Higher Engineering Mathematics
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Lecture – 51
Dulity – II

Hello friends, welcome to my second lecture of Duality. Now, let us first discuss duality principle if the primal and the dual problems have feasible solutions than both have optimal solutions and the optimal value of the primal objective function is equal to the optimal value of the dual objective function that is maximum of Z equal to minimum of W, this is the fundamental theorem of duality, it suggest that an optimal solution to be primal problem can directly be obtained from that of the dual problem and vice versa.

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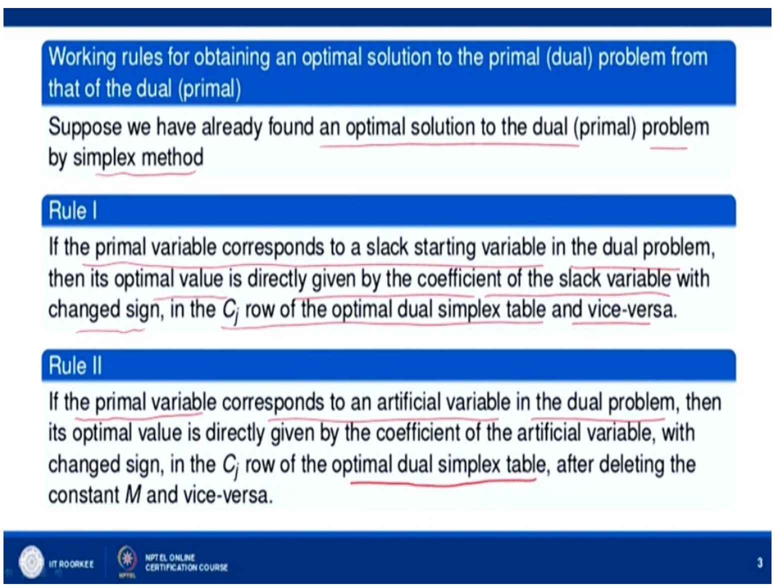

Now, let us discuss working rule to obtain an optimal solution to the primal problem from that of the dual problem. Suppose that we have already found an optimal solution to the dual problem by simplex method, then if the than the rule says if the primal variable corresponds to a slack or starting variable in the dual problem then its optimal value is directly given by the coefficient of the slack variable with changed sign, in the C_j row of the optimal dual simplex table and vice-versa.

Now, the rule two if the primal variable corresponds to an artificial variable in the dual problem, then its optimal value is directly given by the coefficient of the artificial variable with changed sign, in the C_j row of the optimal dual simplex table after deleting the constant M and vice-versa.

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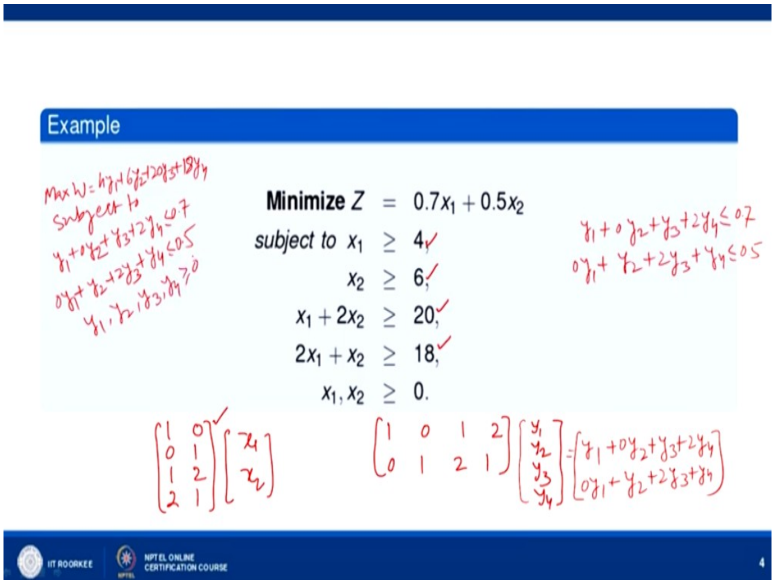

