EXCELing with Mathematical Modeling
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Week – 01
Lecture – 01 (Introduction, Importance and Limitations)

Hello, today we will be starting the course Excelling with Mathematical Modeling.

In this course, you will be learning the basics of mathematical modeling, how to create model or how to develop a model from the scratch.

We will be using mostly the differential equation if the case is a continuous case or difference equation if the case is a discrete case.

We will be using mathematical tools to solve those equations whether it is analytically or numerically.

If it is numerically we will be using Microsoft Excel to solve those equations.

Once you get the solution we will visualize this in the form of a graph and once you get the graph you will understand the dynamics of the model.

Once you get the dynamics of the model you will be able to interpret the solution of the model in terms of the question that you have already posed while making the mathematical model.

So, let's start with excelling with mathematical modeling.

So, the very first definition, what is your mathematical modeling?

So as you can see, it is an application of mathematics to describe real world problems and investigate most important questions that arise for them.

For example, let's take the case of COVID-19.

So, if this COVID-19 comes back again, whether it will be a pandemic, or whether the heart immunity will again help us to overcome the problem.

How many hospital beds we will be requiring in a very worst case scenario?

So, these are the questions that can be posed in this real life problem and mathematical modeling can take care of that.

Let us take another example.

We all have heard of Alzheimer disease. Mostly the elderly people suffers from it.

That means you have a loss of memory with the ages.

Is there any function that can capture this cognitive decline.

So, by cognitive decline, I mean that their memory loss and their ability to do the day to day work.

So, this cognitive decline can it be captured by any mathematical function.

To answer these questions, we can use this mathematical modeling to get the answer.

So, once you pose the question, then you translate that question in the form of a mathematical problem.

And once you get the mathematical problem, you try to find a solution, try to visualize the solution and then interpret that solution in the language of real world problem to make predictions about the real world.

Now let's explain this schematic diagram.

So as you can see here, I have my question.

So the question which I pose now is, say, what is the growth of tumor?

Can there be a mathematical function that can capture the growth of tumor?

So my question is tumor growth, and I am looking for a mathematical function.

Is there any mathematical function that can capture this tumor growth or not?

So once you pose the question, I will come to the mathematical formulation.

So the mathematical formulation, if I take x to be the density of the tumor, then the rate of change of that density of that tumor, this is density of tumor.

So this rate of change will depend on some function, say, let us take it some

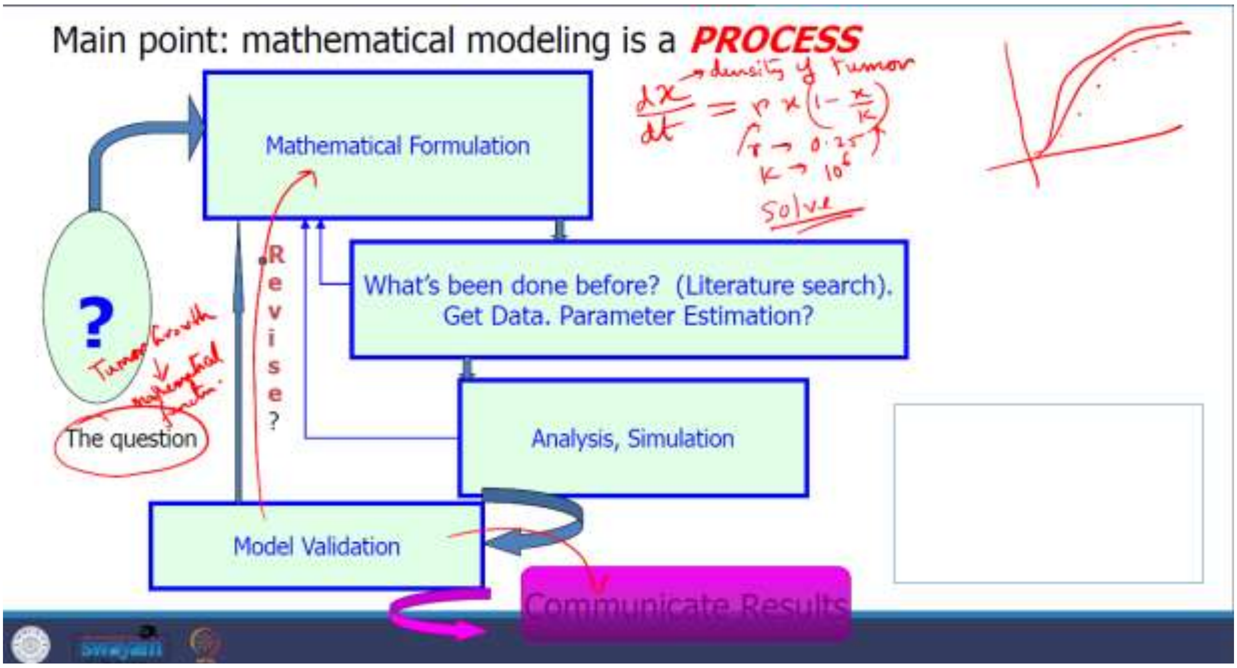
$$\frac{dx}{dt} = r\left(1 - \frac{x}{k}\right)$$

