EXCELing with Mathematical Modeling
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Week – 01
Lecture – 03 (Scaling)

Hello, welcome to the course Excelling with Mathematical Modelling.

Today we will be talking about scaling which is again one of the very important aspect of mathematical modeling.

So, what is this scaling and why we need this scaling?

So, basically you need the scaling to make your dependent and independent variable that appears in your mathematical model to be dimensionless and why we need that to be dimensionless because sometimes you will see that in a mathematical model there are lots of parameters.

So, this scaling they help you to reduce this number of parameters.

and also sometimes you face something numerical problems while solving this mathematical models.

So, scaling reduces those numerical errors and give you a smooth solution of the numerical methods.

So, that is the reason why we go for this scaling in mathematical models.

So as I was telling you it reduces the number of independent and dependent parameters and also it makes the size of the independent and dependent parameters approximately equal to unity.

So for all these reasons scaling is very much recommended for a mathematical model.

So in scaling, what we do is that we introduce a dimensional variable, dimensionless variable

$$Z^* = \frac{Z - Z_0}{Z_c}$$

This Z_0 mostly we take them as 0.

So, all we care that your Z and your Z_c must have the same dimension such that if you divide Z divided by Z_c , you get a Z^* which is a dimensionless quantity.

So, let us proceed with an example and it will be very easy to understand the whole process of scaling through an example.

So, we take this Spruce Budworm outbreak model.

Please note this lecture is not about the model, but it is about how you can scale a differential equation.

The reason we are taking the model so that you can understand the dimensions of various parameters and variables that is appearing in the differential equation.

And for that a little background is needed.

So, this spruce budworm outbreak model

So it is an insect.

It is which attacks the trees or as you can see, pineries, that means the place where the pine trees are grown.

So this insect, they attack the leaves of the trees.

So if you see the picture, so this is the insect and when they attack the trees and eat the leaves, it looks like this.

So basically this insect or spruce budworm, they crawls upon and consume the leaves of the trees and excessive consumption can obviously it kills, it damages the leaves and kills the trees.

So that is why many modelers have studied this.

But however, this bird worm, there is a predation.

So many birds eat those worms.

And one of such bird is Cape May Warbler.

So along with other insects, they also eat these bird worms, which is one of their favourite.

So we have a model where we have a prey, which is the bird worm, and we have a predator, which is this bird.

So the budworms, they prefer larger trees because they have these large leaves and hence it results in a very large outbreak of population on those pine trees.

Now, the figure which you see is, as you can see, it's a bit whitish, but it's supposed to be greenish.

So the reason is that all the trees are infested with this insect and hence this is the aerial view of the forest.

So, for some unknown reason, this budworm population, they just explode and there is a devastating effect on the pine trees.

But then they return to their manageable levels.

But why this management is required?

Because of the timber industry.

So, the woods comes from these pine trees.

And there is a huge loss to the timber industry and hence various management techniques were tried but without success.

So, this problem was studied by many mathematical modelers, several scientists at the University of British Columbia, the big names are Ludwig, Morris, Jones and Holling.

They studied this problem and produced a series of mathematical models.

So this is basically the background of this Spruce budworm outbreak model.

What we will do here is we will take one such model and then I will show you how you can make the whole model dimensionless.

So that is using this process of scaling.

So that is the whole idea.

Now, so what you see on the screen is the differential equation that governs this Spruce budworm outbreak model.

So this B, this is the density of budworm at time t. So the reason this explanation is necessary so that you can understand the dimensions of each of the terms.

So B represents the density.

This r is the intrinsic growth rate.

For the timing you remember this is a growth rate.

We will come to this intrinsic growth rate in the later lectures.

So, growth rate is something per time.

So, now you know the dimension of r that is it will be per time.

This k is called the carrying capacity.

Now what do you mean by this carrying capacity?

Consider the example say a pond.

You have fishes in the pond.

So what do you do is you find that there are say approximately 20 fishes in the pond.

So you buy some fishes, some alive fishes say 20 more and you put them in the pond. Fine.

After a week again you buy 20 more and you put them in the pond and you repeat that in the next week.

But then you see in the coming week, some fish, some dead fish, they float up.

So the meaning is that the pond cannot sustain more than certain number of fish.

It can be 100, it can be 150, but beyond that, it cannot sustain the number of fish.

So that number is called this carrying capacity.

That means the amount which your environment can sustain.

Again this is density because it will be related with the density of Spruce budworm.

Now this part is the predation part.

By predation part I mean where the birds eats the insect.

In this case it is Spruce budworm.

Now β is the rate at which it is eating the insect

So it is the density per unit time.

So this β is called the predation rate and this α it is some saturation constant. Again this represents a density.

