EXCELing with Mathematical Modeling

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Lecture – 45 (Mathematical Model of the Dynamics of Alcohol)

Hello, welcome to the course EXCELing with Mathematical Modelling.

Today we will be talking about a very interesting model, namely the mathematical model of the dynamics of alcohol.

Well, many people, we think that drinking as an integral part of a social life and they believe that the quality of social life is honest by moderate drinking without understanding the risk involved in the consumption of alcohol and as you have noticed most of the traffic accidents actually 45 to 55 percent of the traffic death occurs in India due to the influence because people are driving under the influence of alcohol now to model this dynamics of alcohol that is how this alcohol works inside your body we need to know few things the very first thing is what is BAC that is blood alcohol content.

So, you must have heard or seen even notice that often the traffic police they use some sort of device to measure the alcohol content in a body.

So, you have to take a breath.

So, here this blood alcohol content comes.

So, BAC of 0.1 means 1 gram of alcohol is present in 100 ml of blood.

So the legal limit in India is 0.03% per 100 ml of blood.

And this is easy to remember that 30 milligram of alcohol in 100 ml of blood.

So that is the permissible limit.

And a standard drink if you take 350 ml of beer will raise your blood alcohol content of a 68 kg adult male to 20 to 22 mg of alcohol in 100 ml of blood, which is permissible.

However, the count is different in a female.

So, if you look into the dynamics of alcohol and if you watch the video, you will see that once you first take the alcohol, so it touches the tongue.

So, the very first thing is your alcohol enters the body through the tongue, it absorbs a little bit, but it goes into the stomach and if you have taken some food then it is fine otherwise from the stomach it directly goes to the small intestine and where it is easily absorbed with the blood stream.

So, all the alcohol of one drink is absorbed within 30 minutes.

However, if you take some food and then what happens is that alcohol is trapped with the food and it find is difficult to pass to the small intestine.

So, that is why it is always advisable that when you take alcohol, take lot of food.

In that way, it will affect your body in a much lesser time.

Now, the main content of this alcohol is ethanol.

So, the first is the metabolism of this ethanol by the liver enzymes and the second is the filtration by the kidneys and the average rate that the alcohol leaves the body is 50 milligram per 100 ml per hour.

So, if a person has 75 milligram of ethanol or alcohol in his body at the beginning of the hour, then the body will eliminate 15 mg in the next hour.

So, that is the easy calculation.

Now, if I want to model this one, say I take a_n it is the amount of alcohol in a person's body at the beginning of hour n.

And as mentioned before the person's average elimination of alcohol is 15% from his body each hour.

So, if a_{n+1} is the amount of alcohol in $n+1^{th}$ hour that is an in 1 hour 15 percent is eliminated plus some A.

$$a_{n+1} = a_n - 0.15a_n + A$$

So, what is that A?

It is the amount of alcohol consumed in each time period.

So, it is

$$a_{n+1} = a_n - r\, a_n + A,$$

where r is the fraction filtered by the body set.

Now, here is the thing. So, we have assumed that the alcohol leaves the body at a constant rate of 15 percent. But in reality, this really does not happen because the metabolism of alcohol or ethanol does not happen to be at a constant rate.

It is speed difference; it is different for different people.