So as you can see here, the unknown parameters are r and k. So I need to find the value of this unknown parameters.

So what I will do? I will look into some literature, some references to see if such kind of work is done.

If yes, then I go through that literature and try to find out the values of r and k. Is there any function there? How I can modify that function to get a better results?

So all these sort of things I will look into.



Suppose I didn't get that. Suppose this is the very first problem that I'm trying.

Then I need to get some data.

So what kind of data?

So you can look into any hospital where they have this record of the growth of the tumor with respect to time.

And I'm sure you will get that.

So once you get that data, you will use some mathematical tools to estimate the value of these parameters.

There are many methods, you will slowly learn through them.

So, say I estimate the value to be 0.25 and this is some 10^6 .

So, once you estimate the value of this parameters r and k, you pluck those values here, you solve the differential equation and once you solve the differential equation, you will get the graph.

When you get the graph, you plot the graph and you see whether it matches with the data or not.

If it matches with the data, then your model is validated and you communicate the result.

If not, if you get a curve which does not matches the data, then you go back to the drawing board, you change the model, and then the process follows.

So that's how your mathematical modeling works.

So, what is a mathematical model?

So, basically it is the description of this real life scenario in terms of the mathematical expressions.

Believe it or not, we use this mathematical modeling in our day-to-day life without even knowing that we are using this mathematical modeling.

Suppose, you have a job.

You got a job and you plan to buy a car.

So, you estimate that, okay, I need the amount 5,04,000 to buy the car and you can afford to save rupees 8,000 a month.

How many months you will take?

So, you write a function F(x), where your x is the number of months.

You can save 8000 per month multiplied by the number of months and your target is 5,04,000.

So, if you solve this equation, you can easily get x equal to 5,04,000, divided by 8000 which is 63 months which is approximately 5.25 years.

$$F(x) = 8000\,x = 504000 \;\; \Rightarrow x = \frac{504000}{8000} = 63\,months = 5.25\,years$$

So, some sort of doable thing if you can save this 8000 rupees for 5.25 years you will be able to afford the car.

Now as you can see this model is a simplification of the reality. How?

Because when you are saving this money obviously you are not going to keep this money at home, obviously you want to put it in the bank, and if you want to put it in the bank, you can put it in a recurring deposit which will earn you some interest and you can afford that car in much less time than calculated in this mathematical model.

But, though it is a simplification, it gives you some sort of initial guess that how much time you will be requiring so that you can afford the car. So that is how your mathematical model works in the real life.

Let's take another example.

How much space inside a cardboard box?

See, you buy from Flipkart, you buy from Amazon, and when you get the delivery, you get a cardboard box.

So the aim of these companies is how to maximize the space, how much more space they can get with the same dimension of the cardboard box for a particular product.

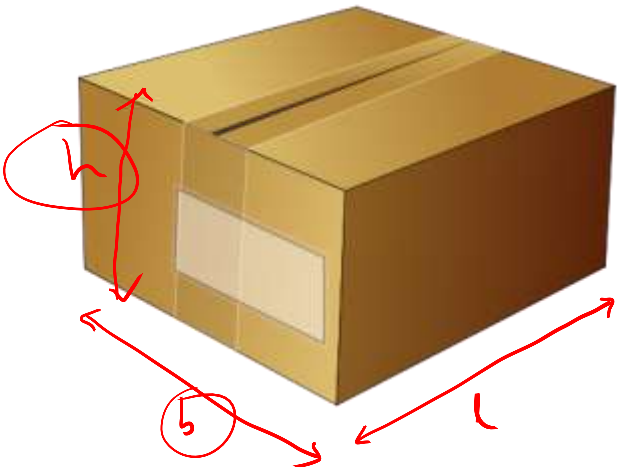
So, that is the aim of the company.

