EXCELing with Mathematical Modeling
Prof. Sandip Banerjee
Department of Mathematics
Indian Institute of Technology Roorkee (IITR)
Week – 01
Lecture – 02 (Units and Dimensions)

Hello.

Today we will be talking about the units and dimensions which are again very important aspect of the mathematical modeling.

So, how do you define the unit? So, you can say that unit is any standard used for comparison in measurements.

So, basically some standard reference is required to express the result of the measurement of certain quantity.

For example, if I say, huge amount of rice or large amount of grain in the godown. This gives you a vague idea.

So, I need to say 30 kilograms of rice or 50 kilograms of grain. So, that gives you a specific idea that how much amount of rice or grain is stored in the godown.

Carl Friedrich Gauss

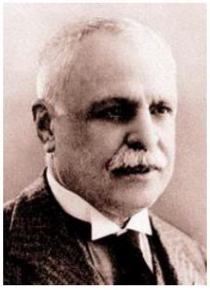
So that is where the units came.

So the first system, it is called the CGS system. It was proposed by this German mathematician Gauss in 1832. The full name as you can see Carl Friedrich Gauss.

But, Gauss chose these units in terms of this millimeter, milligram and seconds.

Later on in 1873, there is a committee of the British Association for Advancement of Science, which includes famous physicists, like, James Maxwell, William Thomson, they recommend that, no, it should be the adoption of centimeter, gram and seconds as the fundamental units. And, hence the name CGS system.

Giovanni Giorgi



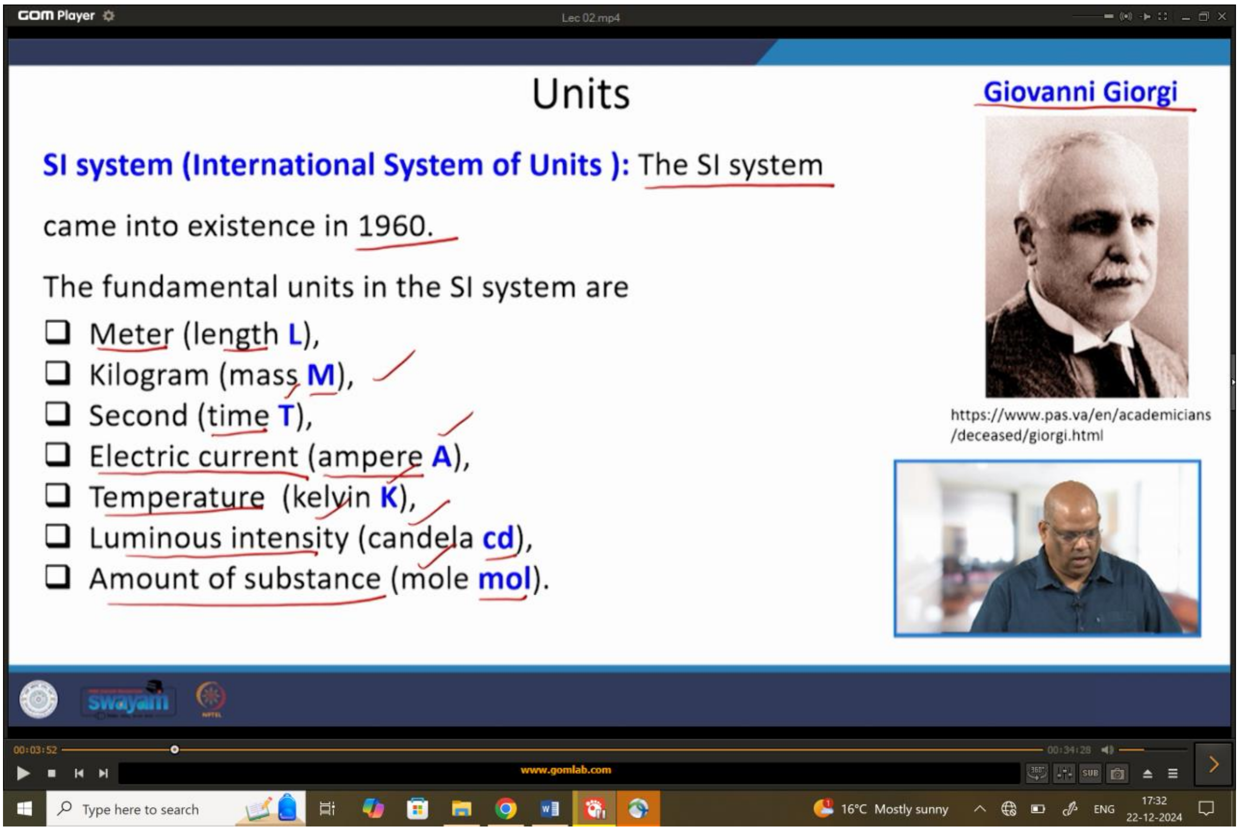
The next system is this MKS system. This was proposed by Professor Giovanni Giorgi.