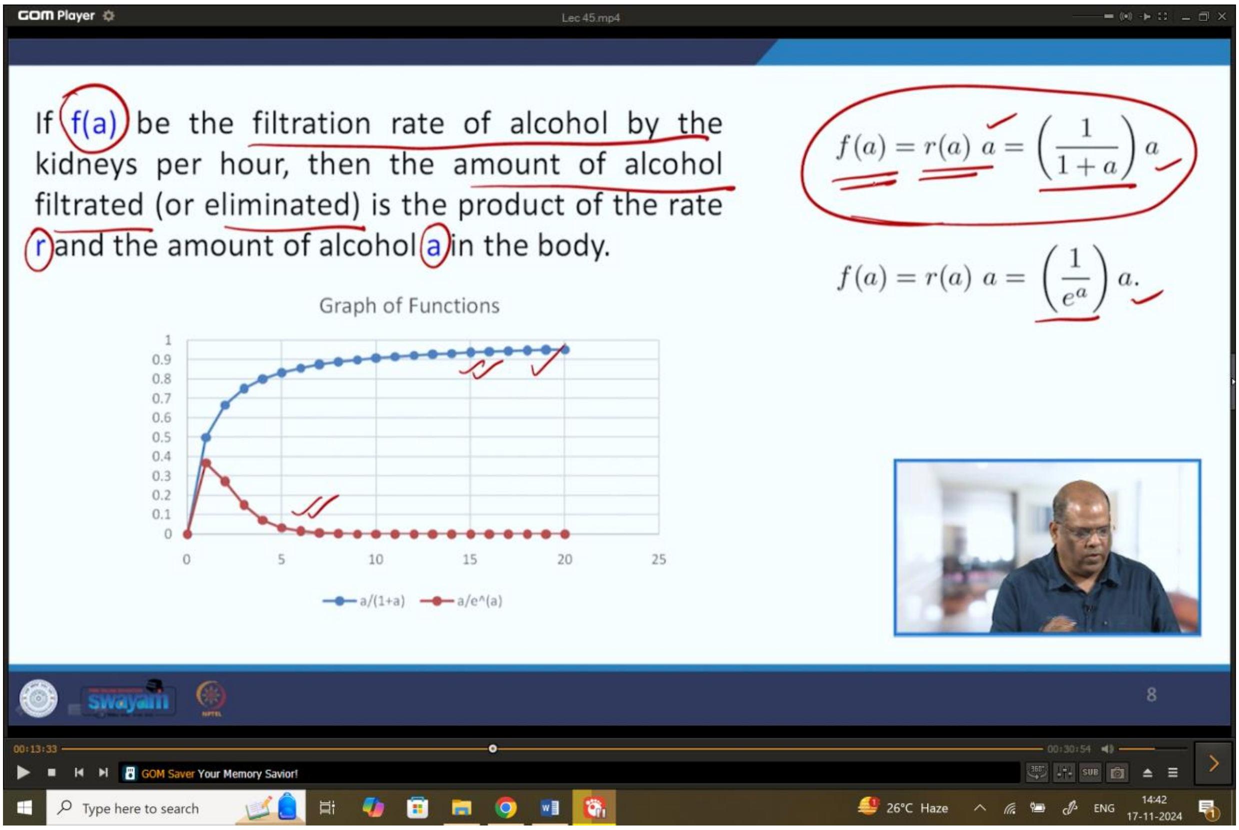
So, what you do is instead of taking r to be a 15% constant you consider it as a function of a that is the amount of alcohol that has been taken.

And we also assume that this $r(a)$ goes to 0 as the amount of alcohol in the body gets larger.

So, basically what happens you take consume more and more alcohol and

The rate you are taking the alcohol; the body cannot get dispose of that.
So ultimately, that rate at which the kidney functions almost becomes negligible.
So that is the assumption. So for such assumption, we consider two such functions.
One is this rational function; another is this exponential function.
Now let us see how these two functions behave.
We will use Microsoft Excel to get the graph of these two functions.
So already I have the file.
Here is the function $r(a)$, the rational function and this is the exponential function.
So first rational function and second rational function.
I will come to what is this effect.
So the first one is this is equal to $\frac{1}{1+a}$ there is B5 and this is equal to 1 by or you can write this as e^{-a}. So, I get both of them.
Let me drag to 20 such values and you get something.
Let me now plot them.
So, if you highlight them using shift key, then go to insert, choose the charts and this
So what you can see is that as you take more and more alcohol your rate will ultimately become less and less the rate at which the body disposes of the ethanol or alcohol.
So both are decreasing function and goes to approximate goes to 0 as time passes.
Now the filtration which is denoted by this $f(a)$. it is defined by the rate at which the alcohol content is disposed of multiplied by the alcohol content.
So this is the rate $r(a)$ and it is multiplied by a in both the cases.
So let us see that how this works.
What kind of graph we get here?
So this is equal to a times which is this multiplied by $r(a)$ which is this and this is equal to again a times multiplied by this $r(a)$.
So let me drag
So, if I now plot these two.
So to highlight this other two columns which is not side by side, you press the control and you press click and then press the shift button and press downward cursor key and they will all be highlighted.
Similarly, you press the control, you press highlight this cell, press shift and down cursor key.
So, both of them are highlighted, go to insert, go to this chart and this curve.
So what conclusion we get from these two graphs?

