EXCELing with Mathematical Modeling
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Week – 06
Lecture – 27 (Vertical Oscillations)

Hello welcome to the course EXCELing with mathematical modeling.

Today we will be discussing about the vertical oscillations.

So, let us consider a string which is attached at the point A. So, this is an elastic string attached at the point A and natural length a. By natural length, I mean that it is not yet stretched, then you add a mass to it.

So, it becomes a bit stretched and this is the mass, say m, this is the point A, this is the point B and this system is in equilibrium.

So, there will be a tension say T_0 and there will be a force due to gravity mg.

So, in the equilibrium position this tension must be equal to the force due to gravity, that is,

$$T_0 = mg$$

Now, there is something called Hooke's law. Hooke's law tells you that the tension in the string will be

$$T = \lambda \frac{\text{increase in length}}{\text{original length}}$$

and this λ is called modulus of elasticity.

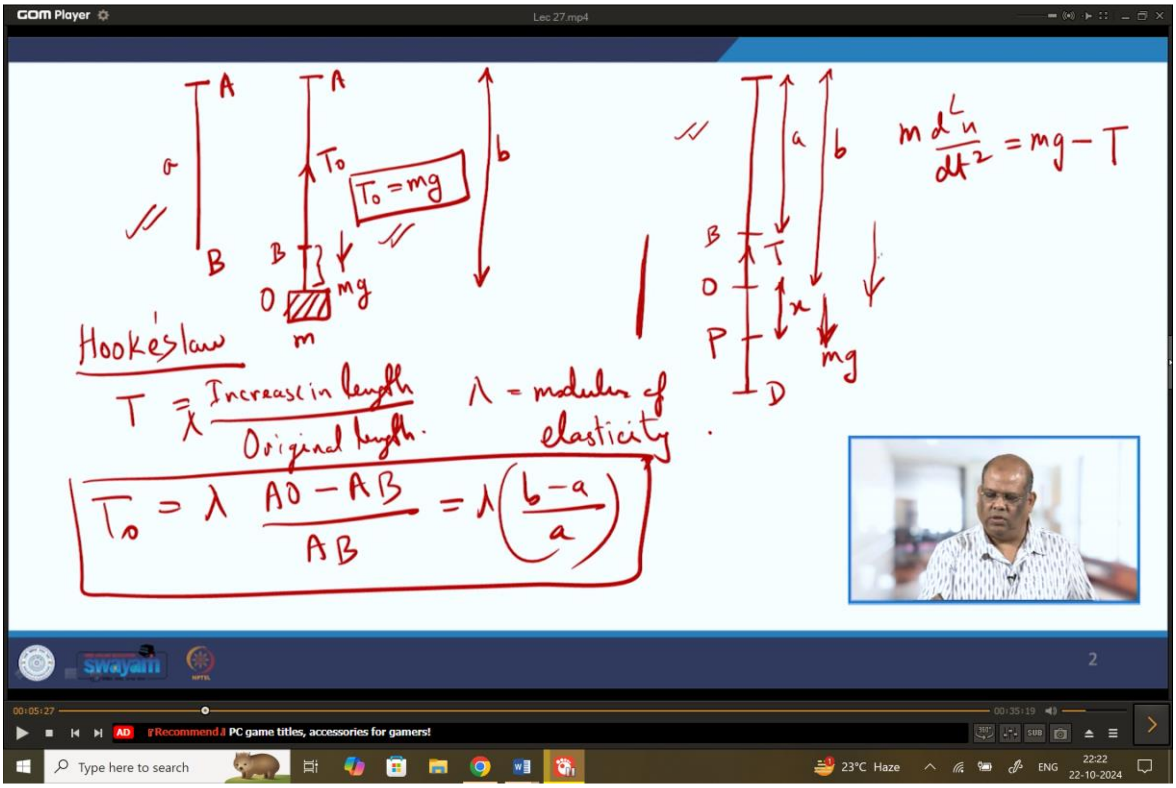
So, if we assume that after you have hung the mass, this length has become b. So, basically if I say this point is O, then

$$T_0 = \lambda \frac{AO - AB}{AB} = \lambda \left(\frac{b-a}{a}\right)$$

Now, once you have this setup, what you will do is, this is say O, you pull the string up to the point D and let it go.

So, this elastic string is going to oscillate just like this and if I say, that P be the position at any time t such that this OP = x. So, this length is a, this length is b.

So, this is the first one where we have only the natural length of the string, this is the second one where we have attached a mass m and we saw what is the tension in the equilibrium position and this is the third one where you have pulled it till the point D and let it go and it is just oscillating. So, what will be the equation of motion?



