wHigher Engineering Mathematics
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Lecture - 52
Dual Simplex Method

Hello friends, welcome to my lecture on Dual Simplex Method. We have seen that a set of basic variables giving a feasible solution can be found by using artificial variables and using big-M method are two phase method.

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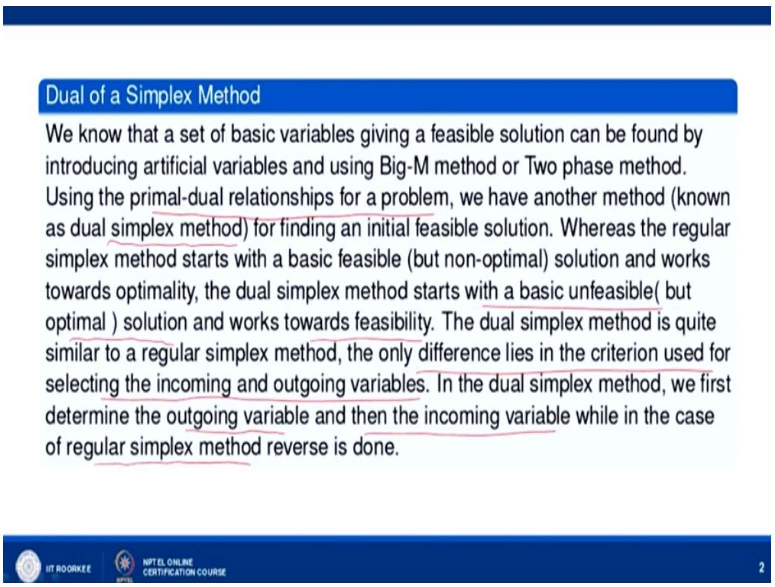

Using the primal dual relationships for a problem we have another method known as dual simplex method for finding an initial feasible solution whereas the regular simplex method starts with a basic feasible but non optimal solution but works towards optimality, the dual simplex method starts with a basic unfeasible but optimal solution and works towards feasibility. The dual simplex method is quite similar to a regular simplex method the only difference is that here we select, the only different lies in the criterion used for selecting the incoming and outgoing variables in the dual simplex method we first determine the outgoing variable and then the incoming variable while in the case of regular simplex method reverse is done that is we first select the incoming variable and then the outgoing variable.

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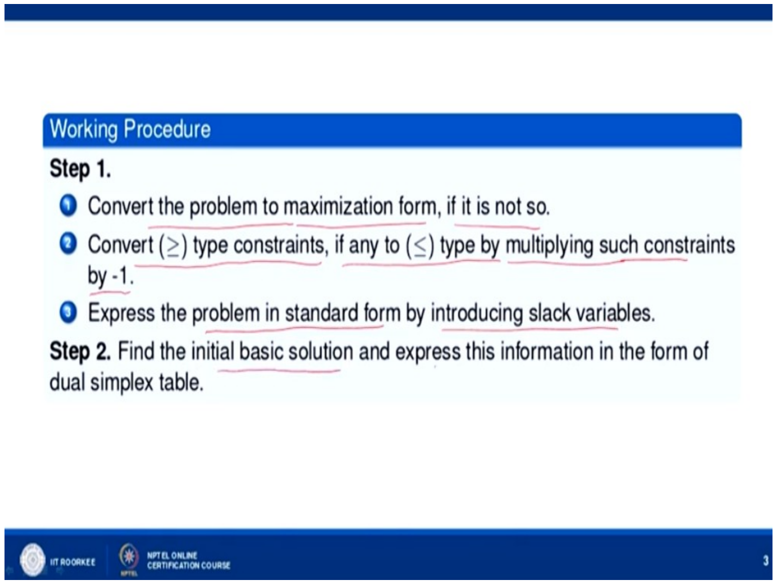

Now, let us consider how we can used this dual simplex method, so first we do this this stubborn where we convert the problem to maximization if it is not in that form that is if it is given to find minimum $Z = \sum_{i=1}^{m} C_i X_i$ then we shall convert it to maximization form. Now, convert greater than or equal to type constant if any to less than or equal to type by multiplying such constant by -1 because for the maximization problem the constant should be of less than or equal to type.

Now, express the problem in the standard form by introducing slack variable because when we have less than or equal to we use slack variables to convert them into equations, now find the initial basic solutions and express this information in the form of dual simplex table.

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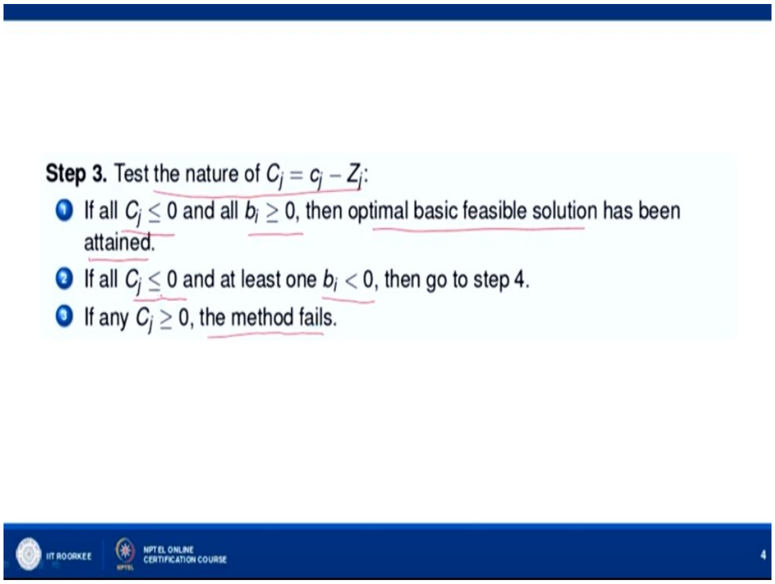

Than the step 3 test the nature of sigma, test the nature $C_j = c_j - Z_j$, if all $C_j \le 0$ and all $b_i \ge 0$ than the optimal basic feasible solution has been attained, if all $C_j \le 0$ and at least one $b_i \neq 0$ then we have to go to step 4.

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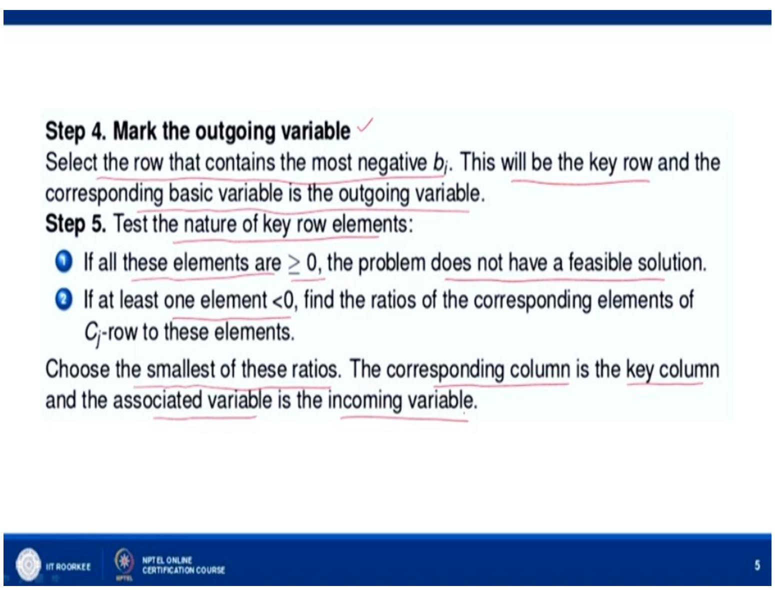

So this is step 4 where we mark the outgoing variable.

