EXCELing with Mathematical Modeling
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Week – 03
Lecture – 12 (Phase Plane Analysis-I)

Hello, welcome to the course EXCELing with Mathematical Modelling.

In today's lecture, we will be learning about the phase plane analysis.

So, to start with, what is a phase plane?

Consider the differential equation

$$\frac{dx}{dt} = ax + by, \qquad \frac{dy}{dt} = cx + dy.$$

So, on the right hand side as you can see the terms are all linear. So the idea is if you have non-linear terms you linearize the system and then write it in this particular form.

If I write them in the matrix form, I will be getting

$$\frac{d}{dx}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and clearly $(0,0)$ is the equilibrium point here.

So, you must be wondering why we are doing all the analysis with $(0,0)$ only. If it is not a $(0,0)$, you can obviously change the origin to $(0,0)$ and they can do all the analysis. Hence, $(0,0)$ is always taken here and also, in this particular case, you can see that (0,0) satisfies this differential equation.

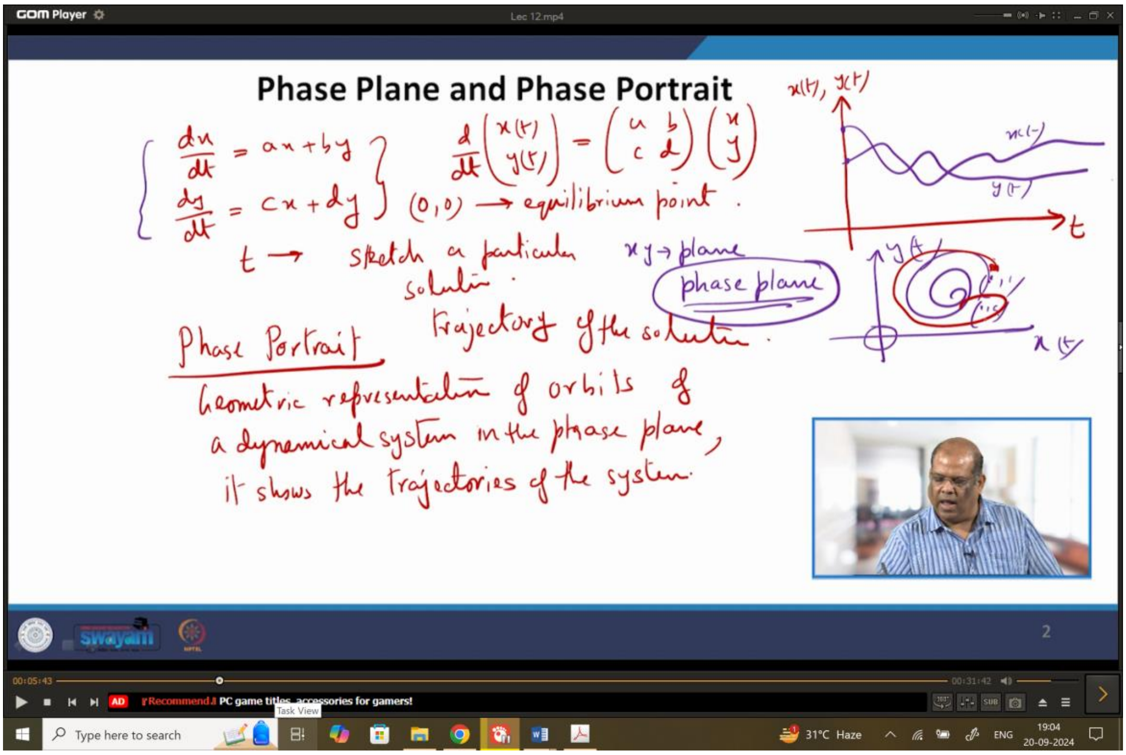
So, now if I want to solve this and plot the graph, I can do it like this, that I can plot it. So, this is my t, this is my $x(t)$ and $y(t)$.

So, if I plot the graph, say, it will start with some initial condition and it will show some dynamics. It will start with some initial condition and it can show some dynamics and go to the equilibrium point.