So, the equation of motion will start with $m\frac{d^2x}{dt^2}$ and you have pulled it in this direction where there is mg and there will be a tension T, and this tension is different from the equilibrium position now, because this is now the equilibrium is disturbed and T is the tension which now depends on this x.

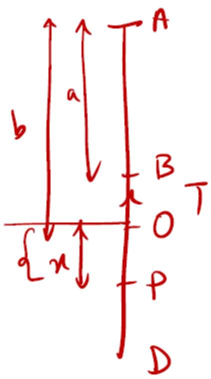
So, in this direction the force is mg, in the upward direction the force is tension T. So, this is our equation of motion.

$$m\frac{d^2x}{dt^2} = mg - T.$$

So, why this mg is taken positive and T is negative?

Because I have pulled the string in this direction, so in the direction of increasing x. So, that is why I have to take all the force in this direction to be positive and against this direction to be negative. So, this is in this direction mg which is positive and against the direction is the tension, that is negative.

So, you have this A, you have this B, you have this O and you have this D and this P is the position. So, this length was the natural length a, this is b, this is your x. So, the tension which is acting upwards will be the increase in length by the original length. So, this is



$$m\frac{d^2x}{dt^2} = mg - \frac{\lambda(b + x - a)}{a} = mg - \frac{\lambda(b - a)}{a} - \frac{\lambda x}{a}$$

Now, you have already shown that

$$T_0 = mg = \lambda\left(\frac{b-a}{a}\right)$$

they cancels and you are left with

$$m\frac{d^2x}{dt^2} = -\frac{\lambda x}{a}$$

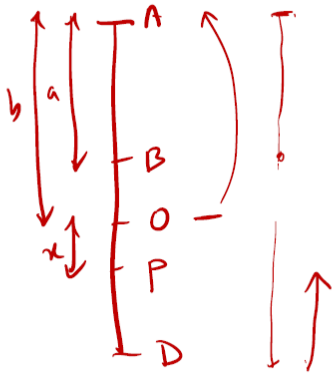
$$\Rightarrow \quad \frac{d^2x}{dt^2} = -\frac{\lambda x}{ma} = -\mu^2 x \text{ (say)}$$

So, the motion is simple harmonic about the centre O and your period of oscillation is

$$\frac{2\pi}{\sqrt{\mu^2}} = \frac{2\pi}{\mu} = 2\pi\sqrt{\frac{am}{\lambda}}$$

So, again this vertical oscillation gives you a simple harmonic motion about this centre O and the period of oscillation is this.

Let us now modify this a little bit. So we modify the situation like this, that you have the same kind of scenario, you have B stretched to O at the point D and let P be the position. So this is your a, this is your b, this is your x.



So, what I do now is I take this mass which is somewhere here at the point A and I drop it. So, if I drop it from the point A this is going to fall down till to some extent D and because of the elasticity again it will move up.

But. when you have dropped it from this point, from this a to b, this is the natural length of the string. So, till the mass come to the point B that elasticity is not working. So, it is just a free fall.

So, from A to B it is going to be a free fall and then from B to D you will have an equation of motion which will take care of this tension.

So, to model this scenario, what we do is, we have from this previous equation, this is your simplified equation of motion. So, if I put this, I will get

$$\frac{d^2x}{dt^2} = -\frac{\lambda}{am}x$$

Now suppose I put, just for the sake of simplification this $\lambda = 2mg$

You can do the whole derivation with λ also but with this you will have a simplification. So if I substitute it there I will get this is equal to

$$\frac{d^2x}{dt^2} = -\frac{\lambda}{am}x = -\frac{2\,m\,g}{a\,m}x = -\frac{2g}{a}x$$

So, this is our new equation of motion with a particular value of modulus of elasticity.

Now to get the whole scenario about the velocity and the time we have to solve this differential equation.

So, if you want to solve this kind of differential equation, one is using this complementary function and particular integral.

However, in this case, we have an easy technique. So, all you have to do is multiply both sides by

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = -\frac{2g}{a}x\frac{dx}{dt}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = -\frac{2g}{a}x\frac{dx}{dt}$$

$$\Rightarrow \int d\left(\frac{dx}{dt}\right)^2 = -\frac{4g}{a}\int x\,dx$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = -\frac{4g}{a}\frac{x^2}{2} + \text{constant} \quad \Rightarrow \left(\frac{dx}{dt}\right)^2 = -\frac{2g}{a}x^2 + \text{constant}$$

Let us see what information you know.

You have dropped it from this point A and it has reached the point B. So, what is the velocity of this mass at the point B?