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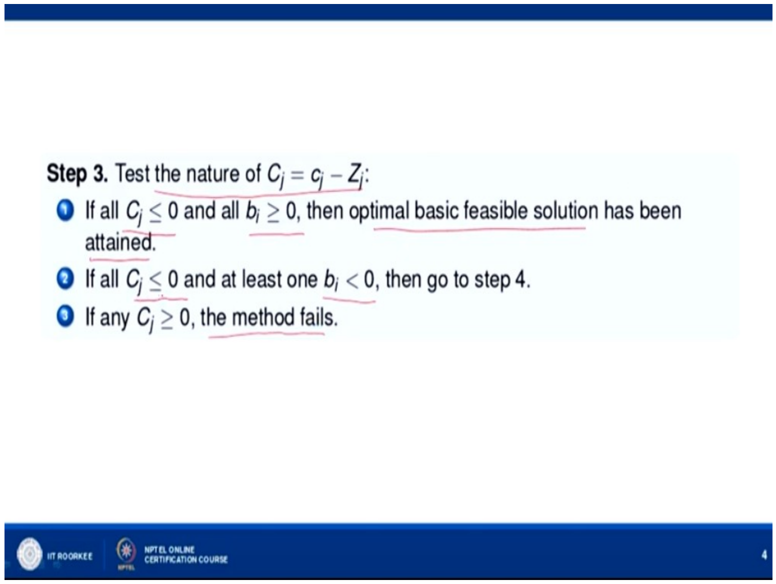

Now if any $C_j \geq 0$ than the method fail.

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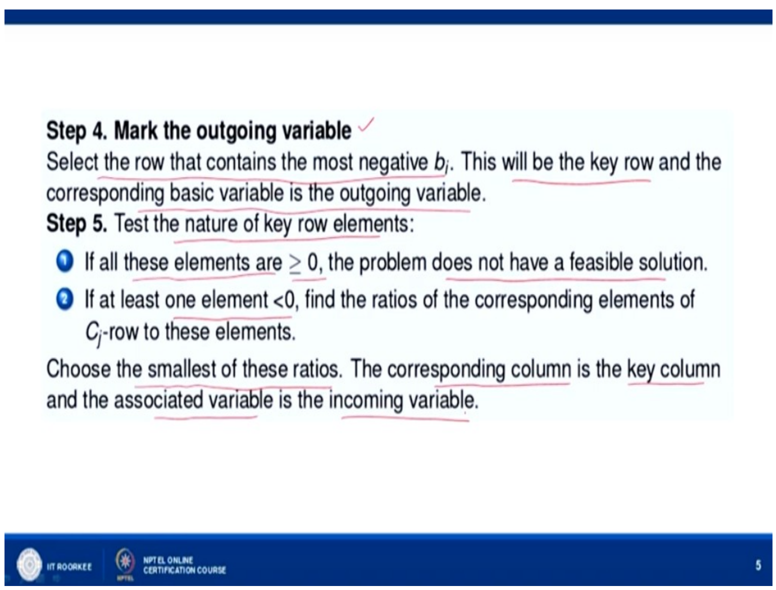

So let us now consider step 4, so in the step 4 we have this $C_j \leq 0$ and at least one $b_i \leq 0$, so this situation we have. So, let us consider the row the that contains the most negative b_i that means the numerically largest b_i we consider for selecting the row this will be the key row and the corresponding basic variable is the outgoing variable.

So if there is at least one b_i than we consider the that b_i which is numerically largest and that particular row in which numerically largest b_i occurs that particular row will be called as the key row and the corresponding basic will be than the outgoing variable test the nature of key row element. Now if all the elements are greater than or equal to 0, let us notice the key row element if they are all negative than the problem does not have a feasible solution, if at least one element is less than 0 that find the ratios of the corresponding elements of the C_j row to these elements and then from these ratios.

We choose this smallest one choose the smallest of this ratio the corresponding column will then be called the key column and the associated variable will be the incoming variable, so first we select the get the outgoing variable and then we get the incoming variable.

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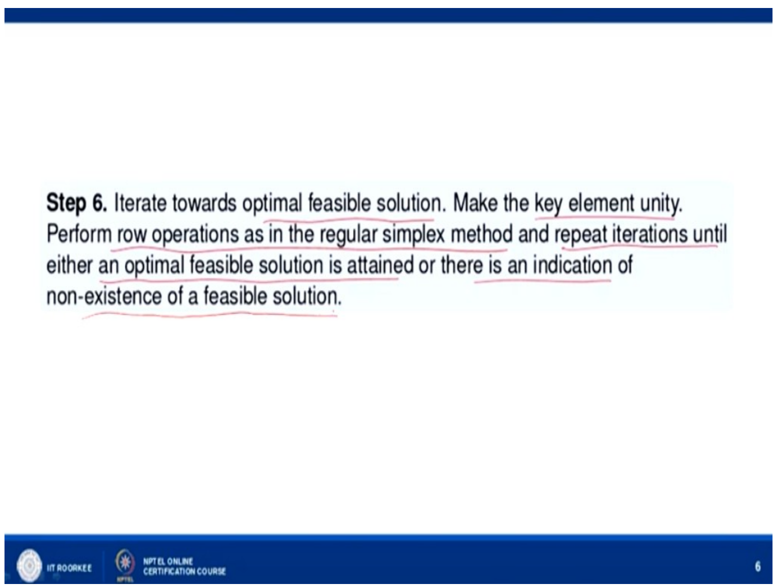

So in the step 6 iterates towards optimal feasible solution, so we iterate towards optimal feasible solution and then what we do for this, we make the element key element unity and then perform row operations as in the case of regular simplex method and repeat iterations until either an optimal feasible solution is attained or there is an indication of non-existence of a feasible solution.

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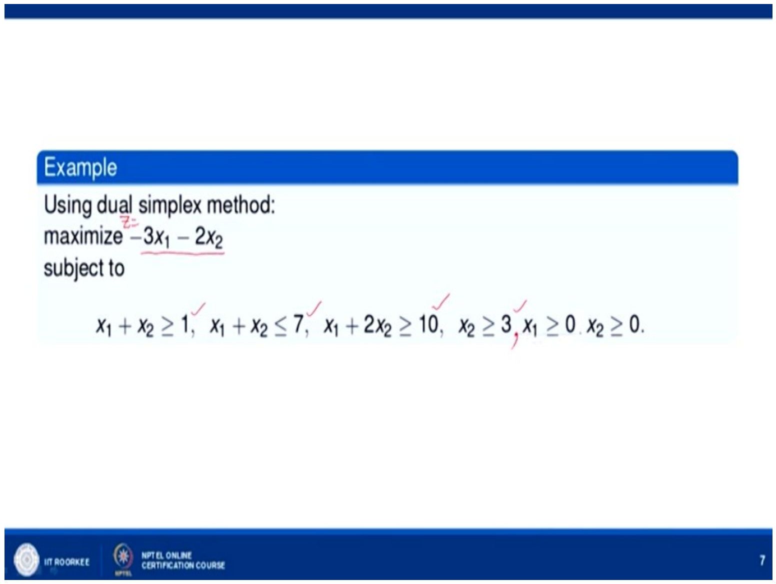

So, let us consider for example this problem we have to maximized -3 x_1- 2 x_2, Z =-3 x_1- 2 x_2 where we are given the constant x_1+ $x_2 \geq 1$, x_1+ $x_2 \leq 7$, x_1+ 2 $x_2 \geq 10$, $x_2 \geq 3$, $x_2 \geq 0$, so $x_1 \geq 0$ and $x_2 \geq 0$.