Not in this case because it must go to $(0,0)$ because that is the equilibrium point where I just took a random diagram. But the idea is that if you now want to plot against $x(t)$ and $y(t)$, then what will you get?

So, you think the solution to this system as the points in this plane xy, and the equilibrium solution will correspond now to this origin and this xy-plane is called the phase plane.

So, basically, you have this system of autonomous differential equation, you solve it, you get a solution, then this is say your $x(t)$, this is say your $y(t)$, and this is your independent variable t.



But now you want the solution in terms of x and y. So this is your x axis, this is your y axis.

You want to plot this is x, this is y, this is (x,y), this is (x,y) and you will get a figure somewhere.

So you want to see what kind of dynamics you get in this xy-plane and this xy-plane is called the phase plane.

So you put one value of t, a particular value and you sketch a particular solution.

So, you have started with some initial condition here and you got a curve, that is called the trajectory, the trajectory of the solution.

So, if you want what is a phase portrait, this is just the geometric representation of orbits of a dynamical system in the phase plane.

So basically what it shows it shows the trajectories of the system.

So you take one initial condition and you get one path, that is, one trajectory you take another initial condition, you get another path, this is another trajectory, you start from here, you get another, you start from here, you get another, and together, all this will be known as a phase portrait.

So, you need to remember what is a phase plane here, that is, the plane on which you get the dynamics, that is, called the phase plane. And for a single value of t or for a particular value of t, the solution which you get is called the trajectory of the solution.

And for many initial conditions, that means many values of t's, you will get a system of trajectories, which will ultimately give you a phase portrait from where you can understand how your system is behaving.

Let us now classify them.

So, this is phase plane diagrams which means phase portrait of linear systems.

So, after you linearize them, you put the equation in the form

$$\frac{dx}{dt} = \lambda_1 x + \lambda_2 y, \qquad \frac{dx}{dt} = \lambda_3 x + \lambda_4 y,$$

which can be written in matrix form as

$$\frac{d\tilde{x}}{dx} = A\tilde{x},$$

where $A = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}$ and $\tilde{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.

It is an autonomous linear system which has been put in this matrix form.

Clearly $(0,0)$ is your equilibrium point because it satisfies the right hand side.

And if this determinant

$$|A| = \begin{vmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{vmatrix} = \lambda_1\lambda_4 - \lambda_2\lambda_3 \neq 0,$$

then you have a unique steady state $(0,0)$, because it becomes a homogeneous system, and the solution can be visualized as trajectory moving in the xy-plane and can be sketched, and those are called the phase portraits, exactly the one which I explained before.

So, basically what you will do? You will get equations like this, you will find the equilibrium point in most cases it will be $(0,0)$, and then you just see where this determinant is not equal to zero and then you plot this solution and you see what kind of dynamics you get in the xy- plane which are known as phase portraits.

Now to explain or understand this better, I take a simplified system of the form

$$\frac{dx}{dt} = \lambda_1 x, \qquad \frac{dx}{dt} = \lambda_4 y$$

So, just in this previous equation, I put $\lambda_2 = 0$, and I put $\lambda_3 = 0$.

$(0,0)$ is the unique equilibrium point. It has been put in the matrix form as

$$\frac{d}{dx}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The characteristic equation of the matrix $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_4 \end{pmatrix}$ is

$$\begin{vmatrix} \lambda_1 - k & 0 \\ 0 & \lambda_4 - k \end{vmatrix} = 0 \Longrightarrow (\lambda_1 - k)(\lambda_1 - k) = 0, \Longrightarrow k = \lambda_1, \lambda_4$$

which are the roots of the characteristic equation, also called the eigenvalues of the matric B.

Now, depending on these signs of the eigenvalues, whether it is positive, whether it is negative, whether it is positive, negative, whether it is complex, we are going to classify the solutions.

So, the first thing is you solve and you get the solution to be like this

$$x = x_0 e^{\lambda_1 t}, \;\; y = \; y_0 e^{\lambda_4 t},$$

with some initial condition or initial values $x(0)$ and $y(0)$.