And, the M stands for the meter, K for the kilogram and S for the seconds.

But this was again promoted and popularized by the electrical engineer George A. Campbell.

George A. Campbell

We now move into our next metric system or system of measurements, namely, the SI system, the international system of units.
So, this SI system, it came into existence in 1960. Again, the credit goes to Giovanni Giorgi.



So, the fundamental units in the SI system are the meter, its dimension is L (length), then, the kilogram which denotes the mass, the dimension is given by M, then, the second, which denotes the time, the dimension is given by T.

Then, we have four more, one is the electric current, measured in ampere and the dimension is A, the temperature measured in kelvin and the dimension is K, the luminous intensity measured in candela, and the dimension is cd and the amount of substance measured in mole and the dimension is mol.

So, with all these seven fundamental units, we form this SI system and this SI system is the most widely used system of measurements.

Now, how do you define this dimension?

So, this dimension of this physical quantity, it means that the power to which the fundamental unit or units can be raised to obtain the unit of that quantity.

In a more simplified language, it means that a physical quantity that can be expressed in terms of the fundamental quantities or namely the mass (M), the length (L) and the time (T).

For example, if I take say volt. The unit of volt is $\frac{kg\frac{m^2}{s^3}}{I}$. If I take the dimension it is M $L^2\ T^{-3}A^{-1}$.

Now, let us calculate the dimension of few known quantities.

The very first is acceleration. So, we know, that is,

$$[\text{Acceleration}] = \left[\frac{\text{Velocity}}{\text{Time}}\right]$$

The square bracket is the notation for the dimension. So, this will be equal to

$$[\text{Acceleration}] = \frac{[\text{Velocity}]}{[\text{Time}]} = \frac{\mathrm{L}T^{-1}}{T} = L\,T^{-2}.$$

So, this is the dimension of the acceleration.

Next, we have force. So,

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$

$$\Longrightarrow\ [\text{Force}] = [\text{Mass}] \times [\text{Acceleration}] = M \times L\,T^{-2} = ML\,T^{-2}$$

So, force will have the dimension $M\ L\ \ T^{-2}$.

Then work or work done, that is, defined as

$$\text{Work} = \text{Force}\ \times \text{Distance}$$

$$\Longrightarrow\ [\text{Work}] = [\text{Force}] \times [\text{Distance}] = ML\,T^{-2} \times \mathrm{L} = M\,L^2\,T^{-2}$$

So, work done will have the dimension $M\,L^2\,T^{-2}$.

Now, the specific gravity, which is defined as

$$[\text{Specific Gravity}] = \frac{\text{Mass of a body}}{\text{Mass of equal volume of water}}$$

$$\Longrightarrow\ [\text{Specific Gravity}] = \left[\frac{\text{Mass of a body}}{\text{Mass of equal volume of water}}\right] = \frac{M}{M} = 1\ \text{(a pure number)},$$

This results in a pure number 1 and hence the specific gravity is dimensionless.

Now, what do you mean by dimensionally homogeneous?

So in mathematical modeling, you will be forming models which will be consisting of differential equation or difference equation.

Now you have a left hand side and you have a right hand side of the equation.

So, the dimension l.h.s equals the dimension of the r.h.s, that is,

$$[\text{left hand side }] = [\text{right hand side}].$$

If you get that, then it is said dimensionally homogeneous.

Let's take an example.

Say, a body is moving in a straight line with a force proportional to the cube of its velocity.

So, if I write F to be the force, it varies as the cube of the velocity v.

So, I can write this as $F = k\ v^3$ where k is the constant of proportionality.