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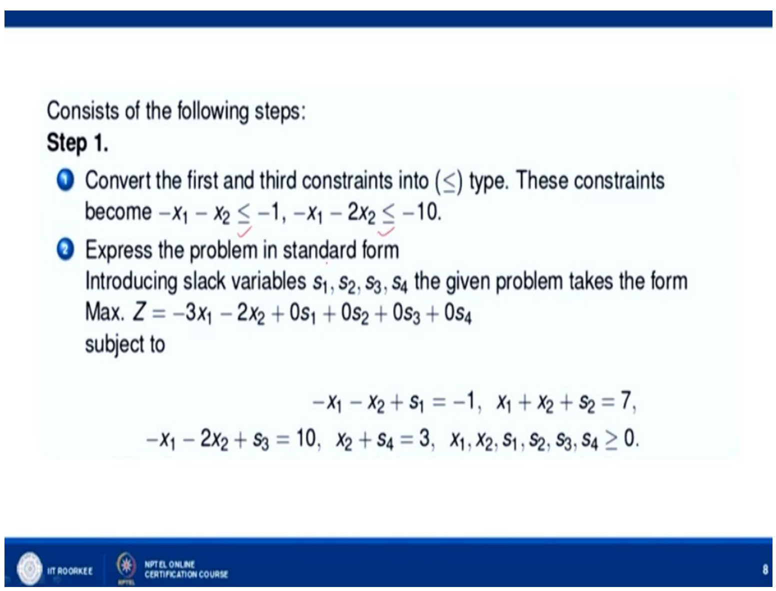

Now, what we do first we will convert the first and third constraint into lest than or equal to type.

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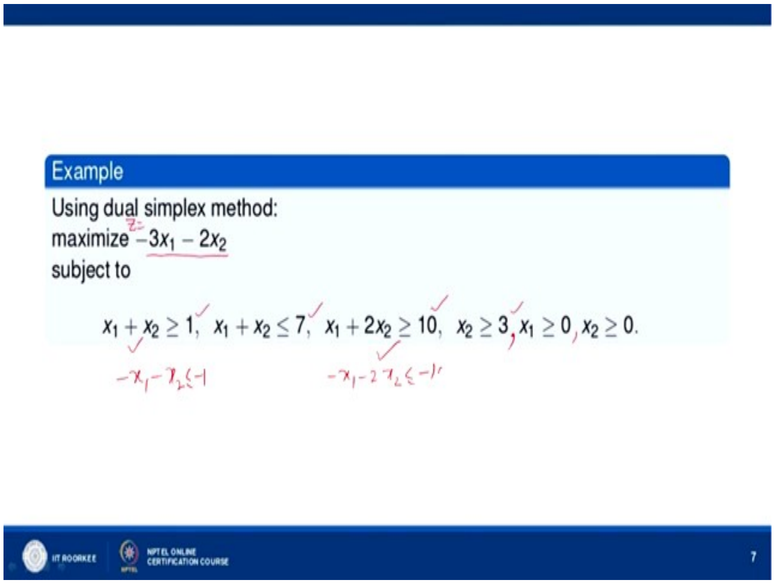
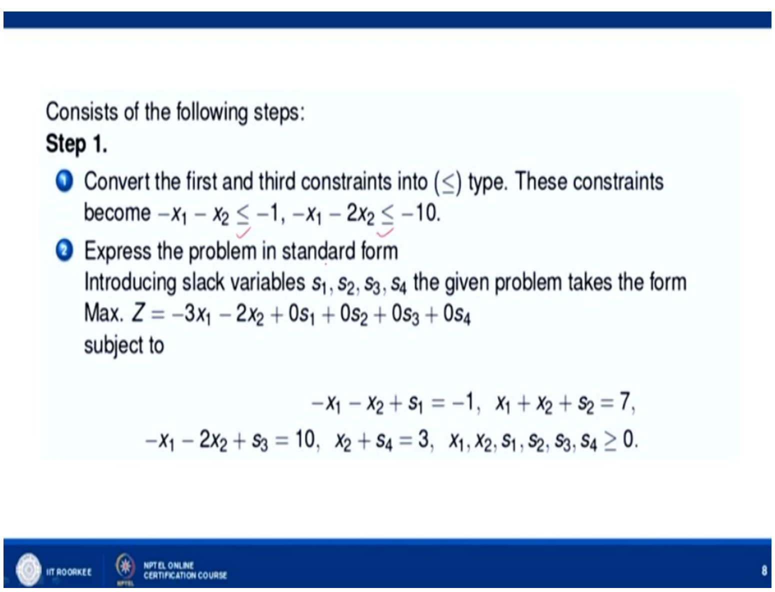


Now you can see the problem is in the maximization form, so we do not have to do anything in this z but here we have to convert all the constraint to less than or equal to type and we notice that the first constraint and the third constraint they are greater than or equal to type, so we multiply there - one to convert them to convert them into less than or equal to type, so first constraint will become $-x_1 - x_2 \leq -1$ and third constraint will become $-x_1 - 2x_2 \leq -10$(Refer Slide Time: 6:45)



So we will have first constraint this and third constraint this. Now express the problem in the standard form, so what we will do we will have to convert this less than or equal to type constraint into equations, so for that we need slack variables.

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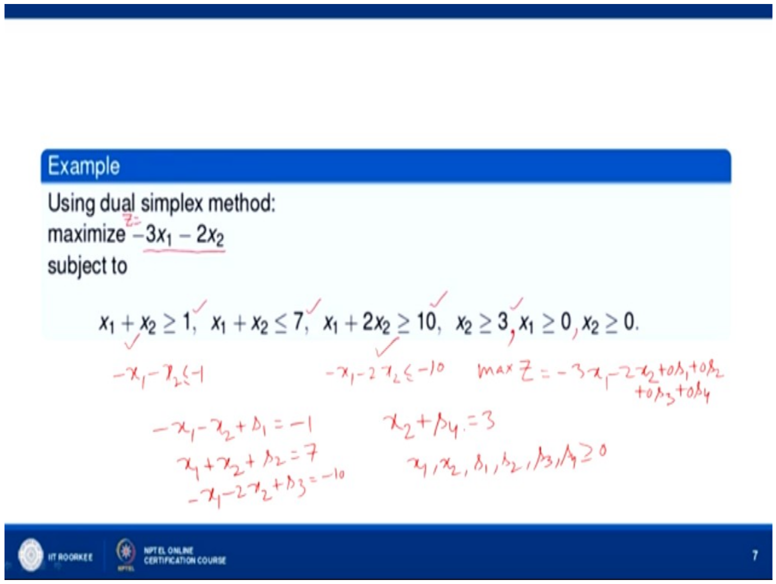

So x_1 + x_2≥ 1 has been converted to -x_1 - x_2≤ -1, so we need a slack variable here so -x_1 - x_2+s_1= -1 we have and then x_1 + x_2≤ 7, so we write x_1 + x_2+s_2= 7and third one is -x_1 -2 x_2+s_3= -10 and the forth one we have a x_2x2, so the forth constant is x_2≤ 3 so we write x_2+ s_4=3, where x_1 x_2 s_1 s_2 s_3 s_4 they are all non-negative.

So we have this situation we have and then objective function will be written as Z will be equal to maximum of Z = -3 x_1- 2 x_2 + 0 s_1+ 0 s_2+¿ 0 s_3 +0 s_4, so we have maximum of Z equal to this where the subjective the equations

- x_1 - x_2 +s_1 = - 1,

x_1 + x_2 + x2 = 7,

- x_1 - 2 x_2 + s_3 = - 10 and

s_2 + s_4 = 3, so we have this situation.

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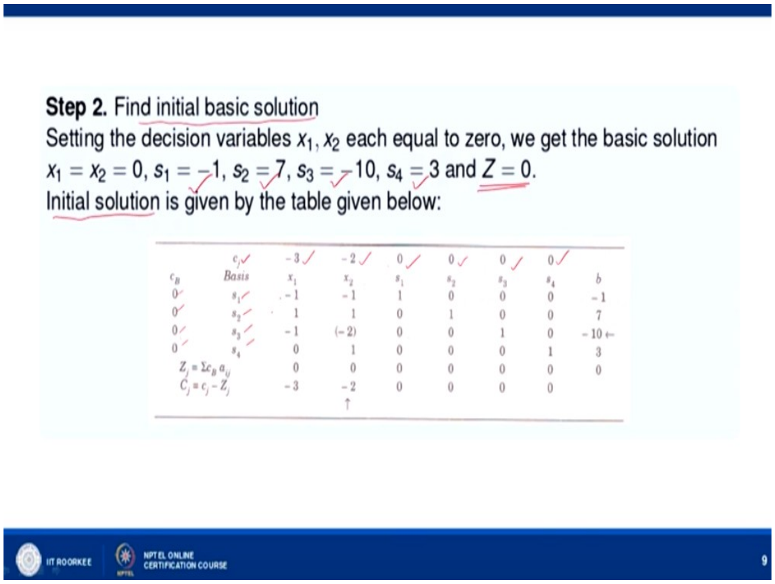

Now, we will write the corresponding simplex table, so we have we have to find initial basic solution first so we have here now 4 equations 1 2 3 4 and the variables are 6.