Now, if both this $\lambda_1, and \;\; \lambda_4$, both these eigenvalues are negative, then what happens?

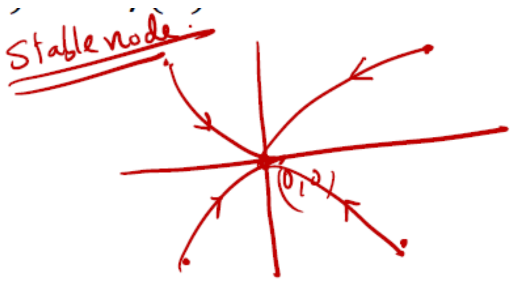
Case I

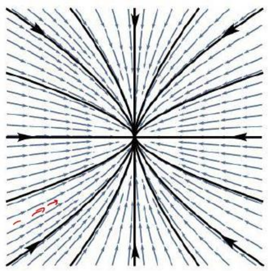
Then you get here that $e^{-\lambda_1 t}$ and $e^{-\lambda_4 t}$. So, as your time becomes large, so this trajectory will approach (0, 0) and it will converge to the equilibrium solution (0, 0), no matter what the initial condition is.

So, as the time increases you can see that $e^{-\lambda_1 t}$ will go towards 0 same with here and your x, y will approach the point (0, 0), which is the equilibrium point here.

And hence, this steady state or this equilibrium point is called a stable node.

So, you have to remember that you have been given a linear system. You find the eigenvalues and if both the eigenvalues are real and negative, what you get is called a stable node, and this is how it looks like.





When you will draw this diagram, all you have to do is you take this, this is where it converges in this case it is (0 0).

So, you take one point here, you take one point here, you take one point here, you take one point here, and all of them will converge to this particular point (0, 0).

The arrow is important, that is, it is moving towards (0, 0) and this is a stable node.

In this plot you can see, these are the vectors, which can be plot with the help of any software and it is reaching the point (0, 0).

Case II

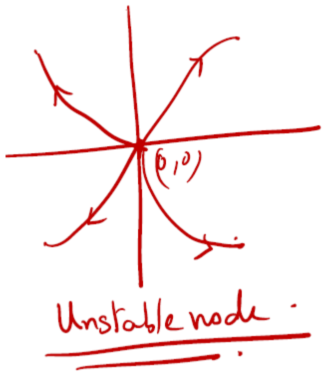
Now, let us see what happens if both of them are positive. So, obviously if both of them are positive then you have

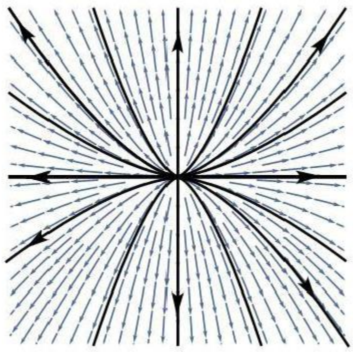
$$x = x_0 e^{\lambda_1 t}, \; y = y_0 e^{\lambda_4 t},$$

so, if both of them are positive this will diverge from the equilibrium (0,0), irrespective of whatever the initial condition is and then your steady state is called an unstable node. This is the figure of the unstable node.

So, it is exactly what you will do is you draw a node, you take one point in each quadrant. This line can be like anything okay. It can be any curve.

However, in this particular case because it is an unstable mode, the arrow will move away from the origin. So, the arrows are very important. This is unstable node.





Case III

Now, let us take the case when one of them is positive and one of them is negative. So, then what happens? So, in this particular case it is

$$x(t) = x_0 e^{-\lambda_1 t}, \; y(t) = y_0 e^{\lambda_4 t},$$

So, λ_1 is less than zero and λ_4 is positive, suppose I put negative sign here.

So, what happens? So, as t increases your $x(t)$ decreases but your $y(t)$ increases exponentially. So,

$$x(t) \to 0 \text{ as } t \to \infty \quad \text{and} \quad y(t) \to \infty \text{ as } t \to \infty.$$

All the solutions in this case will first approach this (0,0) for some time, but when they come close to it, they will move away from the steady state, irrespective of whatever the initial condition is.