Let me go back to the slides.



So, as discussed that we take the function $r(a)$ in such a manner that is the rate at which the alcohol being disposed from the body as

a function rational function $\frac{1}{1+a}$ and

an exponential function $\frac{1}{e^a}$.

So, this is a rational function and this is an exponential function.

If both are plotted, you see that both of them goes to 0 with time.

So, we can choose both the functions at the rate.

Now, so the first one is the exponent is rational one, 1 by 1 plus a and the second one is e to the power minus a. Let us now see what is this $f(a)$. So, it is the filtration rate of the alcohol by the kidney per hour.

And the amount of alcohol filtrated that means eliminated from the body is the product of the rate r multiplied by the amount of alcohol.

So, the function multiplied by the amount of alcohol.

This is the rational one, this is the exponential one multiplied by a. Now if you see the graph.

So, the first one this is for the rational r divided multiplied by a this is for the exponential r multiplied by a. So, the first one says that as the amount of alcohol increases the amount of filtration also increases and this one says as the amount of alcohol increases the amount of filtration decreases.

So, this one seems to be reasonable and that is what we will take that the rational function multiplied by a as our f. So, what we do now is we have taken the rational function as $\frac{1}{1+a}$, but we will make it a generalized one.

So, the generalized function which we take

$$r = \frac{\beta}{\gamma + a_{n-1}}$$

Now, we need to find or estimate the value of this β and γ.

So, β and γ they are constants depending on a particular person and what kind of data you need to calculate β and γ.

So, for example that a blood test is performed on a person who was caught dragging under the influence of alcohol and it is found that 21 grams of alcohol shows 40% filtration and with the same person 36 grams of alcohol shows 25% filtration.

So, now if we put the values here what you will get is that I do not know what is β, I do not know what is γ, but I know what is this a that is 21 grams of alcohol and the filtration rate is 40 percent, so 0.4.

Similarly,

$$\frac{\beta}{\gamma + 36} = 0.25$$

So, from here if I just divide I will get

$$\frac{0.4}{0.25} = \frac{\frac{\beta}{\gamma + 21}}{\frac{\beta}{\gamma + 36}} = \frac{\gamma + 36}{\gamma + 21} = \frac{8}{5}$$

So simply cross multiply

$$5\gamma + 180 = 8\gamma + 168$$

So

$$3\gamma = 12 \Rightarrow \gamma = 4$$

And with $\gamma = 4$ if you substitute it say here, so

$$\frac{\beta}{\gamma + 21} = 0.4$$

and

$$\beta = 0.4 \times 25 = 10$$

So you got your $\gamma = 4$ and $\beta = 10$.

So now your model will look like so your

$$r(a) = \frac{10}{4 + a}$$

and your model is

$$a_{n+1} = a_n - \frac{10a_n}{4 + a_n} + A$$

So, if you recall that the alcohol content in the $(n + 1)^{th}$ hour is the alcohol content the present one minus in one hour we have told that 15% leaves the body but now we say that it is a variable quantity instead of 15% you take the rational function which is $\frac{10a_n}{4+a_n} + A$ which is the constant input of the alcohol.

So this becomes your model now.

This gives the dynamics of the alcohol in a body of a particular person.

Now let us look into the equilibrium solution.

So, the equilibrium solution means there is no change from one generation to another generation.