Now I take the dimension on both sides, that is,

$$[F] = [k][v^3]$$

So, after this step I am going to write the dimension of the force. So

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$

I take the dimension, and get

$$[\text{Force}] = [\text{Mass}] \times [\text{Acceleration}] = [k][v^3]$$

$$\Longrightarrow M \times LT^{-2} = [k][(LT^{-1})^3]$$

I do not know the dimension of k. So, if I simplify them, I will get

$$[k] = ML^{-2}T.$$

So, this is the dimension of this proportionality constant k and if I want to write the unit, the unit is going to be

$$\frac{kg}{m^2\,s}$$

So that is how we solve an equation and find the dimensions using the definition of dimensionally homogeneous.

Next, we calculate the dimension of energy. We have the famous equation

$$E = m\,c^2,$$

given by Albert Einstein, where E is the energy, m is the mass and c is the velocity of light. So, taking dimensions on both sides, we get

$$[E] = [m]\,[c^2] = M \times (LT^{-1})^2 = ML^2T^{-2}\,.$$

So, this is the dimension of energy.

Now, there are various forms of energy, say, potential energy (P.E.), and we know

$$\mathrm{P.E.} = mgh$$
$$\Rightarrow [\mathrm{P.E.}] = [mgh]$$
$$= [m][g \text{ (acceleration due to gravity)}][h \text{ (height)}]$$
$$= M\,LT^{-2}L = ML^2T^{-2}.$$

If you take the kinetic energy (K.E.), then

$$\mathrm{K.E.} = \frac{1}{2}mv^2$$
$$\Rightarrow [\mathrm{K.E.}] = \left[\frac{1}{2}mv^2\right] = \left[\frac{1}{2}m\right][v^2] = M \times (LT^{-1})^2 = ML^2T^{-2}.$$

Again, it is the same. Similarly, you can consider the Heat energy (Q), so

$$[\mathrm{Q}] = ML^2T^{-2}.$$

So, any form of energy will have the same dimension, which is ML^2T^{-2}.

Now if in any model you come across a term like $\cos(\alpha t)$ or $e^{\alpha t}$, where t stands for time and you want to find the dimension of α, such that αt is dimensionless. Then, what you have to do?

So, if you notice that you have $\cos(\alpha t)$, so t is the time which has the dimension T and αt has to be dimensionless. So, α must have the dimension such that it cancels with this T, that is

$$[\alpha t] = [\alpha]\,[t] = [\alpha]T$$

Clearly, $[\alpha] = T^{-1}$, such that $[\alpha t]$ becomes dimensionless.

Similarly, in $e^{\alpha t}$, it is also the same thing. You have to put $[\alpha] = T^{-1}$, such that $[\alpha t]$ is dimensionless.

If your equation involves a derivative, then the dimension of the derivative are given by the ratio of the dimension because the derivative is a limiting ratio of two quantities.

So, if you recall, we have

$$\frac{dy}{dx} = \lim_{\Delta \to 0}\frac{\Delta y}{\Delta x},$$

which is the definition of the limit. So, you can see that it is the limiting ratio of the two quantities $\frac{\Delta y}{\Delta x}$.

Now, let us calculate the dimension of $\frac{dv}{dt}$, then

$$\left[\frac{dv}{dt}\right] = \frac{[v]}{[t]} = \frac{LT^{-1}}{T} = LT^{-2}$$

So, if you notice $\frac{dv}{dt}$ is rate of change of velocity, which is the acceleration and so you can see that it matches with the dimension of the acceleration.

The next is $\frac{\partial v}{\partial x}$. So,

$$\left[\frac{\partial v}{\partial x}\right] = \frac{[v]}{[x]} = \frac{LT^{-1}}{L} = T^{-1}$$

The next one is $\frac{\partial^2 v}{\partial x^2}$. So,

$$\left[\frac{\partial^2 v}{\partial x^2}\right] = \frac{[v]}{[x^2]} = \frac{LT^{-1}}{L^2} = L^{-1}T^{-1}.$$

Next, we have $\frac{\partial u}{\partial x}$, where u is the temperature. So,

$$\left[\frac{\partial u}{\partial x}\right] = \frac{[u]}{[x]} = \frac{K}{L} = K\, L^{-1}.$$

Similarly,

$$\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{[u]}{[x^2]} = \frac{K}{L^2} = K\, L^{-2}.$$

Now, these two quantities may be added if they have the same dimension and quantities of different dimension can be multiplied or divided. These are one of the properties.

Now, pure numbers are dimensionless and if you multiply any physical quantity by a pure number that does not change its dimension.