So now you have an idea that what are the terms and what does they represent so that it will be easy when we write the dimension of each of the variables and the parameters.

t is obviously the time and this is the initial condition that at time t equal to 0 some initial density of the Spruce budworm is present.

So we now give a step by step approach to how to non-dimensionalize this differential equation.

So, step 1, you first identify the variables, the parameters and their dimensions.

So, you have the differential equation this.

$$\frac{dB}{dt} = r\,B\left(1 - \frac{B}{k}\right) - \frac{\beta\,B^2}{\alpha^2 + B^2}$$

$$B(0) = B_0$$

Here B is the dependent variable, t is the independent variable, your r, k, α and β

They are the parameters and I need to know what are the dimensions of these quantities.

We have put it in the tabular form.

Variables	Dimensions
t	T (time)
B	$ML^{-3} = \rho$ (density)
Parameters	**Dimensions**
r (intrinsic growth rate)	T^{-1}
k (carring capacity)	$\rho = ML^{-3}$
α (saturation factor)	$\rho = ML^{-3}$
β (predation factor)	$ML^{-3}T^{-1} = \rho T^{-1}$
B_0 (initial density)	$ML^{-3} = \rho$

So, your variable is t and your dimension is capital T. B, I already told you that is the density.

So, mass by volume. So, $M\ L^{-3}$.

You can denote this by ρ.

Now, the parameters, r, which is the intrinsic growth rate, so per time.

k is the carrying capacity, already explained that it is the density.

α, the saturation factor, again the density.

β is the predation, which is the density per time.

So this is the density, $M\ L^{-3}$ per time, T^{-1}.

And B_0, again the initial density, so $M\ L^{-3}$, which is ρ.

So you can put this quantity as some ρ so that it's easy for simplification.

Otherwise, you have to write this whole thing $M\ L^{-3}$.

So once you have identified all the variables and all the parameters, now you have to introduce a new variable using this method of scaling, which has no dimension.

So what you do is, this is your differential equation.

You introduce a variable

$$B = u\ B^*$$

So, u is the variable which has been introduced, this is a dependent variable and you can write

$$u = \frac{B}{B^*}$$

Similarly, you introduce

$$t = \ z \ t^*$$

So z becomes your independent variable so that

$$z = \frac{t}{t^*.}$$

So here the density of this so the dimension of this B and of this $B^*$the dimension of t and of this t^* must be same so that your u and z are dimensionless quantities and we will be proving that.

So the next part is you have to substitute these in this differential equation.

So for that we use the chain rule. So

$$\frac{dB}{dt} = \frac{dB}{du}\left(\frac{du}{dz}\right)\frac{dz}{dt}$$

So this is our dependent and independent variables.

Now from here $\frac{dB}{du}$ you can get to be B^* from here we keep $\frac{du}{dz}$ as it and $\frac{dz}{dt}$ is $\frac{1}{t^*}$.

Now we substitute this in this differential equation and we will be getting

$$B^* \ \frac{du}{dz} \times \frac{1}{t^*} = r \ u \ B^* \left(1 - \frac{uB^*}{k}\ \right) - \frac{\beta u^2 B^{*2}}{\alpha^2 + uB^{*2}}$$

Now next step we choose $B^* = \alpha$.

So obviously your question will be why we choose this $B^* = \alpha$ why not it is β or k or something else.

Well the reason you are choosing this $B^* = \alpha$ is that from this expression you can see that if I choose $\beta \ B^* = \ \alpha$, I can take this α^2 common.

So, the idea is to make any of one of this term free of a parameter.

So, looking at it then you have to decide what is going to be the value of B^* there is no hard and first rule it will just come with practice.

So if I put $B^* = \alpha$ then I get

$$\alpha \frac{du}{dz}\frac{1}{t^*} = r\,u\,\alpha\left(1 - \frac{u\alpha}{k}\ \right) - \frac{\beta u^2 \alpha^2.}{\alpha^2 +\ u^{*2}\alpha^2}$$

I write this as

$$\frac{du}{dz} = r\ \ t^* u\left(1 - \frac{u}{\frac{k}{\alpha}}\right) - \frac{\beta t^* u^2}{\alpha\,(1+u^2)}$$

Now, so our next step I have to put some value of this t^*.

Now as I told you the aim is to make one of the term parameter free.

So if I substitute $t^* = \frac{\alpha}{\beta}$ so this is going to cancel and this term will be free of any parameters.

So, this will be $\frac{u^2}{(1+u^2)}$ only.

So, that is why we choose the $t^* = \frac{\alpha}{\beta}$.