And in this particular case, this steady state will be called a saddle point.

So, in this case what happens is that one of them is positive and one of the root is negative. So, due to the negativity, one solution will approach the origin due to positivity, one solution will move away from the origin.

So, the dynamics is that after reaching close to this (0,0), it will just come off of the system and moves away.

If you want to look into the dynamics, the phase portrait, so this is the phase portrait of a saddle.

Now, how to draw this? This is very important.

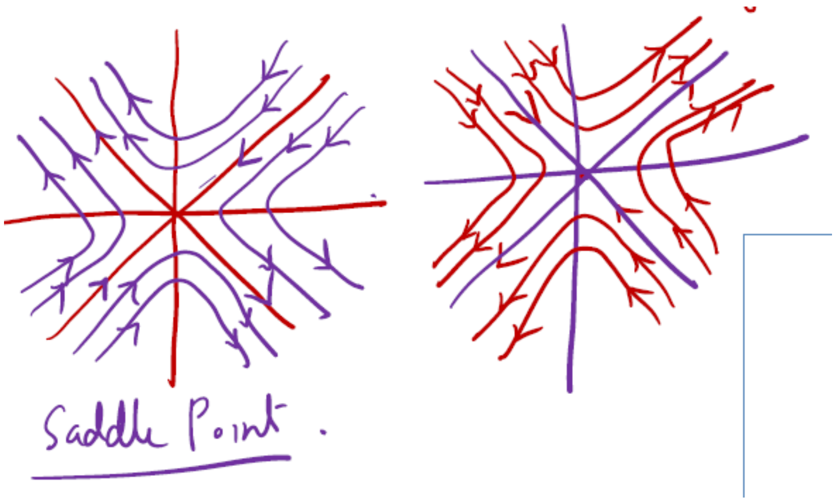
So, first thing is you draw this and then you draw two lines. So, first you draw this axis and then you draw these two lines.

First is in one line you just take the arrow inside and in one you take the arrow outside.

So, inside means in that particular case, the solution is moving in because of the negativity of one of the eigenvalue and moving out is because of the positivity of the other eigenvalue.

The rest of the solution will be like this. So, this is coming from this side and moving like this. So, coming from this side and moving like this. So, you put an arrow like this. Coming from this side and moving like this, put an arrow like this, put an arrow like this, here also put an arrow like this.

In the similar way, you just follow the path of these arrows and you just draw the figure like this. So, this becomes a saddle point.



Let me draw this one more time. So, first is you draw the axis and you draw two straight lines passing through the origin and then you just take one line where the arrow will go move towards the origin and another line where the arrow will move away from the origin.

So once you get these four arrows, you just see from this side it is moving to this side. So arrow is like this. Similarly, here also from this side it is moving this side.

So arrow is like this. From this side to this side, arrow is like this, and from this side to this side, the arrow is like this.

So, the figure of the saddle point.

Let us move to

Case IV

So, now you have complex eigenvalues, $\lambda_1 = a + ib$ *and* $\lambda_4 = a - ib$,

And since it is a quadratic equation, we know that the values will be complex conjugates and

your dynamics will depend on this value of a. So, your solution is in this form

$$x(t) = x(0)e^{at}(cosbt + i\, sinbt),$$

$$y(t) = y(0)\, e^{at}(cosbt - i\, sinbt)$$

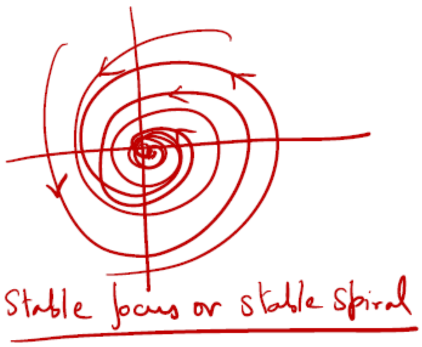
and suppose a is less than zero.

So, if a is less than zero, both these eigenvalues will have negative real part.

So, if it is a negative real part, then obviously this will decay because as t becomes large, this will decay and all these trajectories will spiral towards (0,0), irrespective of the initial condition and it is called either a stable spiral or a stable focus.