Since from A to B, since the natural length is also a, that elastic property is not acting when the particle falls or the mass falls from A to B, only after it reaches the point A then that elasticity of the string will start working.

So, from A to B, it is a free fall. So, if it is a free fall and it starts from the point A, your initial velocity that is u = 0. So, if your initial velocity is equal to zero, then we know that

$$v^2 = u^2 + 2g\,x \;\Rightarrow\; v^2 = 0^2 + 2ga \;\Rightarrow\; v = \sqrt{2ga}.$$

It is positive because it is moving in the direction of increasing x. So, I know at the point B the velocity is $\sqrt{2ga}$. So, at the point B,

$$\frac{dx}{dt} = \sqrt{2ga}$$

and then what the value of this x. Now, if you want to calculate this value of x, then we have this

$$T_0 = mg \Rightarrow \lambda\left(\frac{b-a}{a}\right) = mg \;(\text{since}, \lambda = 2mg)$$

$$\Rightarrow 2mg\left(\frac{b-a}{a}\right) = mg \Rightarrow b - a = \frac{a}{2}$$

So, now if your particle is at the point B and you want to calculate because x is this distance but your x has been calculated from this point and it is moving towards this point.

So, from O in the downward direction you have calculated the x. However, if you want to calculate the x in the upward direction, I will get the distance, that is, from O to B, the distance is $\frac{a}{2}$, but since it is in the opposite direction, this x is going to be $-\frac{a}{2}$.

So, one more time what is happening is that you have calculated this x from the point O and it is moving in the downward direction and that direction is taken to be positive in the direction of increasing x.

However, if you want to calculate this x again from the point O in the upward direction it is fine you will get the value which is from O to B it is $\frac{a}{2}$ but since it is moving in the upward direction and not in the downward direction this we have to consider as $-\frac{a}{2}$.

So, our initial condition is that at the point B,

$$\frac{dx}{dt} = \sqrt{2ga}, \qquad x = -\frac{a}{2}$$

You substitute it in $\left(\frac{dx}{dt}\right)^2 = -\frac{2g}{a}x^2 + \text{constant}$, and you calculate the value of the constant, we get

$$2ga = -\frac{2g}{a}\frac{a^2}{4} + \text{constant}$$

So, the

$$\text{constant} = 2ga + \frac{2g}{a}\frac{a^2}{4} = = \frac{5ga}{2}$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = -\frac{2g}{a}x^2 + \frac{5ga}{2}$$

Now, if I want to find what is the greatest extension?

So, the greatest extension is that from this point A you have dropped it and the string has come up to this point where your velocity becomes zero.

So, that is the greatest extension and then move up. So, if I want to calculate this whole value, your value of x that means I have to put at the point $\frac{dx}{dt} = 0$.

So, because it has started from the A and it has come up to the point D. So, to get this much length I have to put at the D, the value is zero and I calculate that particular x. So, if I put

$$\frac{dx}{dt} = 0 \Rightarrow \ x^2 = \frac{5a^2}{4} \ \Rightarrow \ x = \frac{\sqrt{5}\,a}{2}$$

So the greatest extension is that from the point B up to the point D. So BD is your greatest extension and that is equal to BO + OD.

Now, BO you have already calculated, it is

$$BO = \frac{a}{2}$$

OD you just calculated which is

$$OD = \frac{\sqrt{5}\,a}{2}$$

and the total extension becomes

$$\frac{\sqrt{5}+1}{2}a.$$

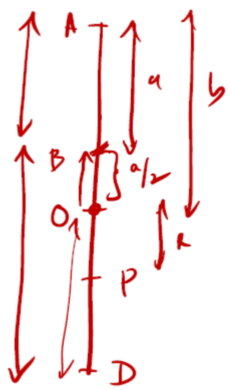
So, one more time, the greatest extension means that you have dropped it from the point A and then this mass will fall like this, at this point the elasticity of the string will come into action, it will go down against there is a tension and at one point that tension will be equal to the $force = mg$ and then it will stop. So, it means the velocity will be equal to zero. So, in the velocity equation you put that equal to zero, like here and calculate the value of x. So, that x is gives you this much value and this is the extension due to the mass. So, this plus this, will give you the greatest extension, which is $BO + OD$, and you got this value.

Next if you want to consider how much time it will take from falling here up to the point this.

So, if you want to do that, so basically, you have a time t_1, which is time from B.

Let me quickly again draw the figure, this is A, this is B, this is O and this is D. This is your a, this is your b and this is your x and this much is $\frac{a}{2}$.