Now, the dimension of this integral $\int_a^b p\, dq$ it is given by $[p] \times [q]$.

And, the important index law which says that if A, a physical quantity has the dimension

$$M^{a_1} L^{b_1}\, T^{c_1}.$$

whereas another physical quantity B has the dimension

$$M^{a_2} L^{b_2}\, T^{c_2}.$$

Then by index law, the dimension of

$$[AB] = M^{a_1+a_2}\, L^{b_1+b_2}\, T^{c_1+c_2}$$

Now, let us take an example.

You have given a differential equation

$$\frac{dp}{dr} = \rho \frac{v}{r^2}$$

where p is your pressure, ρ is your density, r is your radial distance and v is the velocity. You have to check if the equation is dimensionally homogeneous or not, that is, the dimension of the left hand side must be equal to the dimension of the right hand side.

So, let us start with the pressure. So you have to write the definition of pressure (p), which is

$$\mathrm{p} = \frac{\text{Force}}{\text{Area}}.$$

Now

$$\text{Force} = \text{Mass} \times \text{Acceleration},$$

so

$$\mathrm{p} = \frac{\text{Mass} \times \text{Acceleration}}{\text{Area}}$$

$$\Rightarrow [p] = \frac{[\text{Force}]}{[\text{Area}]} = \frac{[\text{Mass}] \times [\text{Acceleration}]}{[\text{Area}]} = \frac{M \times L\,T^{-2}}{L^2} = ML^{-1}T^{-2}$$

Next comes the density ρ. So,

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}.$$

$$\Rightarrow [\text{Density}] = \frac{[\text{Mass}]}{[\text{Volume}]} = ML^{-3}.$$

Dimension of velocity we already have done, velocity is distance per unit time and the dimension of r which is again the radial distance which is L. So, once you get the dimension of all this, let us start with the left hand side, that is, $\frac{dp}{dr}$. So,

$$\left[\frac{dp}{dr}\right] = \frac{[p]}{[r]} = \frac{M\,L^{-1}T^{-2}}{L} = M\,L^{-2}T^{-2}.$$

Let us take the right hand side, which is $\rho\,\frac{v}{r^2}$. So,

$$\left[\rho\,\frac{v}{r^2}\right] = [\rho\,]\,\frac{[v]}{[r^2]} = \frac{ML^{-3} \times L\,T^{-1}}{L^2} = ML^{-4}T^{-1}.$$

So, as you can see the dimension of the left hand side and the dimension of right hand side, they are not equal and hence the equation is not dimensionally homogeneous.

Let us take another example. So, the problem is distance travelled by a particle in time t is given by the equation $s = a + bt + ct^2 + dt^3$. You have to find the dimension of a, b, c and d.

Obviously, we will be using the definition of dimensionally homogeneous. So, I write the equation,

$$s = a + bt + ct^2 + dt^3,$$

Here, s is the distance, t is the time, I do not know what a, b, c, d's are, I have to find their dimensions. So,

$$[s] = [a] + [bt] + [ct^2] + [dt^3].$$

$$\Longrightarrow [s] = [a] + [b]T + [c]T^2 + [d]T^3$$

Now, if this equation has to be dimensionally homogeneous the dimension of the left hand side must be equal to the dimension of the right hand side.

As you can see the dimension of the left hand side is L

And, the dimension of the right hand side, if $[a] = L$, then it is dimensionally homogeneous.

Now, if $[b]T$ has to be L, then the $[b] = LT^{-1}$, so that if you multiply it by T, it gives you L. With the similar logic, $[c] = LT^{-2}$, so that if you multiply by T^2, the dimension is L.

And similarly, $[d] = LT^{-3}$. So, if you multiply by T^3, it is dimension L. So, to maintain that the equation is dimensionally homogeneous,

$$[a] = L, \qquad [b] = LT^{-1}, \qquad [c] = LT^{-2} \qquad \text{and} \qquad [d] = LT^{-3}.$$

We next look into the following example:

Example: Fourier's law, relating heat flux to temperature gradient is given by

$$J = -\alpha \frac{\partial u}{\partial x}$$

where **J** is the heat flux, **u**is the temperature, and **x** denotes distance. Determine the dimension of α.