If I want to find this solution, this will look somewhat like this that it will start from somewhere, and then it is some sort of damped oscillation like this.

The oscillations are due to the presence of sine and cosine and the damping is because this is negative and it slowly decays out. So, the trajectories which you get or the phase portrait looks like this.



This is a stable focus. If you just want to draw them, you just take and round and round and round.

So, the thing which you have to remember is that this point will never enters this (0 0), it will just spiral round and round about this (0 0).

So, this is one point, I can take another point here, I can do like this. I can take another point here, I can do like this.

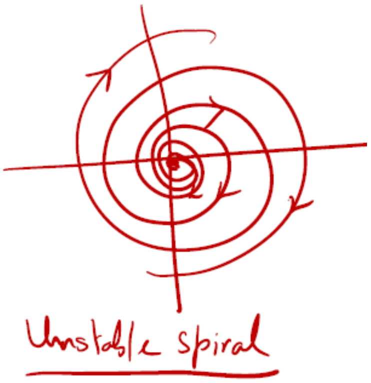
And I get what is called a stable spiral. And as usual, the arrow is important. You are moving towards the origin.

You are moving towards the origin. It's moving towards the origin. So, this will be a stable focus or a stable spiral.

And, when you have $a > 0$, obviously the eigenvalues have a real positive part, in this case e to the power at glows exponentially and the trajectory will spiral away from your (0 0), irrespective of whatever the initial condition may be, and it is called unstable spiral or unstable focus.

So, to draw this again, you draw the spiral and I will just arrow is outside, it is moving out. You can draw one more, so you start it from here, so it moves out.

So, this is your unstable spiral or unstable focus.



Now, what happens when a = 0?

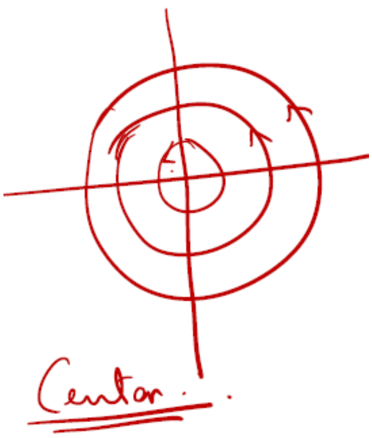
So, when a equal to zero, your eigenvalue becomes purely imaginary.

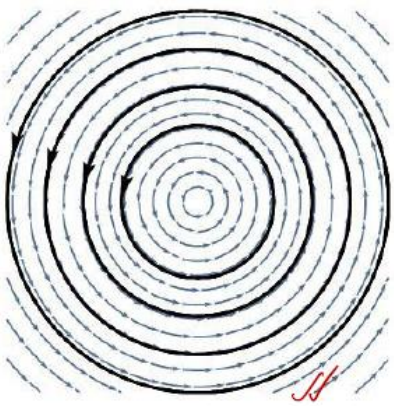
So, in that case is all trajectories will form a closed orbit like this, and the solution is periodic because you will have only the sine and the cosine functions and your solution will look like this, just a periodic solution.

So, this is your t and this is your x or y.

So, if you want to draw this, it is just concentric circles which is steady state will be called the center.

So, you just draw some concentric circles and mark the arrow in one direction. So, this is called a center.





To sum up, I put it in the tabular form. So, you have the system, linear system, you find the eigenvalues and then you see whether it is real or imaginary.

Nature of the eigenvalues λ, μ	Nature of the steady state (0,0)	Stability of the steady state (0,0)
Real, unequal, same sign:		
(i) $\lambda < 0$, $\mu < 0$	Node	Asymptotically stable
(ii) $\lambda > 0$, $\mu > 0$	Node	Unstable
Real, unequal, opposite sign:		
(iii) $\lambda < 0$, $\mu > 0$	Saddle point	Unstable
Real, equal, same sign:		
(iv) $\lambda = \mu = \lambda^* < 0$	Node	Asymptotically stable
(v) $\lambda = \mu = \lambda^* > 0$	Node	Unstable
Complex conjugates (a+ib, a-ib):		
(vi) $a < 0$	Spiral point	Asymptotically stable
(vii) $a > 0$	Spiral point	Unstable
(viii) a=0	Center	Stable but not asymptotically stable

So, if you remember this chart, I think your job is done and if the equilibrium point is not stable, it is then unstable.