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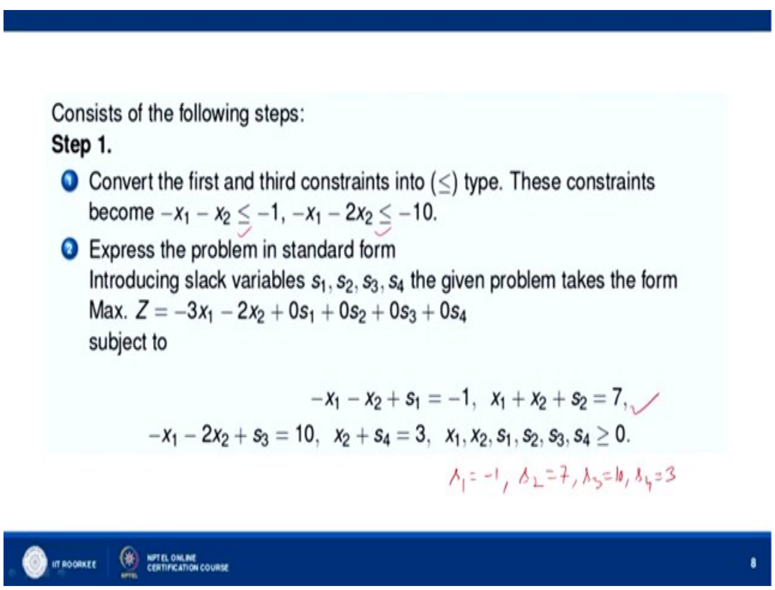

So let assume the 2 dissent variable x_1 and x_2 to be equal to 0, 0, when we take $(x_1\ x_2)$ =(0, 0) we get s_1 = - 1, s_2 = 7, s_3 = 10 and then s_4 = 3.

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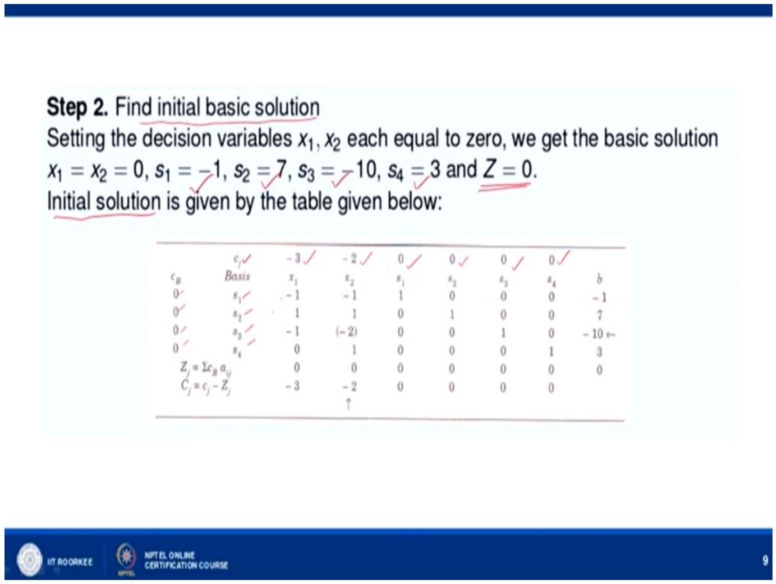

So we get s_1 = - 1, s_2 = 7, s_3 = 10 and s_4 = 3. 3 and Z = 0 because Z = -3 x_1- 2 x_2 + 0 s_1+ 0 s_2+¿ 0 s_3 +0 s_4, so $x_1 x_2$ we have assume to be 0 the value is of s_1 s_2 s_3 s_4 are -1, 7, 10, 3.

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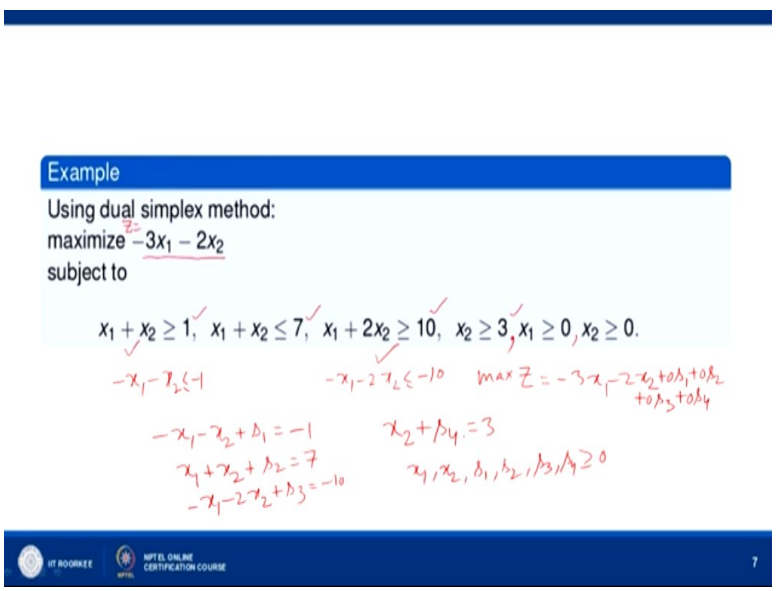

So when we put them here we get Z = 0.

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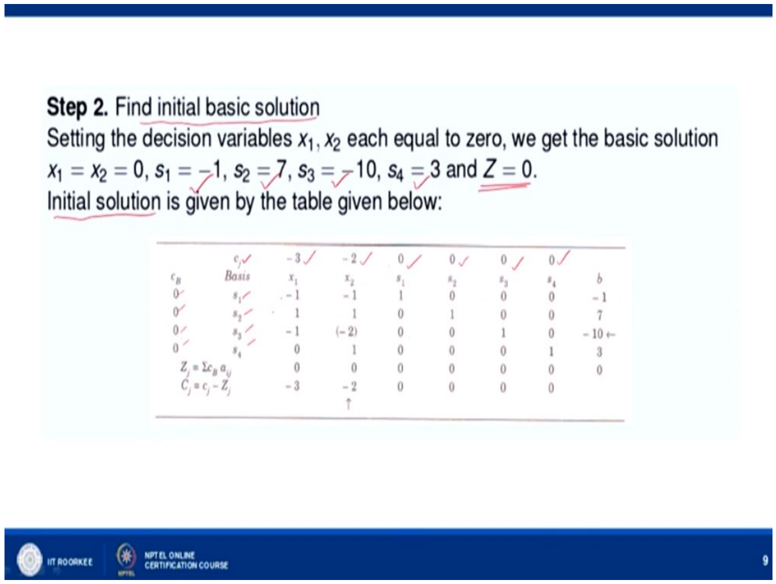

Now initial solution, let us find initial solution so initial solution you can see here, what initial solution we have got, x_1 = 0, x_2 =0, s_1 = - 1, s_2 = 7, s_3 = 3 and Z equal to 0 is illustrated here in this table, so we have C_j is are the coefficients of the variables x_1 x_2 s_1 s_2 s_3 s_4 in the objective functions, so coefficient of x_1 is - 3 coefficient of x_2 is - 1 coefficient of s_1 is 0 coefficient of s_2 is 0 coefficient of s_3 is 0 coefficient of s_4 is 0 and basis variables are s_1 s_2 s_3 s_4. C_B column, C_B column contains the coefficient of the basis variable in the objective function, so coefficient of the basis variables in the objective function are.