So to do that, we first need that how much space generally a cardboard box holds.

And when we talk about space, obviously, we talk about the volume.

So, if this is our length (l) and this is the breadth (b) and this is the height (h), and we know that the volume is defined as the length × breadth × height.

$$\text{Volume} = l \times b \times h$$



Suppose our length is 350 millimeters, breadth is 250 millimeters and height is 150 millimeters.

Then, it can be easily calculated that your volume is the product of the length, breadth and height and that product is

$$\text{Volume} = l \times b \times h = 350 \times 250 \times 150 = 13125000 \text{ mm}^3$$

Now, what we miss here see this cardboard box have their thickness we have ignored the thickness, but then if you want a real life scenario you need to modify the model, I need to put the thickness here. So, the thickness, say, I take it as x.

So, if I take the thickness of this cardboard box, now you can see the top part, it has a thickness x and the bottom part, this also has a thickness x. So, basically 2x is deducted from this height h. Similarly, the same 2x will be deducted from the length (l) and from the breadth (b).

So now your new formula for the volume, if I denote it by V_1, it is

$$V_1 = (l - 2x)(b - 2x)(h - 2x)$$

So, if you include the thickness of this cardboard box, your new volume is given by V_1.

Suppose, I take the thickness to be 4 millimeters. So our new volume is going to be

$$V_1 = (350 - 8)(250 - 8)(150 - 8) = 342 \times 242 \times 142 = 11752488 \text{ mm}^3$$

Fine, now let us play with the model.

So, the company's objective is to maximize the profit.

So, what they will try to do, they will try to put more product into the same cardboard box so that ultimate their cost of sending the product is less.

So, their aim is to optimizing space for the profit.

So, one of the research division guy says, OK, let's make our thickness x to be 3 millimeters.

Let us see how much we can gain.

So, if your thickness is now 3 millimeters, your new volume V_2 is going to be

$$\begin{aligned} V_2 &= (350 - 6)(250 - 6)(150 - 6) \\ &= 344 \times 244 \times 144 \\ &= 12086784 \text{ mm}^3 \end{aligned}$$

So, what is the difference?

The difference that is saved is $V_2 - V_1$.

So, your $V_1 = 11752488 \text{ mm}^3$.

If I take the difference, the difference is

$$12086784 - 11752488 = 334296 \text{ mm}^3.$$

So, this much amount of volume is saved.

If you see the percentage, the

$$\%\, change\ in\ volume = \frac{334296}{11752488} \times 100 = 2.844.$$

So, the percentage change is 2.844, which looks quite small number.

But, then if you consider the volume of the cardboard boxes that is sent by these companies, like Amazon or Flipkart, it's huge.

And if they can save 334296 mm^3 of volume, they can put more product.

And at the end of the day, the profit margin is huge.

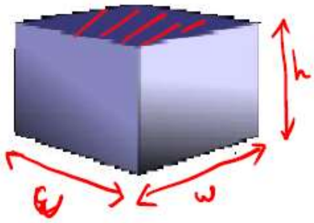
So that's how your real life problem works in case of mathematical modeling.

Let's look into another real life problem.

So here the box, the volume is already given.

So you have this box which can hold 30 cm^3 of some material. It can be a box of chocolate, it can be a box of candles, any kind of product.

The box has a square base so the base or the width they are square so let us take them to be w and let us put this as height (h).



Now one thing you have to note it is given that box should have a square base, that is number one, and double thickness on top and bottom. The meaning of this is, if this is one top there should be another cardboard sheet on it.

So, the thickness become double.

Similarly, at the bottom if it is one cardboard sheet it should be replaced by two cardboard sheets.

So, that the thickness is doubled and the cost of the cardboard is 0.25 per square centimeter.

So, what is going to be the economical size of such cardboard boxes which can hold your 30 cm^3 of some material.

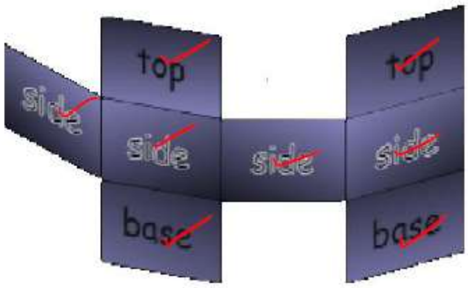
Now, obviously we will start with the volume. So, the volume of this is going to be that

$$V = w^2 h$$

and the volume remains constant, which is 30 cm^3.