So, we replace this

$$a_{n+1} = a_n = a^*$$

$$a^* = a^* - \frac{10a^*}{4 + a^*} + A \Longrightarrow \frac{10a^*}{4 + a^*} = A$$

$$10a^* = 4A + Aa^* \Longrightarrow (10 - A)a^* = 4A \Longrightarrow a^* = \frac{4A}{10 - A},$$

and since it is the alcohol content you must have

$$10 - A > 0, \text{ that is, } A < 10.$$

So that is the condition for the existence of a positive equilibrium solution.

So once you get the equilibrium solution then you go for the stability analysis here the model is

$$a_{n+1} = a_n - \frac{10a_n}{4 + a_n} + A$$

So, let some

$$g(a) = a - \frac{10a}{4 + a} + A$$

where your

$$a^* = \frac{4A}{10 - A}$$

and this is your equilibrium solution.

So, let us look into the stability at the point a^*. So, you have to find

$$g'(a) = 1 - 10\frac{4 + a - a}{(4 + a)^2} = 1 - \frac{40}{(4 + a)^2}$$

Therefore,

$$g'(a^*) = 1 - \frac{40}{\left(4 + \frac{4A}{10 - A}\right)^2}$$

$$= 1 - \frac{40(10 - A)^2}{(40 - 4A + 4A)^2} = 1 - \frac{(10 - A)^2}{40}$$

Now, for the system to be stable at the point a^* you must have

$$|g'(a^*)| < 1 \Rightarrow \left|1 - \frac{(10 - A)^2}{40}\right| < 1$$

$$-1 < 1 - \frac{(10 - A)^2}{40} < 1$$

So, let us quickly solve the inequality. Now,

$$1 - \frac{(10 - A)^2}{40} < 1 \Longrightarrow (10 - A)^2 > 0 \Longrightarrow A < 10.$$

and

$$-1 < 1 - \frac{(10 - A)^2}{40} \Longrightarrow \frac{(10 - A)^2}{40} < 2 \Longrightarrow (10 - A)^2 < 80$$

and this will imply

$$-\sqrt{80} < 10 - A < \sqrt{80}$$

which is

$$-4\sqrt{5} < 10 - A < 4\sqrt{5}$$

$$\Rightarrow -4\sqrt{5} - 10 < 10 - A - 10 < 4\sqrt{5} - 10$$

$$\Rightarrow 10 - 4\sqrt{5} < A < 10 + 4\sqrt{5}$$

$$\Rightarrow 1.06 < A < 18.94$$

From the existence, we already have $A < 10$. So combining these two, we will have

$$1.06 < A < 10$$

So, the conclusion is for this model which captures the dynamics of alcohol, the content of alcohol every hour must lie between 1.06 to 10 units such that the system is stable.