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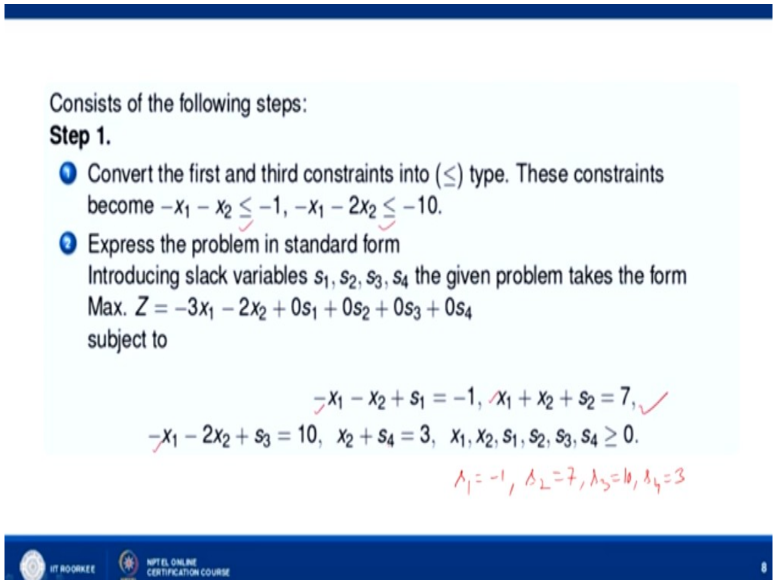

Now, this is body matrix so we have x_1 the coefficient of x_1 here is - 1, here it is 1, here it is - 1, and here it is 0 in the fourth equation.

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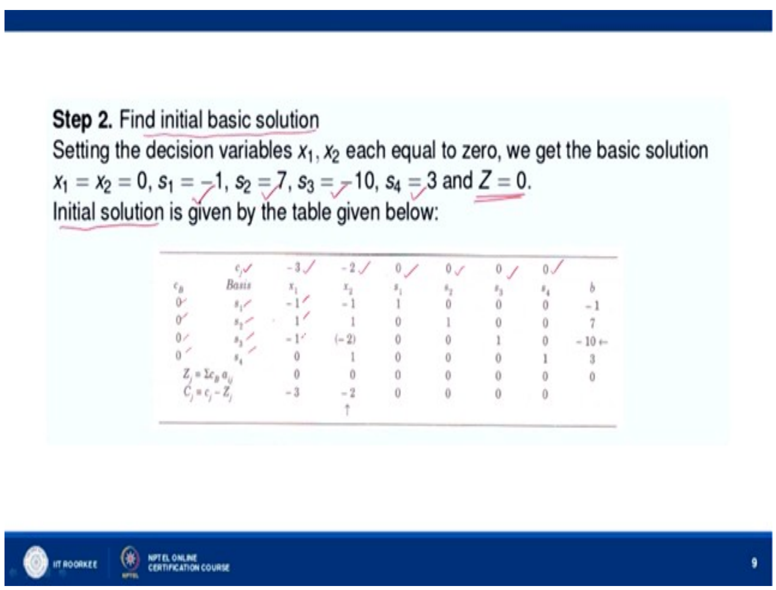

So we have the first column of the body matrix as - 1 1 - 1 0.

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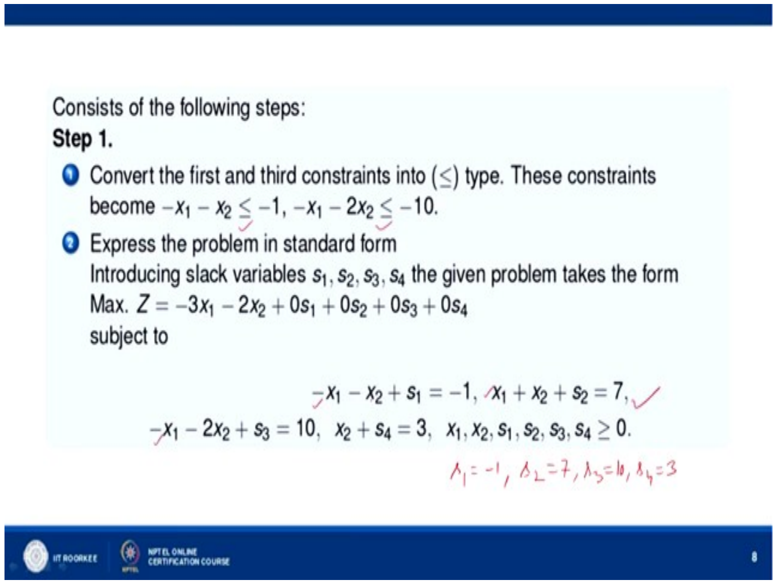

Second column corresponds to the variable x_2 so here the coefficient of - 1 here it is 1 here it is - 2 here it is 1.

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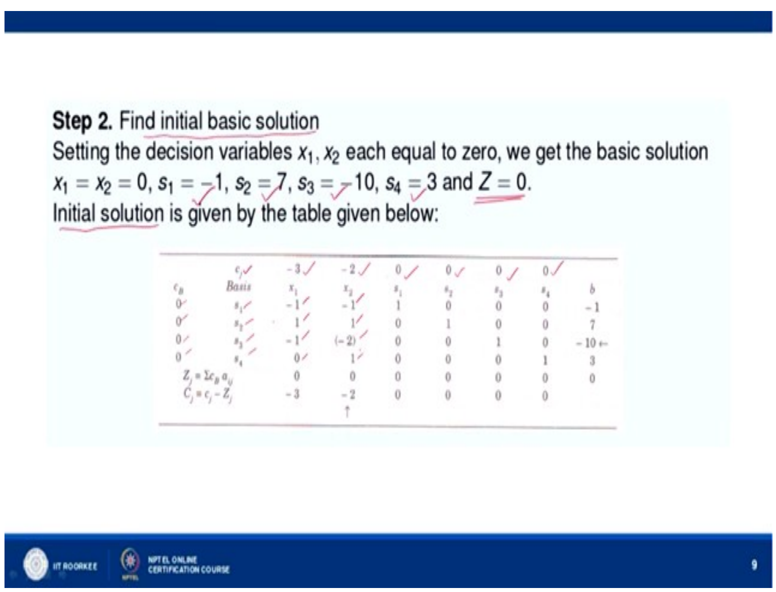

So this is the column corresponding to x2 - 1 1 - 2 1.

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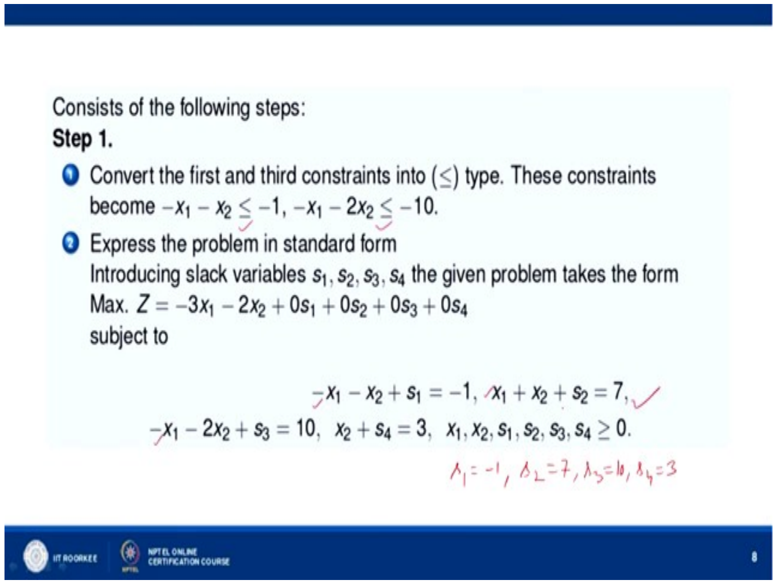

And the coefficient of s_1 coefficient of s_1 is 1 here here it is 0 here it is 00 here it is 00, so we have 1 00 00 00 coefficient of s_2 is 00 1 0 0 coefficient of s_3 is 0 0 1 0 coefficient of s_4 is 0 0 0 1.