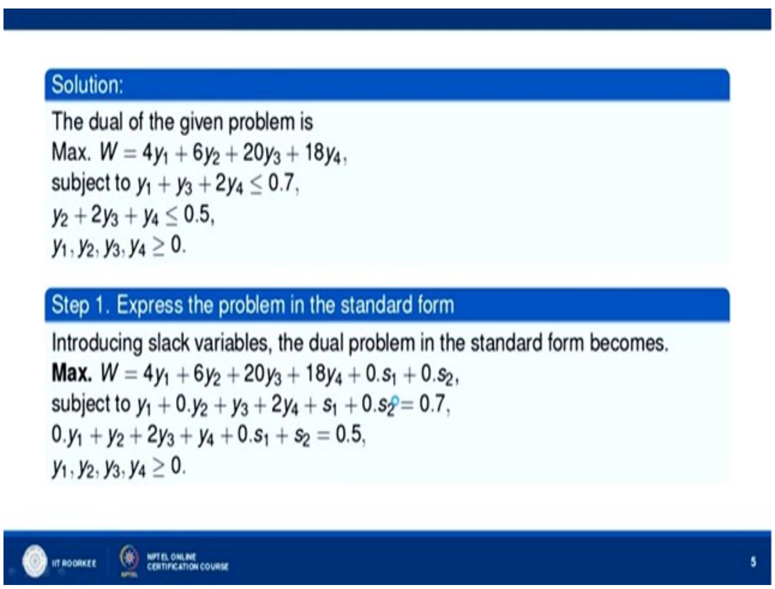

Say for example, let us consider the LPP,

Minimize $Z = 0.7\,x_1 + 0.5\,x_2$

subject to $x_1 \geq 4$, $x_2 \geq 6$, $x_1 + 2x_2 \geq 20$, $2x_1 + x_2 \geq 18$, x_1, $x_2 \geq 0$, so we will write the dual of this problem first.

So, dual of this problem will be maximize W, because this is minimization problem, so dual will be maximize W = $4y_1 + 6y_2 + 20y_3 + 18y_4$, that is $b_1y_1 + b_2y_2 + b_3y_3 + b_4y_4$, so= $4y_1 + 6y_2 + 20y_3 + 18y_4$, and then here the $x_1 \geq 4$, so the matrix is this 0 1 0 than 0 1, 1 2, 2 1, we have the 4 constriants x_1, x_2, $x_1 \geq 4$, $x_2 \geq 6$, $x_1 + x_2 \geq 20$, and $2x_1 + x_2 \geq 18$, so the left hand side of these constants of the result of the multiplication of the matrix 1 0 0 1 1 2 2 1 by the vector x_1 x_2.

So, when we write dual then we take the transpose of this matrix, so transpose of this matrix will be 1 0 1 2 0 1 2 1, this is transpose, transpose of this matrix so into $y_1 y_2 y_3 y_4$and then greater or equal will be change to less than or equal to.

So we will have, so the left hand side of the constants will be then this will be equal to $y_1 + 0y_2 + y_3 + 2y_4$, and the second row will be $0y_1 + y_2 + 2y_3 + y_4$, so$y_1 + 0y_2 + y_3 + 2y_4 \leq$ C1.

So, we will have the constant $y_1 + 0y_2 + y_3 + 2y_4 \leq$ C1 that is 0.7 and the second one will be $0y_1 + y_2 + 2y_3 + y_4 \leq$ C2 that is0.5, so the dual of this problem will be maximize W = $4y_1 + 6y_2 + 20y_3 + 18y_4$, 18subject to subject to the constants $y_1 + 0y_2 + y_3 + 2y_4 \leq 0.7$ and $0y_1 + y_2 + 2y_3 + y_4$, ≤ 0.5, so this is our dual problem y1, y_2, y_3, $y_4 \geq 0$.