So, if you want to calculate the time, you have to consider the equation



$$\left(\frac{dx}{dt}\right)^2 = -\frac{2g}{a}x^2 + \frac{5ga}{2}$$

$$\Rightarrow \frac{dx}{dt} = \sqrt{\frac{2g}{a}}\sqrt{\frac{5\,a^2}{4} - x^2}$$

$$\Rightarrow \int_0^{t_1} dt = \sqrt{\frac{a}{2g}}\int_{-\frac{a}{2}}^{\frac{\sqrt{5}\,a}{2}} \frac{dx}{\sqrt{\frac{5\,a^2}{4} - x^2}}$$

So if I now want to put the value of x, so x is calculated from O but it has started from B. So if I calculate this distance, it is $a/2$ but since it is in the opposite direction this I have to take $-a/2$. And from here to here I have already calculated that value which is $\frac{\sqrt{5}\,a}{2}$.

So if you integrate this, this is going to be

$$\int_0^{t_1} dt = \sqrt{\frac{a}{2g}}\ \sin^{-1}\left(\frac{x}{\frac{\sqrt{5}\,a}{2}}\right)\Bigg|_{-\frac{a}{2}}^{\frac{\sqrt{5}\,a}{2}} \quad \Rightarrow \quad t_1 = \sqrt{\frac{a}{2g}}\left[\frac{\pi}{2} + \sin^{-1}(1/\sqrt{5})\right]$$

This is a simple one, just substitute the values, and you will get this one.

Now what will be the time from a to b and please remember from A to B it is just a free fall. So, if t_2 is the time from A to B.

So, there you will use

$$x = ut_2 + \frac{1}{2}g{t_2}^2$$

So, at the point B, your x = a, initial velocity is zero. So,

$$x = \frac{1}{2}g{t_2}^2 \quad \Rightarrow \quad {t_2}^2 = \frac{2x}{g} \quad \Rightarrow \quad t_2 = \sqrt{\frac{2a}{g}}.$$

So, your time from A to D is $t_1 + t_2$ where your t_1 is this quantity, and your t_2 is this quantity.

If you want that what will be the time from falling from A to D and back, all you have to do is multiply this time two times, that is, 2 $(t_1 + t_2) = 2\left(\sqrt{\frac{a}{2g}}\left[\frac{\pi}{2} + \sin^{-1}(1/\sqrt{5})\right] + \sqrt{\frac{2a}{g}}\right)$.

Let us now look into the numerical solution of this vertical oscillation.

So we have the equation

$$m\ddot{x} = mg - T$$

and from here we derived that

$$\ddot{x} = -\frac{\lambda}{ma}x$$

Now we have to solve this equation numerically. As you can see, it is a second order differential equation.

So, what we do is we make it a system of first order differential equation.

And how to do that?

So, you put $\dot{x} = y$ and then $\dot{y} = -\frac{\lambda}{ma}x$.

So, if you put $\dot{y}$, this becomes $\ddot{x} = y$, because the dot means you are differentiating with respect to time.

So,

$$\dot{y} = -\frac{\lambda}{am}x$$

So, if you simplify this, instead of this $\dot{y}$, if I now substitute this equation, so $\dot{y}$ from here is $\ddot{x}$

So, you get

$$\ddot{x} = -\frac{\lambda}{ma}x,$$

which is nothing but this original equation.

So, this original equation can be written in the system of as a system of equations, first order equations.

And I need an initial condition. So, I take say at time t = 0, I have

$$\frac{dx}{dt} = 0, x = 11$$

So, when I say $\frac{dx}{dt}$, which is $\dot{x}$, which is actually the value of y to be 0.

And let us take some numerical values of λ, a and m. So, we will be using Microsoft Excel to solve this system of first order differential equations.

Now, I already have the files open.

So, as you can see that you have

$$\frac{dx}{dt} = y\,, \qquad \frac{dy}{dt} = -\frac{\lambda x}{am}$$

We have taken $\lambda = 2, a = 10, m = 20, h$, which we will be using for Euler's numerical method is 1. So, I have my t, I have my x and I have my y. So, at t=0, the value of x=11 and the value of

$$y = \frac{dx}{dt} = 0.$$

So the velocity at time t=0, y which is $\frac{dx}{dt}$, that is the velocity is 0 and the initial value of the distance x=11.

So let us now calculate.

So this is equal to $0 + 1$ and let me drag them. The reason we are taking so many values because there will be an oscillation and when you take more values the oscillations becomes much clear so we take 400 of these values.

Next we calculate this x so x is equal to so we are solving this is equal to this is equal to $x_0 + h$, which is a constant multiplied by y. Similarly, we calculate y, this is equal to $y(0)$ +h which is a constant.