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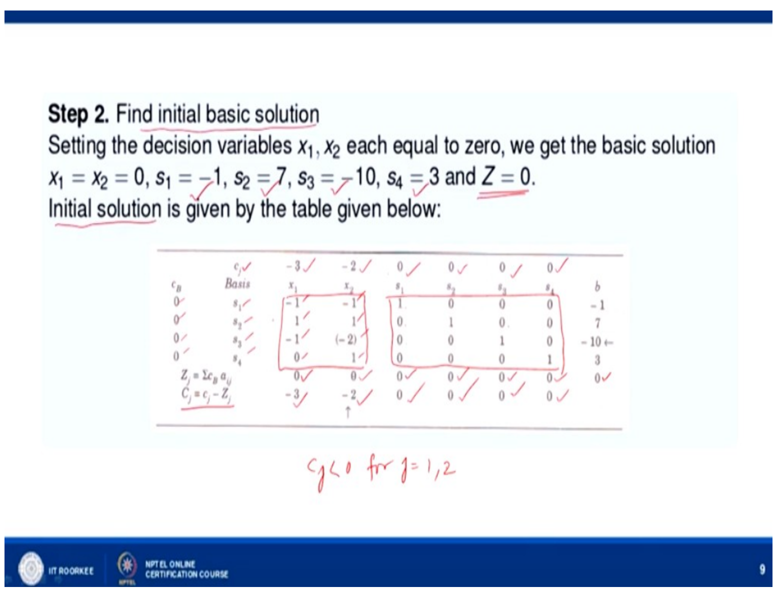

So we have unit matrix here you see 1 0 0 0 0 1 0 0 0 0 1 0 0 0 1, so this is unit matrix it is unit matrix and this one is body matrix, this one is body matrix. Now, so what we will do we will calculate first Z_j equal sigma a_{ij} like we do in the regular simplex method, so we multiply column to the coefficient of x_1 that is first column, so 0 into - 1 is 0 0 into 1 is 0 0 into - 1 is 0 0 into 0 is 0 total is 0 and then similarly Z_b column when the elements of $C_b$column are multiplied to the corresponding element of $s_2$column, we summed up we get 0 here also we get 0 here 0 here 0 and here 0, and the value of Z, value of Z that is the C_b column elements when are multiply to B column elements again give 0 so Z equal to 0 here.

Now, we determine capital C_j is c_j - Zj, so - 3 - 0 is - 3 - 2 - 0 is - 2, 0 - 0 is 0, 0 - 0 is 0, 0 - 0 is 0, 0 - zeo is 0, 0 - 0 is 0. Now, we notice that $C_j \le 0$ in fact Cj ≤ 0 for j = 1 and j = 2, so the solution that we have got not optimal solution so we see that sorry.

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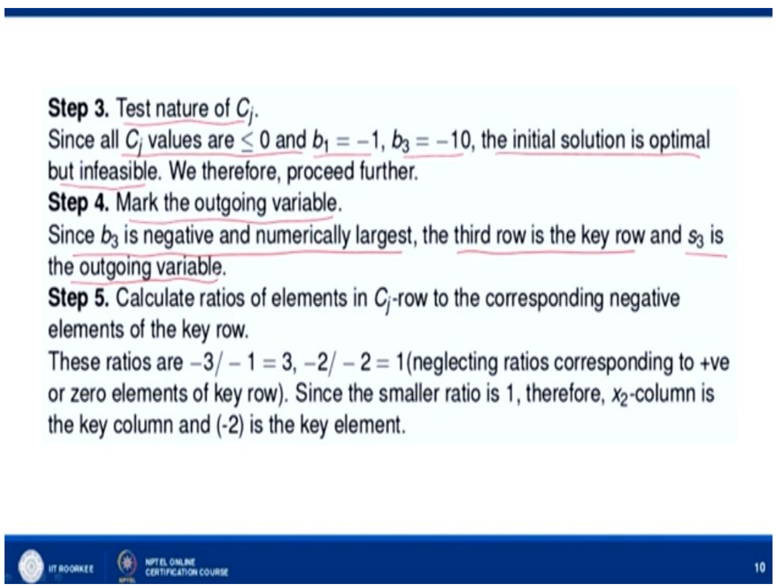

So since all C_j values are less than or equal to 0 the initial solutions are optimal but in feasible because b_1 is - 1 b_3 is - 10 and therefore, the values of s_1 is -, s_3 is - 10, so the solution is optimal due to the fact that optimality condition satisfied C_j is less than or equal to 0 for every j but s_1 and s_3 are taking negative values so the solution is not feasible, so we say that we have got an initial optimal solution but it is infeasible. Now, less therefore go to step 4 we have to mark the outgoing variable.

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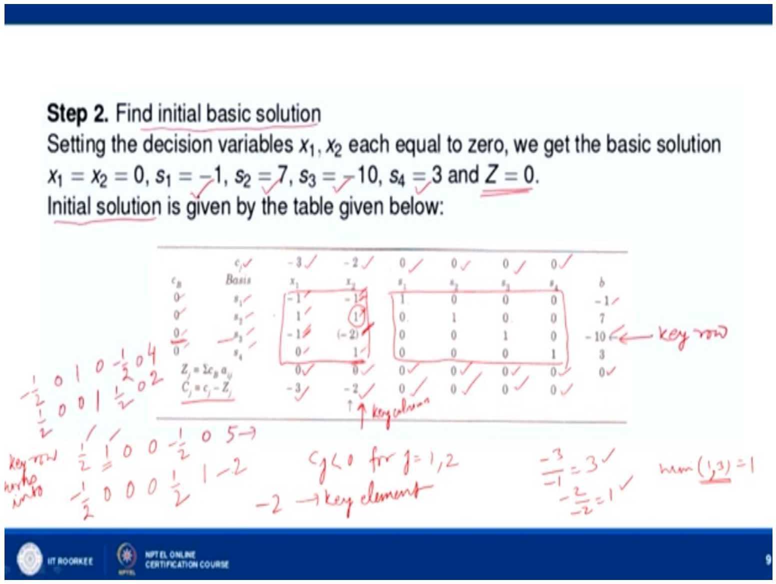

So, since, now let us see in the elements corresponding to V column we see that which one is most negative which is numerically largest, so - 1 is there b_1 is - b_2 b_3 is - 10, so this is numerically largest so the key row is then this row, this row is the key row, this row is key row, s_3 will be outgoing variable now we have to find incoming variable, so what we do since b_3 is negative and numerically largest the third row is the key row and s_3 is the outgoing variable. Now, let us calculate the ratios of elements in C_j row to the corresponding negative elements of the key row.

So we see that the key row elements are only we have to consider those elements which are negative in the key row the element which are 0 or positive will not be consider, so there are two elements which are negative - 1 - 2, so let us find the corresponding ratios C_j row elements divided by key row element, so - 3 divided by - 1 so that is equal to 3 and then - 2 divided by - 2 equal to one and then we have to consider minimum of the ratios minimum of this 2 ratios, so minimum of 1 3 is equal to 1, so what we have we have to pin point the key column, so which the ratio - 2 by 2 is equal to 1, so - 2 is here.

So this element - 2 when divides this - 2 it gives us 1 which is the minimum ratio, so this column is the key column this is key column and at the intersection of key and key column we get the key elements so - 2 is key element, then we will make, will then write the next table where s_3

will be going out in place s_3 x_2 will come in and in the C_b column we shall write the coefficient of x_2 and place this 0, that is - 2, so let us see so we have pin pointed the column which is key column and key column turns out to be x_2 column and - 2 is the key element.