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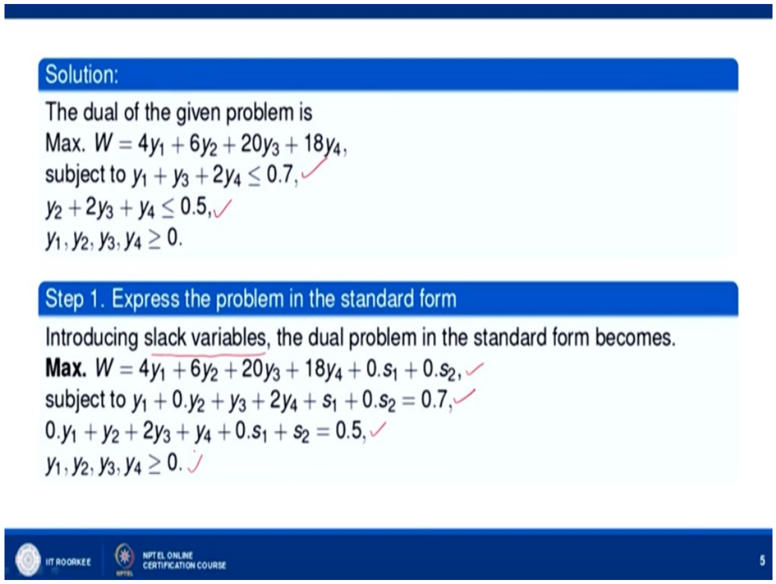

So, we have this maximum of W $=4y_1 + 6y_2 + 20y_3 + 18y_4$, subject to $y_1 + y_3$, this is $y_1 + y_3 + 2y_4 \leq 0.7$, $y_2 + 2y_3 + y_4 \leq 0.5$, $y_1, y_2, y_3, y_4 \geq 0$.

Now, let us express this problem in the standard form. So, here we have the constants of the type $\leq$ so we will use slack variable, and when we use slack variable there are two constant of $\leq$ type, so we will need two slack variables s_1 s_2.

So objective function will be written as W equal to $4y_1 + 6y_2 + 20y_3$, $+ 18y_4 + 0\ s_1 + 0\ s_2$., subject to these constant will be converted into equations by means of $s_1.s_2$. so $y_1 + 0y_2 + y_3 + 2\ y_4 + s_1 + 0\ s_2 = 0.7$ and then $0\ y_1 + y_2 + 2y_3, + y_4, + 0\ s_1 + s_2. = 0.5$, $y_1, y_2, y_3, y_4 \geq 0$, so now our problem dual problem has been expressed in the standard form.

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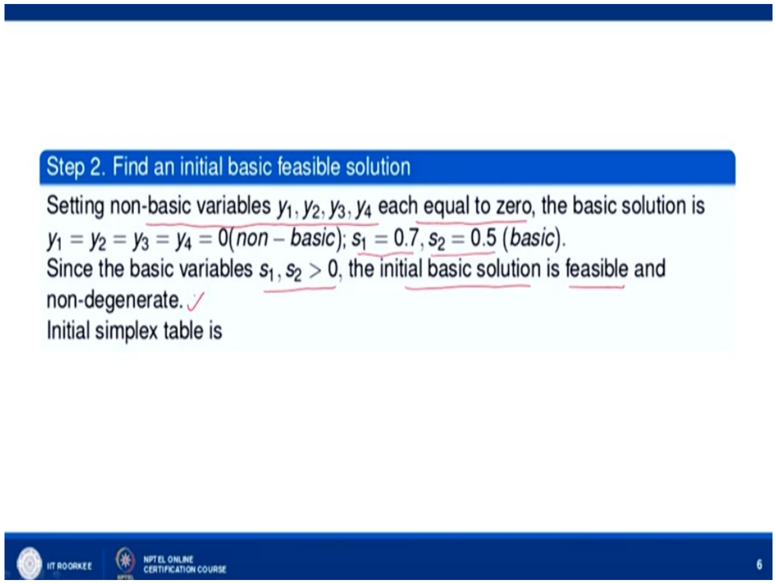

