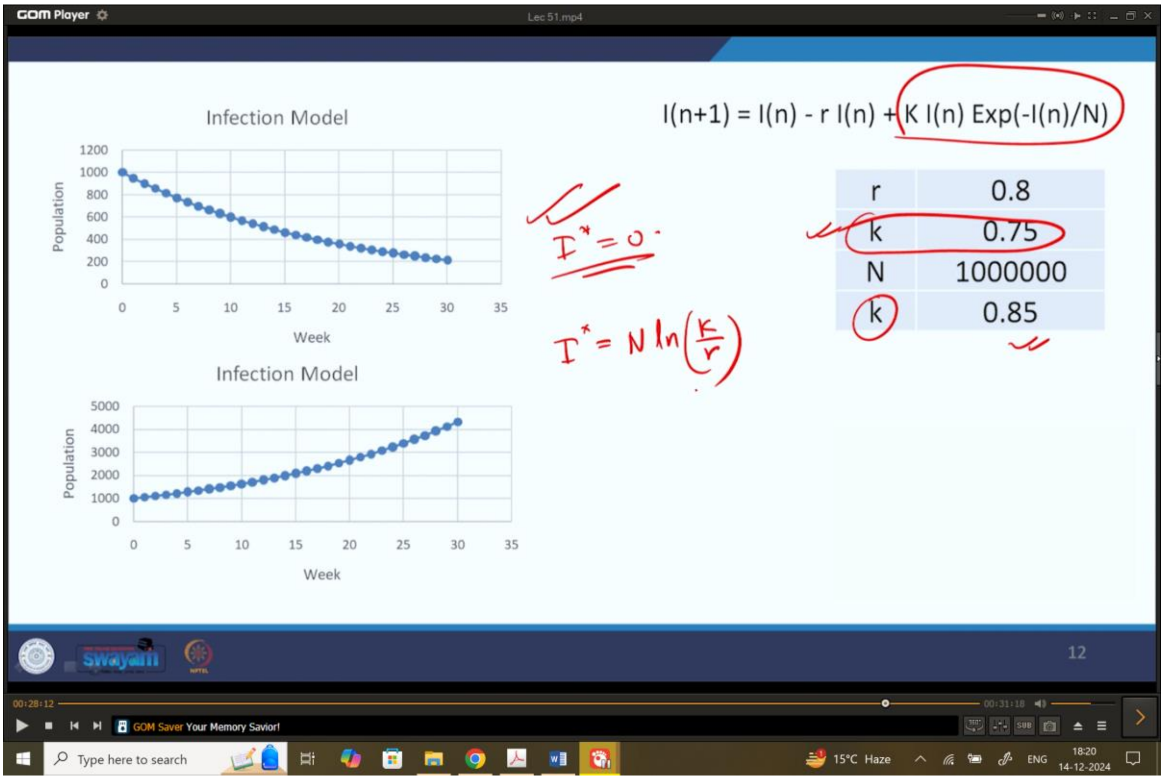
So, it depends on the initial condition whether this particular model will be stable or unstable at the point $I^* = 0$.

But for this value of k it is stable about this point $I^* = 0$.

So, with this model and with this rate of recovery the infectious population will come down to zero. So, everybody will be cured of the disease.



Now, where we have the law changes to K times I(n) multiplied by $e^{-\frac{I_n}{N}}$. So, in that particular case, I have taken the value of K to be 0.75 and this is the graph which I get.



So, it starts with 1000 and with K equal to 0.75, I see that the system is stable about the equilibrium point $I^* = 0$. But if I change the value of K to 0.85, I see that the graph rises, that is, the infectious population rises.

So, it depends on the initial conditions where you will be able to calculate the value of K whether the system is stable about $I^* = 0$ or the system is stable about the equilibrium point $I^* = N \ln\left(\frac{K}{r}\right)$.

So, what we are doing here is we have the equation

$$I_{n+1} = I_n - rI_n + KI_n e^{-\frac{I_n}{N}}$$

For steady state solution we must have

$$I_{n+1} = I_n = I^*$$

$$\Longrightarrow\ I^* = I^* - rI^* + KI^* e^{-\frac{I^*}{N}} \ \Longrightarrow\ I^*\left(-r + Ke^{-\frac{I^*}{N}}\right) = 0$$

$$\Longrightarrow\ I^* = 0 \quad \text{and} \quad -r + Ke^{-\frac{I^*}{N}} = 0$$

$$\Longrightarrow\ I^* = 0 \quad \text{and} \ \ e^{-\frac{I^*}{N}} = \frac{r}{K} \ \Rightarrow -\frac{I^*}{N} = \ln\frac{r}{K} \Rightarrow \frac{I^*}{N} = -\ln\frac{r}{K}$$

$$\Longrightarrow\ I^* = 0 \quad \text{and} \ \ \Rightarrow \frac{I^*}{N} = \ln\left(\frac{r}{K}\right)^{-1} \ \ \Rightarrow \frac{I^*}{N} = \ln\left(\frac{K}{r}\right)$$

$$\Longrightarrow\ I^* = 0 \quad \text{and} \ \ I^* = N\ln\left(\frac{K}{r}\right).$$

which gives the two steady states and coincides with the values, mentioned before.

So, with this we come to an end of this particular lecture on infectious model for discrete case.

In my next lecture, we will take an interesting model, that is a smoking problem or a smoking model.

Till then, bye-bye.