To solve the given problem, we need the definition of the heat flux. So, heat flux is the heat energy per unit area per unit time. So,

$$\text{J} = \frac{\text{Eneregy}}{\text{Area} \times \text{Time}}.$$

$$\Longrightarrow \ [\text{J}] = \frac{[\text{Eneregy}]}{[\text{Area}] \times [\text{Time}]} = \frac{[\text{Work done}]}{[\text{Area}] \times [\text{Time}]} = \frac{[\text{Force}][\text{Distance}]}{[\text{Area}] \times [\text{Time}]}$$

$$= \frac{[Mass][Acceleration][Distance]}{[Area] \times [Time]} = \frac{MLT^{-2}L}{L^2T} = MT^{-3}.$$

Now, we take the equation

$$J = -\alpha \frac{du}{dx}$$

$$\Rightarrow \; [J] = [\alpha]\frac{[\partial u]}{[\partial x]} = [\alpha]\frac{[u]}{[x]}$$

(ignoring the negative sign)

$$\Rightarrow MT^{-3} = [\alpha]\frac{K}{L} \Rightarrow \; [\alpha] = MLT^{-3}K^{-1}$$

So, dimension of α is $M\,L\,T^{-3}Q^{-1}$.

We now look into dimensional analysis.

So, first we define what is a dimensional equation.

So, when the dimension of a quantity is obtained and it is expressed in the form of an equation, that equation is simply called the dimensional equation.

And then what is dimensional analysis?

So, it is a method of studying the relationship between physical quantities with the help of dimensions and units of measurements.

For example, if we consider the frequency of a stretched string, now that depends on the tension of the stretched string, the length of the stretched string and the mass per unit length of the string.

If you want to find a formula or want to find a relation between the frequency and the tension, the length and the mass per unit length of the string, here where this dimensional analysis comes.

So, how we proceed?

Let us take the example: consider the frequency n that depends on the tension t_s, the length l and the mass per unit length. So, I can write frequency n as

$$n = k\, t_s^a\, l^b\, m^c.$$

Now you have to be very careful here, it is still that mass per unit length while calculating its dimension. This is not only mass, it is mass per unit length.

If I now calculate the dimension of this tension. So, tension it is nothing but force.

$$\text{Tension} = \text{Force} = \text{Mass} \times \text{Accerleration} = M \; \times LT^{-2} = [m] = MLT^{-2}$$

The length l is L but M is mass per unit length. So,

$$m = \frac{Mass}{length} \Rightarrow [m] = ML^{-1}.$$

and n is the frequency which means vibration per unit time $\Rightarrow [n] = T^{-1}$.

Now, I need to find the values of these real numbers a, b and c and I will be using the index law. So, I take the dimension of the quantities

$$[n] = [k][t_s{}^a[l^b][m^c].$$

Now this is again

$$[n] = k\,(MLT^{-2})^a(L)^b(ML^{-1})^c$$

$$\Longrightarrow\ [n] = k\,(M)^{a+c}(L)^{a+b-c}(T)^{-2a}$$

$$\Longrightarrow\ T^{-1} = k\,(M)^{a+c}(L)^{a+b-c}(T)^{-2a}$$

$$\Longrightarrow (M)^0(L)^0(T)^{-1} = k\,(M)^{a+c}(L)^{a+b-c}(T)^{-2a}$$

Now, I use the index law and equate the power from both sides, which gives

$$-2a = -1 \Longrightarrow a = \frac{1}{2};$$

$$a + c = 0 \Longrightarrow c = -a = -\frac{1}{2}.$$

So, I got the value of a and I got the value of c and finally, we get

$$a + b - c = 0 \Longrightarrow b = c - a = -\frac{1}{2} - \frac{1}{2} = -1$$

$$\Longrightarrow\ c = -1.$$

So, now you put these values in

$$[n] = [k][t_s{}^a[l^b][m^c],$$

and get

$$n = kt_s{}^{\frac{1}{2}}l^{-1}m^{-\frac{1}{2}}\ \Longrightarrow n = \frac{k}{l}\sqrt{\frac{t_s}{m}}\ .$$

So, this is the formula or the relation between the frequency n which depends on the tension, the length and the mass per unit length.

With that we come to the end of our today's lecture on units and dimension. In our next lecture we will be hearing about the scaling.

So using this scaling we can make a equation dimensionless which is again a very important aspects of mathematical modeling.

By making an equation dimensionless you can reduce the number of parameters and also it helps in the numerical solutions of the model it makes it error free.

So, till then bye bye.