So, we insert the \$ multiplied by $-\frac{\lambda}{am}x$

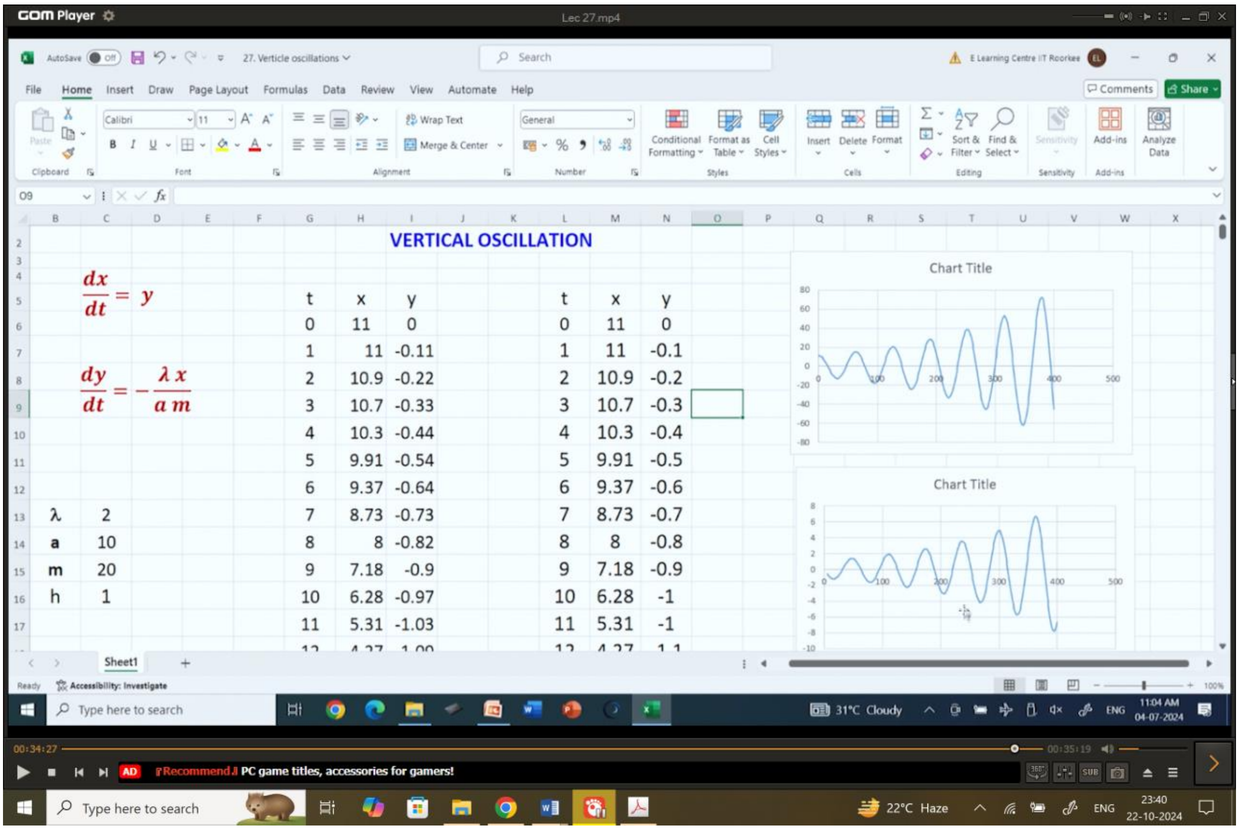
Since these are constants, we put the $ inside, put this thing under a bracket, denominator and closed. So, this is multiplied by x and now we just drag them to 400 values like this.

So I select them, I make the data font size a little bit large, generally take them as 20 and align them.

So this is now the calculated value. So let us first plot between t and x.

So we will be plotting two graphs once with t and x another risk between t and y and we have sort of 400 values here and we go to insert the charts and add the oscillation.

So this is one graph the another one is between t and y. Now to select the third column what you do is you press control you highlight this you press shift and the down key so it will select all the values and then again you go to insert and the graph.



So, the conclusion is that now this one is of x that is $\dot{x}$=y and this one is the graph for y. So, ultimately if I substitute y this is $\frac{d^2x}{dt^2}$.

So, this is basically the velocity and this is basically the distance which you solve from here.

And both the curves are periodic in nature which confirms that there is a vertical oscillation.

So, with this we come to the end of the discussion of this particular problem about the vertical oscillations.

In our next lecture, we will be talking about some epidemic models. Till then, bye-bye.