Let us take some examples now, say, we start with

$$\frac{dx}{dx} = -x + 4y = f(x, y)$$
$$\frac{dy}{dt} = -x - y^3 = g(x, y)$$

So, I have taken nonlinear not the linear one.

So, the matrix A, so if I put this is as some f, and this is some g, is

$A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -1 & -3y^2 \end{pmatrix}$ at the equilibrium point (0,0).

because clearly (0,0) satisfies this equation and (0,0) is the equilibrium point.

So, at (0, 0) this value becomes $\begin{pmatrix} -1 & 4 \\ -1 & 0 \end{pmatrix}$. So, I need to find the eigenvalues. So,

$$|A - \lambda I| = 0 \Longrightarrow \quad \begin{vmatrix} -1-\lambda & 4 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\Longrightarrow \lambda(\lambda + 1) + 4 = 0 \Longrightarrow \quad \lambda^2 + \lambda + 4 = 0$$

$$\Longrightarrow \quad \lambda = \frac{-1 \pm i\sqrt{3}}{2}$$

So, I have a complex root with a negative real part and hence it is going to be a stable focus.

So you have a complex root with negative real part implies the equilibrium point (0,0) is a stable spiral or stable focus and the phase portrait is, you take like this and since it is stable you just put the arrows in, like this. So, this is a stable focus.

Let us take another example.

$$\frac{dx}{dt} = x(3 - x - 2y) = f(x, y)$$
$$\frac{dy}{dt} = y(2 - x - y) = g(x, y).$$

This time let us take other equilibrium point, which will be other than the origin.

So, equilibrium points, this I leave it to you, I have already calculated that so this will be (0,0) then it is (0,2), it is (3,0) and it is (1,1).

So these are the 4 equilibrium points you can easily check you put these equal to zero and solve them and you will get these values. The next thing is I will calculate A.

$$A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}$$

at the equilibrium point (x^*, y^*).

So let us start with $(x^*, y^*) = (0,0)$, so if I put it, I get, $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

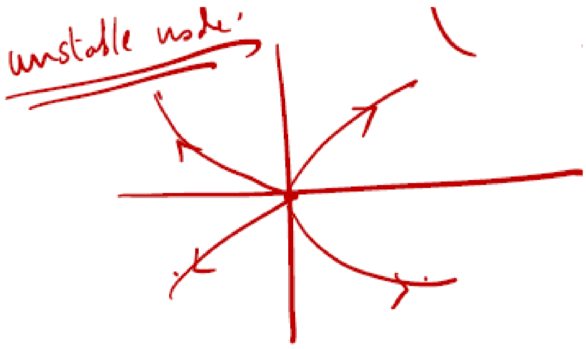
Clearly the eigenvalues are 2 and 3. So, they are real and positive, and hence it is unstable node.

If you want the face portrait or face plane diagram, you choose four initial points.

This is your origin. You just bring the curve here and it is unstable.

So, it is moving out.

That is the important part, you have to remember the arrows. So, this is your unstable node.



In the similar manner, you can check for the other equilibrium points.

The only thing you have to note here is when you are say checking for this (1,1), you will see that this is going to give you a saddle point.

Now, when you draw this figure and if I draw like this, do not label this axis because now this point is your (1,1), that is the equilibrium point, not the origin.

So, I will not give the exact position of the axis.

So, what does this mean?

So, I will write that this is not the (0,0), I will just write (1,1) and since it is a saddle point, then I draw these two lines, like this it is moving away and you draw your curves.

So, it is wise that you do not level your figure, you just get the dynamics and you write what kind of dynamics you are getting.

So, we end our lecture today and in our next lecture we will be taking this to the next level where I will explain that why when the two eigenvalues are positive, why you are getting that as a node, why not it is a focus.

Till then, bye-bye.