And if you do that you will get

$$\frac{du}{dz} = r\frac{\alpha}{\beta}\ u\left(1 - \frac{u}{\frac{k}{\alpha}}\ \right) - \frac{\beta}{\alpha}\ \frac{\alpha}{\beta}\ \frac{u^2}{(1+u^2)}$$

So, this cancels I have under highlighted this parameter.

So, in the next step what we will do is

we will write

$$\frac{du}{dz} = au\left(1 - \frac{u}{b}\right) - \frac{u^2}{(1+u^2)}$$

where

$$a =\ r\frac{\alpha}{\beta}$$

and

$$b =\ \frac{k}{\beta}$$

So, now you see the equation which you get of using this method of scaling has two parameters a and b. So, from four parameters it is now reduced to two parameters.

The next thing what you have to check is whether the equation which you have just got whether it is a dimensionless or we have some dimension there.

That part you have to check very carefully.

Now, to check the dimension, let us start with u. So, u we have defined as

$$\frac{B}{B^*}$$

So, if I take the dimension of u, it is dimension of B divided by dimension of B^* and B is the density. So, it is $M\,L^{-3}$,

B^* we have put to be α.

So, it is the dimension of α and as you can see from here the dimension of α is again density.

So, this is $\frac{M\,L^{-3}}{M\,L^{-3}}$ which is 1 a pure number and hence dimension view is dimensionless basically.

$$[u] = \frac{[B]}{[B^*]} = \frac{M\,L^{-3}}{[\alpha]} = \frac{M\,L^{-3}}{M\,L^{-3}} = 1$$

Let us now take z. So

$$z = \frac{t}{t^*}$$

$$[z] = \frac{[t]}{[t^*]} = \frac{T}{[\alpha/\beta]}$$

Now dimension of α so α is just the density so instead of $M\,L^{-3}$ I am now putting this ρ and dimension of β which is the density per time so it is $\rho\,T^{-1}$.

$$[z] = \frac{[t]}{[t^*]} = \frac{T}{[\alpha/\beta]} = \frac{T}{[\alpha]/[\beta]} = \frac{T}{\rho/\rho T^{-1}} = 1$$

So, again z is your dimensionless quantity.

So, the variables which you have introduced u and z, you have now proved that they are dimensionless.

Now, let us take the parameters. You have the parameter

$$a = \frac{r\alpha}{\beta}$$

So, dimension of a

$$[a] = \frac{[r][\alpha]}{[\beta]}$$

Now, dimension of r, it is per time, so T^{-1}, dimension of α, it is $\mathrm{M}L^{-3}$ and dimension of β, it is density per time, so $\mathrm{M}L^{-3}\ T^{-1}$.

$$[\mathrm{a}] = \left[\frac{\mathrm{r}\,\alpha}{\beta}\right] = \frac{[\mathrm{r}]\,[\alpha]}{[\beta]} = \frac{T^{-1}\ \mathrm{M}\,L^{-3}}{\mathrm{M}\,L^{-3}\ \mathrm{T}^{-1}} = 1$$

So, your a is again dimensionless.

Our second parameter

$$b = \frac{k}{\alpha}$$

So, dimension of b is $\frac{[k]}{[\alpha]}$ and dimension of k is the density

dimension of α is again density we are cancelling and 1.

$$[\mathrm{b}] = \left[\frac{\mathrm{k}}{\alpha}\right] = \frac{[\mathrm{k}]}{[\alpha]} = \frac{\rho}{\rho} = 1$$

So, now you have proved that the equation which you got after scaling whether it is the variables or whether it is the parameters they are all dimensionless.

So, now let us look into the initial condition.

So, what happens to the initial condition?

So the initial condition is given to be $B(0)$ equal to some B_0.

Now if I substitute because your B equal to $u\,B^*$, so your $B(0)$ is going to be $u(0)\,B^*$, because they are functions of t. So here it is $u(0)\,B^*$.

that is equal to B_0.

So, your u(0) which is your initial condition now which is $\frac{B_0}{B^*}$.

Now B_0 is the density because it is initial value from here.

So, it is $\mathrm{M}L^{-3}$ if I take the dimension and your B^* is α.

and dimension of α is again the density.

So, $\frac{\mathrm{M}L^{-3}}{\mathrm{M}L^{-3}}$ and it is 1.

$$[u(0)] = \frac{[B_0]}{[B^*]} = \frac{\mathrm{M}L^{-3}}{[\alpha]} = \frac{\mathrm{M}L^{-3}}{\mathrm{M}L^{-3}} = 1$$

So, your initial condition is again also dimensionless.

So, you get your equation of the form

$$\frac{du}{dz} = au\left(1 - \frac{u}{b})\right) - \frac{u^2}{1 + u^2}$$

$$u(0) = \gamma$$

So, I take this expression to be γ.