Here we need to find the surface area, that means the how much cardboard box you are going to use.

So if I open this cardboard box, if I open the sides, so this will be 1, 2, 3, 4 sides and the top part.



The top part is this one and the base part is this one. Now why you are why there is two because I need the double thickness and hence there is another top and another base.

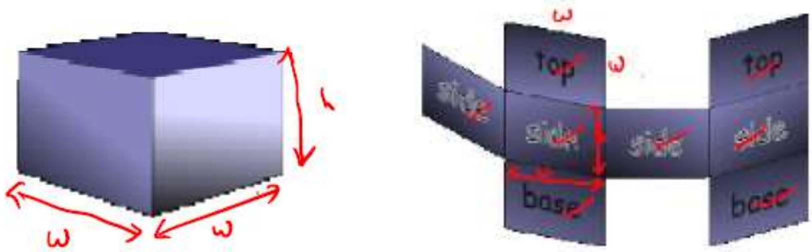
Now from here, so my aim is that to express or to find a function in terms of a single variable which will be w in this case and then find the minimum value of that w so that my cost is also minimum. So, from this equation I can get that the

$$h = \frac{30}{w^2}$$

Now, next we calculate the cost.

So, for that I need the surface area, how much area of cardboard box I need because my product cost per square centimeter.

So, as you can see that this is my base, which is square and this is my height. So, if I consider here this will be the height and this is the base.



So, I have four such sides, and the area of this is, w h, four such sides, hence, 4wh, plus I have the square base and the square top. So, this part is w, this part is w, the area is w^2, 1, 2, 3 and 4, so, 4 w^2.

This gives me the area of the cardboard = 4 w h + 4 w^2, that we will be requiring to create this cardboard box which can hold 30 cm^3 of space.

Now, I already have h in terms of w. So, I will replace this, this will be

$$\left(4\,w\,\frac{30}{w^2} + 4\,w^2\right).$$

So, I have taken 4 common, one w cancels and I get

$$4\left(\frac{30}{w}+w^2\right)$$

So, this gives me the area of the cardboard.

What is the cost? The cost is 0.25 per centimeter square. So, the cost, if I denote it by C,

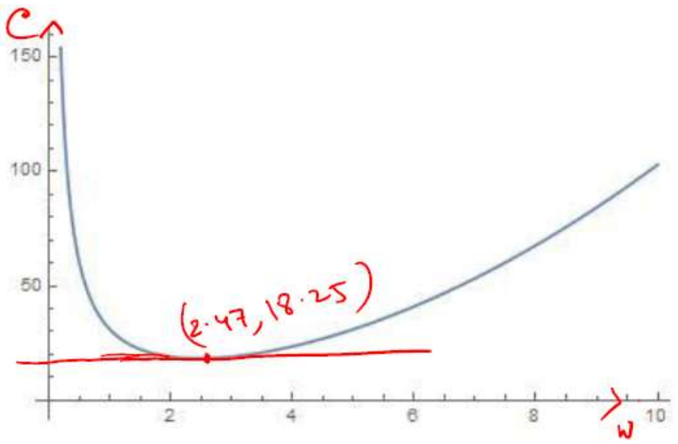
$$Cost\ (C) = 0.25 \times 4\left(\frac{30}{w}+w^2\right).$$

This gives me 1.

So, I get a function $\left(\frac{30}{w}+w^2\right)$ like this.

So, the next step is that I will be plotting this function and if you plot the graph you will get the curve something like this, where this is your w, and this is your c. So, the minimum value of this w as you can see is the bottom of this curve.

So, if I draw a tangent the bottom part. So, somewhere here I will get the minimum value of w and where it intersects the y axis will give the minimum value of the cost and the value for this is (2.47,18.25).



This value you can directly calculate from the graph itself.

So, with the minimum value of w=2.47 you can get the cost to be 18.25.

Of course, you can vary this value a little that is say you can make this value to be 2.45 to 2.49, cost will increase or decrease a bit but still you get the same effect in the box.

So, that is in the hand of the modeler and in the hand of the company to see the realistic scenario while making this dimension of the box.

So, this is again another example where how your mathematical modeling can create a scenario which match with the real life scenario.