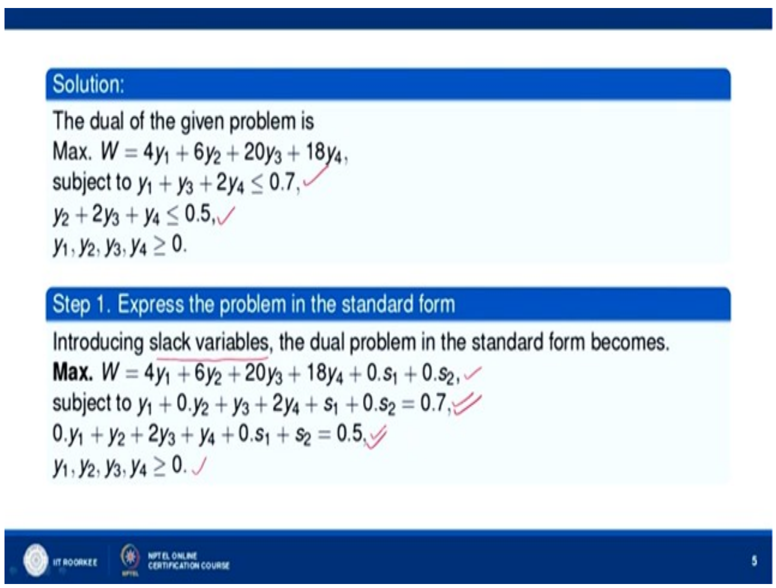

Now, let us find an initial basic feasible solution. So setting, now there are two equations, this equation and this equation, there are two equations, and there 6 variable so we can take any 4 variable equal to 0, let us take y_1,y_2, y_3, y_4 =0 and then we get the value of s_1 =0.7 s_2= 0.5.

So we get, taking non basic variables as y_1,y_2, y_3, $y_4$each equal to 0 the basic solution is y_1,y_2, y_3, $y_4$equal to 0 non basic, and s_1 = 0.7, s_2 = 0.5 basic variables. So since the basic variables are greater than 0 the initial basic solution is feasible and non-degenerate.

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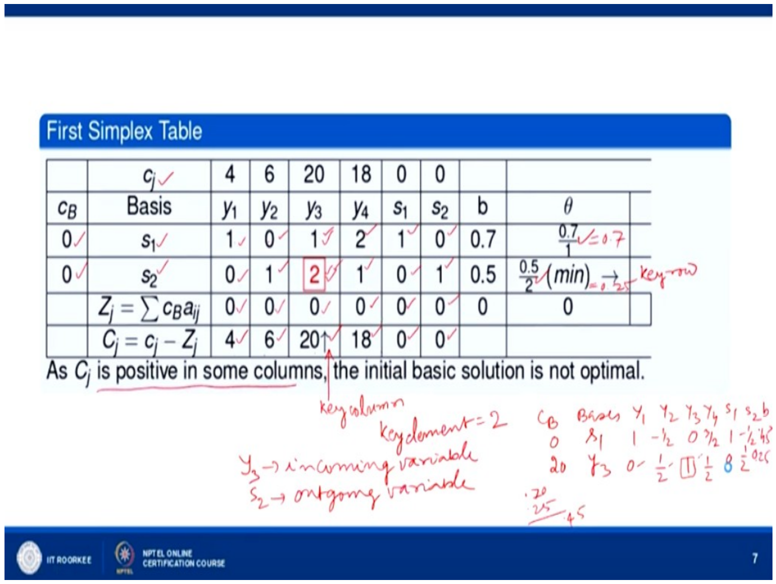

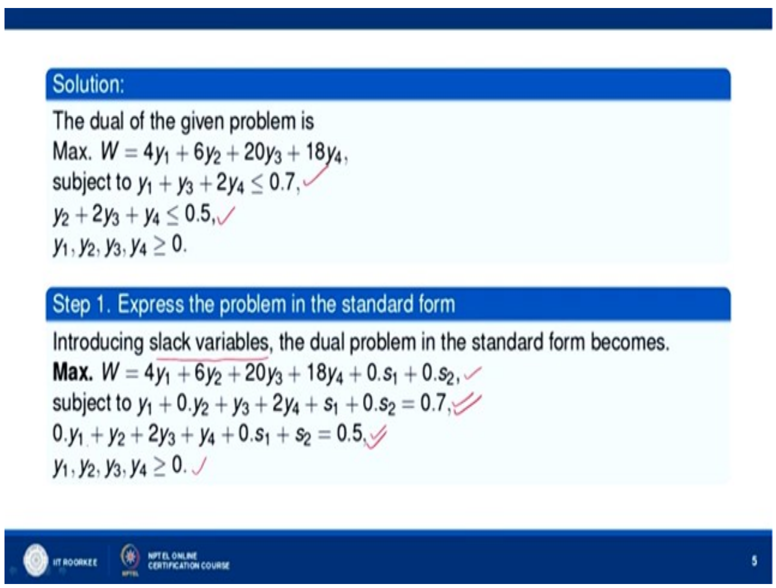

Now, initial simplex table is as follows. So this is initial simplex table, c_j row, this is smallc_j row contains the coefficients of the variablesy_1,y_2, y_3, y_4, s_1, s_2 which are 4, 6, 20, 18, 0, 0 and basic variables are s_1 and s_2, their coefficients in the objective function are 0, 0 so that constitute the cB column.