So, this is now your differential equation of the given model which is now dimensionless.

Now, the question is that whether this is the unique one or we can have a separate one also.

So as I told you there is no hard and first rule of choosing the values of this B^* and t^*.

So if you want an alternative method, so let us see if we can find any alternative method of this alternative values of this B^* and t^* so that we can get a different dimensionless equation of the same model.

So, what do you do is we rewrite the equation.

So, you have when you have substituted $B = u\,B^*$ and $t = zt^*$ you substituted in the original equation and you got the resulting equation to be

$$\frac{du}{dz} = r\,t^*u\left(1 - \frac{u\,B^*}{k}\right) - \frac{\beta B^*t^*}{\alpha^2 + B^{*2}u^2}$$

In the previous one, we took that $B^* = \alpha$.

But let us now look into this equation.

If I take $B^* = k$ instead of $B^* = \alpha$, then what we will get?

We will get

$$\frac{du}{dz} = r\,t^*u\left(1 - \frac{u\,k}{k}\right) - \frac{\beta B^*t^*u^2}{\alpha^2 + k^2u^2}$$

it will cancel and

$$\frac{du}{dz} = r\,t^*u(1 - u) - \frac{\beta B^*t^*u^2}{\alpha^2 + k^2u^2}$$

So this part is free from any parameter.

Now if I choose my

$$t^* = \frac{1}{r}$$

then this will cancel and this whole term will be free of any parameter.

So, you can see that that is how it works.

You have to first put some value of this dependent variable or even independent variable and then decide what will be the value of the other variables.

So, if I simplify here, I will get

$$\frac{du}{dz} = u(1-u) - \frac{\beta k \frac{1}{r} u^2}{\alpha^2 + k^2 u^2}$$

So, this is what you get once you put this $\beta^* = k$. Now, I will do a little simplification here.

So, I will write this as u(1-u) I will take this k^2 common.

So a big simplification and you will get

$$-\frac{\frac{\beta}{rk} u^2}{\frac{\alpha^2}{K^2} + u^2}$$

So if I take this k^2 that k^2 is going to cancel with this k and you are going to get $\frac{\beta}{rk}$ and this side will be $\frac{u^2}{\frac{\alpha^2}{\alpha^2} + K^2}$

Now this part I can substitute as some capital A and this part I can substitute as some capital B and I get

$$\frac{du}{dt} = u(1-u) - \frac{Au^2}{u^2 + B^2}$$

where

$$A = \frac{\beta}{rk}$$

and your

$$B = \frac{\alpha^2}{k^2}$$

Now, so you got another equation where a first part is free of parameters again instead of 4 parameters you have 2 parameters what you have to check is whether they are dimensionless or not.

So, already we have checked that your u and z are dimensionless.

Now, you have to check whether your A and B are dimensionless or not.

So, if I take the dimension of A, this will be dimension of beta divided by dimension of k and dimension of r. So, as you can see dimension of beta from here, it is the density that is $ML^{-3}\ T^{-1}$.

Dimension of k is only density so ML^{-3} and dimension of r is per time so T^{-1} both are identical they cancels and A is a dimensionless quantity because this is a pure number.

$$[A] = \frac{[\beta]}{[r][k]} = \frac{\mathrm{M}L^{-3}\,T^{-1}}{T^{-1}\;\;\mathrm{M}L^{-3}} = 1$$

In the similar manner if I take the dimension of B it is dimension of α by dimension of k. So, B sorry this is alpha by k and this is equal to dimension of alpha is again density $\mathrm{M}L^{-3}$ dimension of k again than density $\mathrm{M}L^{-3}$ and a pure number hence B is also dimensionless.

$$[B] = \frac{[\alpha]}{[k]} = \frac{\mathrm{M}L^{-3}}{\mathrm{M}L^{-3}} = 1$$

So, you get an equation where an alternate equation through scaling where your parameters capital A and capital B is again dimensionless.

So, as you can see that this transformation using this scaling is not unique.

You can choose various values such that you get the differential equation to be dimensionless and it is in the hand of the modeller which differential equation he or she will take that will facilitate your modelling process.

And for this obviously if you can just consider the initial condition that is

$$B(0) = B_0$$

This is again already proved that this is dimensionless.

So, you have your differential equation of the form

$$\frac{du}{dz} = u(1-u) - \frac{Au^2}{u^2 + B^2}$$

with the initial condition

$$u(0) = \gamma$$

So, that is how this scaling helps you in reducing the number of parameters in a particular model.

So, we end our lecture with this and in the next lecture we will be talking about how to build a mathematical model or how to create a mathematical model from the scratch.

Till then bye-bye.