Now, what are the benefits of this mathematical modeling? Why we need to use this?

So, these are used by many scientists, engineers, mathematicians, biologists, economists and even social science guys.

So, what are the benefits?

The first benefit is because it is math based, it gives you something a precise thing and which allows you to develop some ideas and some assumptions.

So, that is one of the benefits.

Once you have this math-based thing, it gives you a direction. It gives you an idea when you try to solve that particular problem.

While solving the problem, if you can solve in an analytical method, it is fine.

But at the same time, you need to perform numerical results.
So, once you get this numerical results, you can use a calculator, you can use a software. In our case, we will be using Microsoft Excel to get this numerical solution and hence the numerical visualization of that solution.

So, once you get the numerical visualization of the solution which will be in the form of the graph, so from the graph you will be able to understand the dynamics of the mathematical model and once you get the dynamics of the mathematical model, you will get an in-depth understanding that how your mathematical model or how that system works.

Once you get the idea that how your system works, then it can help you to make the decisions of the question that you have already posed while creating this mathematical model.

So, that is, these are the benefits of this mathematical modeling, but there are obviously some limitations.

One of the limitation is the you simplify the process while you build your model.

And, when you simplify the process, there is a possibility that you may ignore or you may not take an important parameter or important aspect of the mathematical model that you are proposed.

And, if you do that, then obviously the result which you will get, will not match with your real life scenario. That is one of the huge drawback of this mathematical model.

Also, once you formulate that model, so it will be a mathematical process and which has some specific rules. So, you need to follow that rules to get the solution of that mathematical model which may not be true for that real life scenario.

And, the third one which is the most important one, your model must validate the real life scenario and to validate that you will be needing data.

So, if you do not have a data if you have an incomplete data or if you have incorrect data, then that will create a huge problem in the output of the mathematical model.

So, you have to be very careful when you are dealing with the data, while doing your mathematical modeling and estimating those parameters.

So, as I told you before, these models are the approximate or they are sort of modified version of the real life phenomena.

We cannot take all the aspects of the real life scenario.

So, we take those aspects, these are really important and while forming these models.

And hence, these models are some sort of inexact and sometimes it is wrong.

So, that is one of the limitations.

Now, if you do not understand the actual real life scenario or actual phenomena, what is happening it is very difficult to formulate this mathematical model.

So, if the understanding is less the mathematical description is going to be imperfect and hence the outcome of that mathematical model will not be what you see in the real life scenario.

Now, again, the importance is on the parameters.

So, while you collect the data, you will see that either you get an incomplete data or you get a data from the experiments, which may not be collected in a proper manner.

So, all these kind of difficulties it gives you in the end to calculate or to estimate the model parameters in an incorrect way.

So if the model parameters are incorrect, so obviously the results which you are going to get will not fit the actual data and hence your model won't be able to be validated.

And, another important thing is when you solve this model, in our case, it will be the differential equation or the difference equation you will be needing initial conditions, you will be needing boundary conditions, and those initial and boundary conditions may be known may not be known.

If they are known, then it is fine if they are not known then it poses a problem. Because, if you put a wrong initial condition that may results in the wrong output of the mathematical model. So, you have to be very careful while choosing this initial conditions.

But despite all these limitations we still depend very much on this mathematical models.

Why?

Because it gives you a very powerful tool to represent this natural processes.

You consider a biological experiment, say, as we have talked before you want to find the growth of the tumour, or you find to find the how many doses of the drugs to be given to a particular patient so that this tumour is eradicated.

Now, to do that experiment, it's a huge cost, it's a huge time consuming thing.

Instead, if you can construct a model and you can solve that model, you can simulate that model and get a future prediction.

So, if you get a future prediction, then on the basis of that, your experiment can be simplified time can be saved, cost can be reduced.

So, that is one of the greatest advantage of this mathematical modeling.

So, you get the result, you extrapolate that to some large spatial scales and then you predict the future.

While doing that, you will face some uncertainty and to quantify this uncertainty, there is something called sensitivity analysis which we will be doing in our later lectures.

So, you can determine the sensitivity of the output, you can see which of the model parameters are most sensible and which are not that sensible, and like that while taking or while estimating the values of those parameters you have to be bit cautious.

So, that is the end of our lecture today.

In the next lecture, we will be learning about the units and dimensions.

Units and dimensions are one of the important aspects of this mathematical modelling and we will be doing in our next lecture.

Till then, bye-bye.