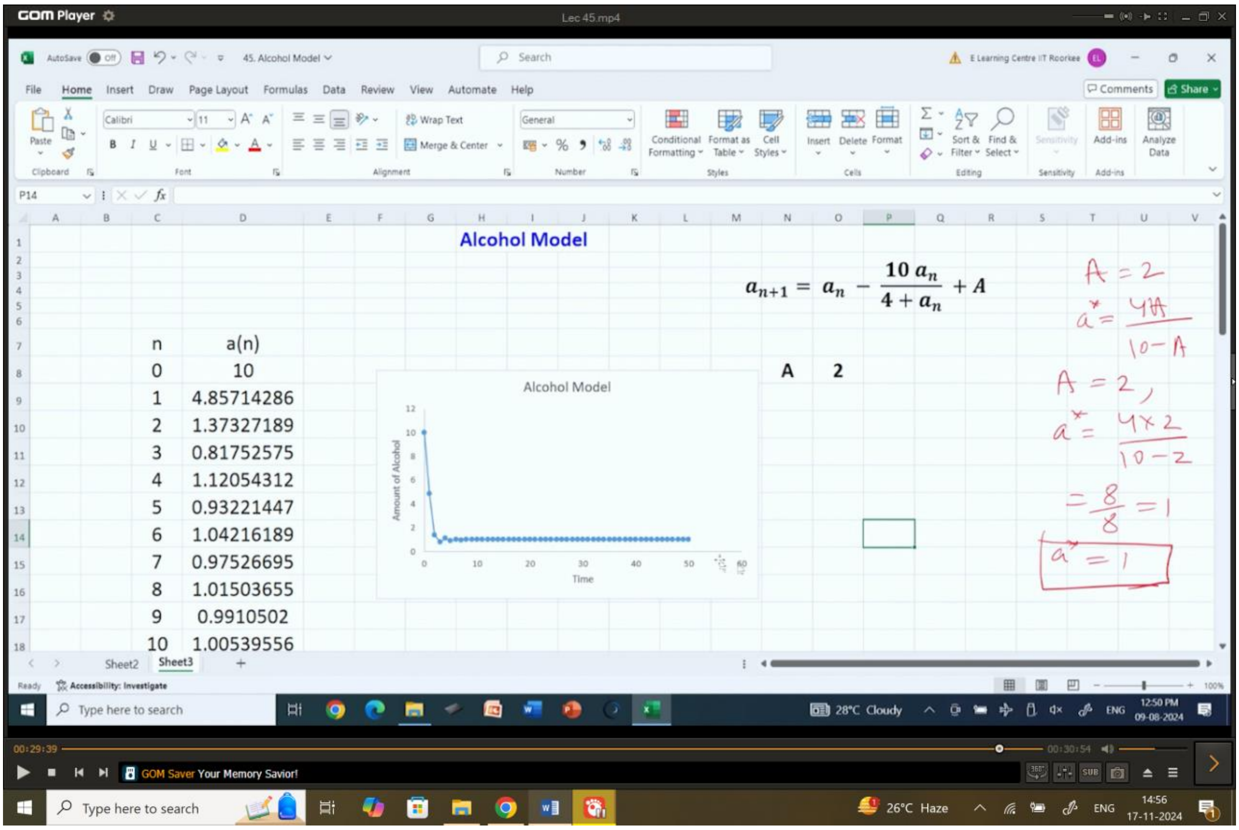
We now look into the numerical solution of this alcohol model where we have substituted the value of A to be 2 and as we have shown before that

The equilibrium point of this model is given by a star equal to 4A divided by 10 minus A. So if we substitute your

$$A = 2, a^{*} = \frac{4 \times 2}{10 - 2} = \frac{8}{8} = 1$$

So your $a^{*} = 1$.

So with this information, we now solve this particular alcohol model using Microsoft Excel where we have taken the initial condition to be 10.



So we first put this is equal to 0 plus 1.

So we give an increment of 1 and drag it so up to say 30, 25, a little more.

Now this initial value we have taken to be 10 you could have taken any number so this will be equal to a n this is the initial value a 0 minus 10 times again a 0 divided by 4 plus again A0, put again this bracket plus A and the value of A is constant.

So, I put dollar sign and you calculate this value.

So, I drag it a little more because I already know the value will be 1.

So, let me drag this also a little bit.

let us make it 50.
So, to plot this we highlight them select all the 50 values go to insert this chart and click this.
So basically you see that this equilibrium point it reaches the value of 1 which matches with your analytical solution and the system is stable because the value of A lies within the range of stability which we have proved before.

So this is alcohol model, if you want remove the grid lines, I want an axis title, this is basically this is time and this is the amount of alcohol.

So you get the model where you have shown that your numerical results matches with the analytical results because the system is stable and it reaches the equilibrium point A star equal to 1.

So summing up today we have learned about an interesting model namely the dynamics of alcohol in the body where we saw mathematical functions which controls the amount of alcohol in the body.

We also consider a model which shows the dynamics and it is investigated, we find the equilibrium point, the stability condition depending on value of some constant A and we see the results numerically also and we found that the analytical result matches with the numerical results.

So, in my next lecture, we will be talking about some linear prey predator model and see the dynamics among them.

Till then, bye-bye.