So, we get this instead ofx_3. Now we get x_3 here s_3 has gone out and the coefficient of s_2 is - 2, so now what we do, we then divide the key row element by key element, so divide the key row elements by key element, so key row elements will become key row elements will become 1 by 2 we are dividing by - 2 so 1 by 2 1, key row turns into - 1 divided by - 2 that is 1 and then 1 here this is 1 by 2 than 1 here than 0 than 0 than 1 - 1 by 2 this is divided by - 2 - 1 by 2 is 0 and we get here - 10 divided by - 2, so we get 5. Now with the help of this element which is 1, we have to make this element the other elements of the x2 column 0.

So this is - 1 here, so - 1 will can be made 0 with help this row if we add this row to this row, so let us add this row to the first here, the first row will become than - 1 + half, so - half, - 1 + 1 will become 0 and then 1 + 0 will become 1, 0 + 0 will become 0 and then, so this is s_1 s_2 $s_3$is 0 here so 0 - half is - half, s_4 is 0 here, here also 0, so we get 0, and then - 1 + 5, so we get 4 here so this is first row. In the second row we have in the x_2 column 1 here, so this 1 can be made 0 if we subtract this new row from that one.

So, let us subtract this row from this one so 1 - half is half and then 1 - 1 is 0 then we get 0 - 0 0, then we get 1 - 0 equal to 1 then we get 0 - - half so we get half here, and then we get 0 - 0 equal to 0 and we get 7 - 5 equal to 2. So and this is the row this one, this row is here already. Now we make this element 1 0, so we again what we will do, we will subtract this row from this one, so 0 - half so we get - half here and what we will get here, 1 - 1, 0 here, and then 0 here, 0 - 0 0, then 0 here, 0 - 0 0, then we get 0 - - half so we get 1 by 2 and then we get one - 0 1, and then we get 3 - 5 equal to - 2.

So, let us compare these rows with the rows given in the next simplex table.

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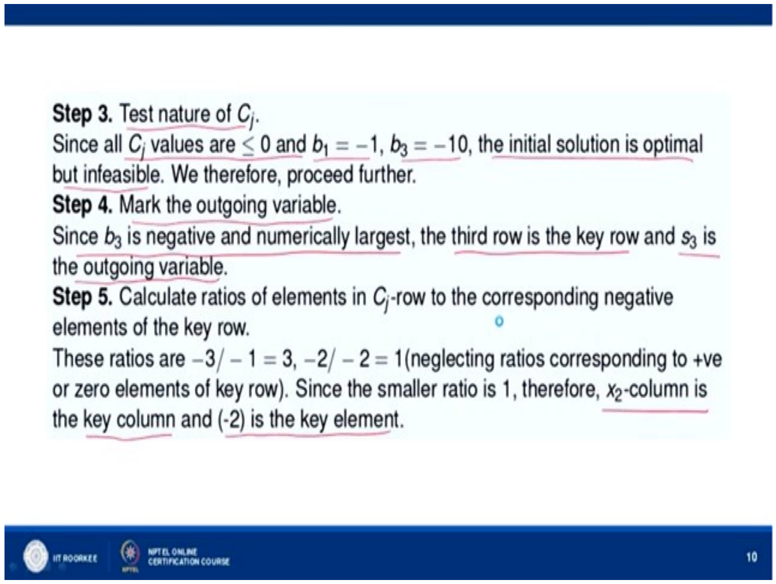

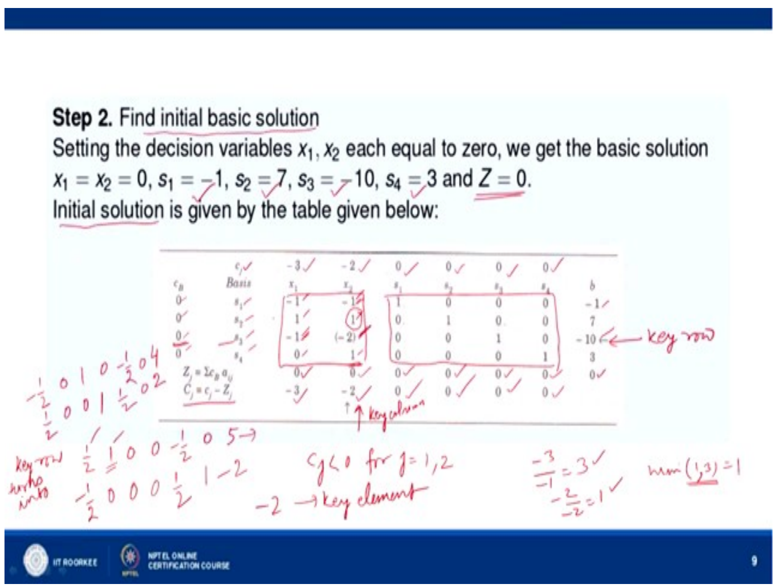

So we can see here - half 0 1 0 - half 0 1 0 - half 0 4 so first row matches second row is also where second row is this one 1 by 2 0 0 1 1 by 2 0 half 2, so 1 by 2 0 0 1 1 by 2 0 2 and then third row is half one double 0, half 1 double 0 - half 0 5 - half 0 5 and the row corresponding to this s4 is - half then 0 0 0 - half 0 0 0 half 1 half 1 - 2 half 1 - 2.

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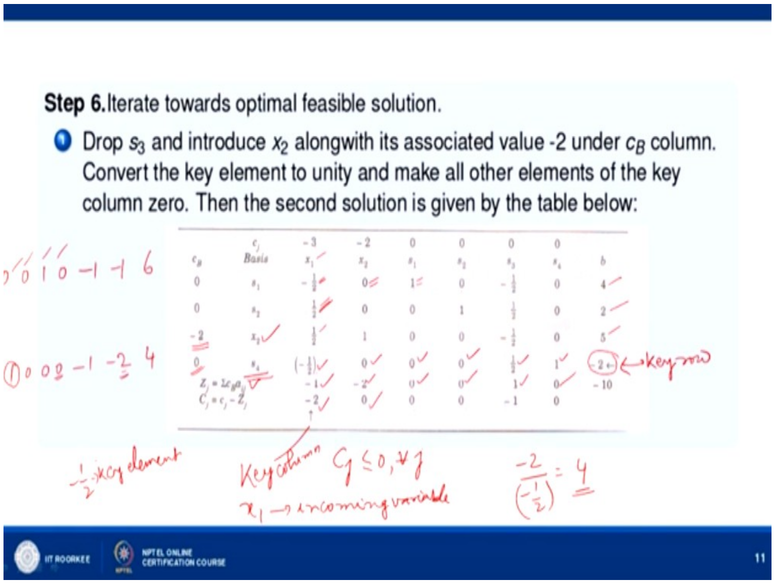

Now, what we will do, we will again find Z_j while multiplying C_b column element with the column corresponding to elements of the column corresponding to x_1 x_2 s_1 s_2 s3 s_4, so when we do this for x_1 column and what we get 0 into - half 0 0 into half 0 - 2 into half is - 1 0 into - half is 0, so we get - 1 here similarly for this x_2 column we get - 2 and for this s1 column we get 0 s_2 column gives 0 and s3 s3 is 0 into this 0 into this 0 - 2 into - half is + 1 and 0 into half 0, so we get + 1 here and similarly for s_4 we get 0 here. Now, capital C_j, let us find capital C_j is c_j- Z_j so - 3 + 1 so we get - 2 - 2 + 2.

So we get 0 here 0 - 0 is 0, 0 - 0 is 0, 0 - 1 - 1 - 1, 0 - 0 is 0, so we get this, now we can see $C_j \leq 0$ for all j and what we are getting here s_1 = 4, s_2 = 2, x_2 = 5, s_4 = - 2, so $C_j \leq 0$ means optimality condition is satisfies but the value of s_4 is coming out to be - 2, x_1 is 0 s_1 =4, s_2 = 2, x_2= 5, s_4= - 2, so the solution is infeasible, although it is optimal but it is infeasible.

So, what we will do let us proceed further and we have to now mark the outgoing variable, so let us see which value of the B column is negative and, so this value this 4 2 5 - 2 there is only one value which is negative, so this column if there are more than one value than we find the value which is numerically largest and among the negative values the value that is numerically largest that gives us the key row but here it there is only one negative value so - 2 this row is the key row, so key row elements means this key row means s_4 is outgoing variable.

Now, let us find incoming variable what we will do, we have to see the key row elements, key row elements are - half 0 0 0 half 1, so what we will do, we will divide the negative value of C_j by the negative values of this key row, only negative values will be divided so here - 2 will divided by - half the 0 of the key row and the positive values are not considered for ratio, so - 2 divided by - half gives you 4.

So if there are more than one negative values in key row then we will consider all ratios of C_j row elements divided by the negative elements of the key row and then we will consider the minimum of all such ratios, so whichever ratio is minimum that particular column will give us the key column here there is only one negative value in the key row - half and corresponding ratio is 4 so this column is key column.

So x_1 is incoming variable and at the intersection key row and key column we get the key element, so - half is key element this is key element. Now so what we will do in the next simplex table this s_4 will be replace by x_1 in the C_b column in place of 0 here we shall write the coefficient of x_1 that is - 3 and the elements of key row will then be divided by the key element to make the key element unity.

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The third solution is given by the table below:

	c_j	−3	−2	0	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	0	0	1	0	−1	−1	6
0	s_2	0	0	0	1	1	1	0
−2	x_2	0	1	0	0	0	1	3
−3	x_1	1	0	0	0	−10	−2	4
Z_j		−3	−2	0	0	3	4	−18
C_j		0	0	0	0	−3	−4	

Since all $C_j \leq 0$ and all $b's \geq 0$, therefore the solution is optimal and feasible. Thus the optimal solution is $x_1 = 4$, $x_2 = 3$ and $Z_{max} = -18$.

So what we will do let us go to the next simplex table, so this you can see here C_b column 0 0 the coefficient of x_2 is - 2 and in case s4 now we are writing x_1 the coefficient of x_1 is - 3 here and we have divided the elements of key row y - half.

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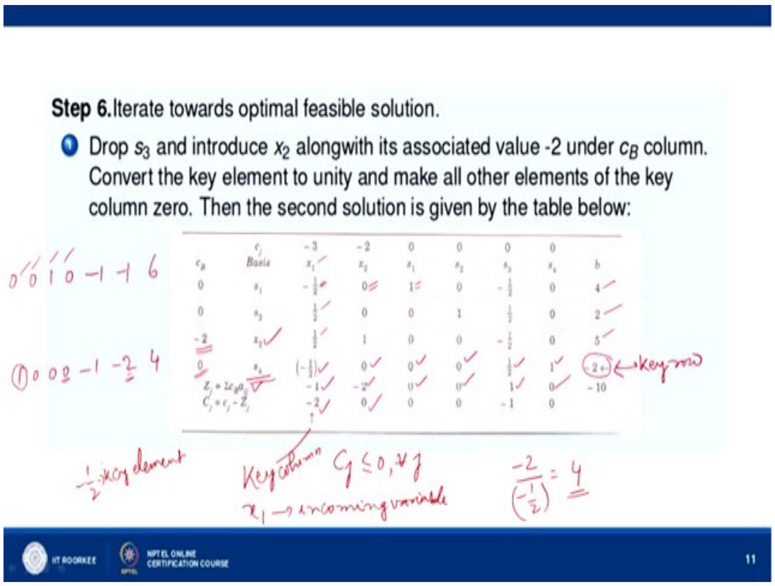

So this becomes 1 0 0 1 0 0 0 - 1 1 divided by - half is - 2 and then - 2 divided by - half so that becomes 4 - 2 divided by half becomes 4, now this has becomes 1 after it has becomes 1 we make the elements this one this one this one of the x_1 column 0, so what we will do with the help of this one we have to make this element 0.

So we multiplied it by half and add to this row, first row, so we will get 0 here, we are multiplying it by half and adding here so 0 and then half multiplied by 0 and added to 0 gives you 0 half multiply by 0 and added to 1 gives you 1, so we have got this element this element this element. Now we go to the s_2 so half multiply to 0 and added to 0 give you 0 half multiplied by - 1 is - half - half we add to - half to get - 1 and then we add B multiply by half, so this gives you - 1, - 1 we add to s4 we get - 1 and then we multiply half to this row so 2 4 into half is 2, s is added to 4 we get 6 here, so let us see we get this row as the first row.

So 0 0 1 0 0 1 0 - 1 - 1 6, so - 1 - 1 6 so this is how we make this the element this element this element here as 0 by elementary row operation multiplying this new key row by half and adding to the first row. Similarly, we can make this element 0 by multiplying by half and add subtracting form there and this elements can also be made 0 by multiplying this new key by half and subtracting from third.

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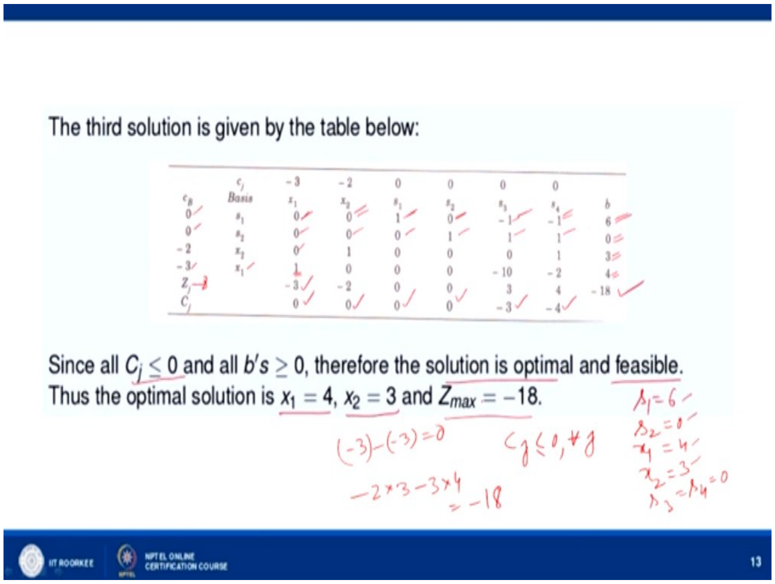

c_B	c_j Basis	-3 x_1	-2 x_2	0 s_1	0 s_2	0 s_3	0 s_4	b
0	s_1	0	0	1	0	-1	-1	6
0	s_2	0	0	0	1	1	1	0
-2	x_2	0	1	0	0	0	1	3
-3	x_1	1	0	0	0	-10	-2	4
Z_j		-3	-2	0	0	3	4	-18
C_j		0	0	0	0	-3	-4	

So after that after we have done that this is the new simplex table, so 0 0 1 0 - 1 - 1 we have verified this and this is the new row after the elementary row operations, so this elements are become 0s with the help of this we have made this this this element 0. Now, let us determine Z_j so Z_j is than 0 into 0 0 into 0 - 2 into 0 - 3 into 1 so - 3 and - 3 when subtracted from - 3 - 3 - - 3 gives you 0, so this is 0 similarly, we calculate capital C_j for x_2 column, this is 0 here capital C_j for s_1 column 0 here s_2 0 s_3 column - 3 s_4 column - 4.

So we can notice that $C_j \leq 0$ for all j and the values of in the V column values in the V column are now 6 0 3 4, so ¿ 6,s_2=¿ 0, x_1 = 4, x_2 = 3, so the basic variables x_1 x_2 s_1 s_2 have values 4 3 6 0 and s_3 s_4 are 0, so x_1 x_2 s_1 s_2 s_3 s_4 all are greater than or equal to 0 and $C_j \geq 0$, so therefore, the solution is optimal and also feasible and thus the optimal solution is x_1= 4, x_2 =3 and Z maximum value of Z is you see, we B column elements multiplied to C_b column, so 0 into 0 0 into 6 0 into 0 - 2 into 3.

So - 2×3 =- 6 and then we get - 3×4 = - 18, so maximum Z maximum value of Z is - 18, so this is how we apply the dual simplex method, to solve the X linear programming problem. That is all in this lecture thank you very much for your attention.