And the body matrix is, this is the body matrix, so o4 1 0 1 2 0 1 2 4 that is body matrix, so we have this body matrix 1 0 1 2 0 1 2 1 and the unit matrix is 1 0 0 1. Now, let us calculate let us solve this by using simplex method so we find Z_J, Z_J=$\sum C_B a_{ij}$. So 0×1 is 0, 0×0 is 0, so Z_J is 0 for J = 1, and then 0×0 is 0, 0×1 is 0, so again we get Z_2 = 0, and then 0 ×1 is 0, 0×2 is 0, so we get 0, here also we get 0, here 0, here 0.

Now 4-0 is C1, so that is 4, 6 -0 is 6, 20- 0 is 20, 18- 0 is 18, then 0 -0 is 0, 0 - 0 is 0.

Now, let us see c_j is positive in some columns, 1, 2, 3, 4 in 4 columnsc_j is positive, so let us find the maximum value of c_j, maximum value of c_j is 20, so this is our key column. Then we find the ratios of the values in the B column by the values in the corresponding values in the key column.

So, 0.7 divided by 1, 0.5 divided by 2, and this is 0.7, this is 0.25, so of the two positive values 0.7 and 0.25, 0.25 is minimum, so this row is our key, so this row is our key row at the intersection of key row and the key column we have the key element, key element is 2. so s_2 is out going variable, y_3 is incoming variable, so y_3 incoming variable, and s_2 is out going variable. So in the next simplex table s2 will be replaced by y_3, so y_3 is the incoming variable and s_2 is out going variable, so in the next simplex table we will have basis (s)s_1, s_2 will be replaced by y_3 and CB column will have coefficient of s_1 in the objective function is 0, coefficient of y_3 in the objective function is 20, then we have y_1,y_2, y_3, y_4, so they are (1 0), so before that let us now divide the key row elements by the key element.

So, we have 0, 1 by 2, we are dividing by 2, the key row elements, so then we have y_3 as 1 and then 1 by 2 that is y4 so than s1 is 0, s2 is 1 by 2 and b becomes 50.5 by 2 that is 0 0.25 so that is b.

Now, with the help of this element 1, this element 1 we make this element equal to 0, so we subtract this row, the new key row after we have got it divided by 2, this row we subtract from the first row, so 1 - 0 is 1, than 0 minus half so we get minus half, then $y_3$this becomes 1-1 0, y4 this is 2 here, and here we have 0, so we get 2, and then s_1 is 1 - 0 so we get 1, s_2, s_2 is 0 minus half, so we get minus half, and then 0.7 - 0.25, 0.7 - 0.25, so we get 40.45, so this is 40.45.

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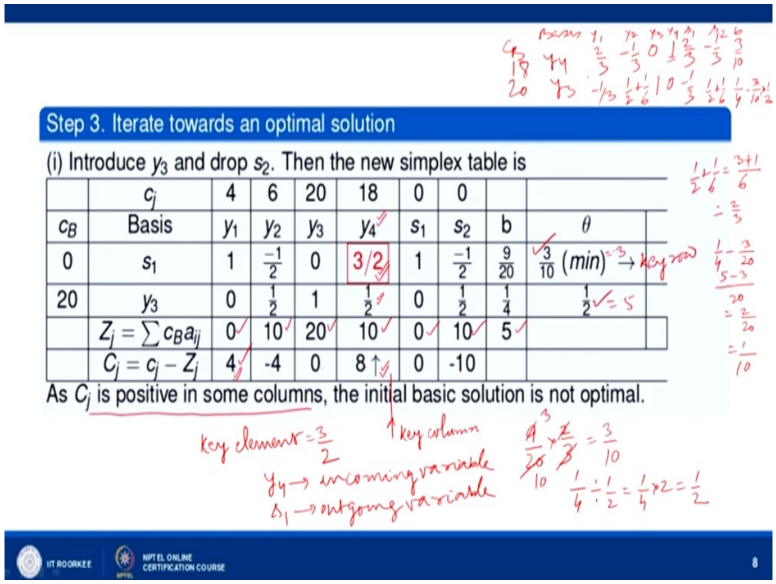

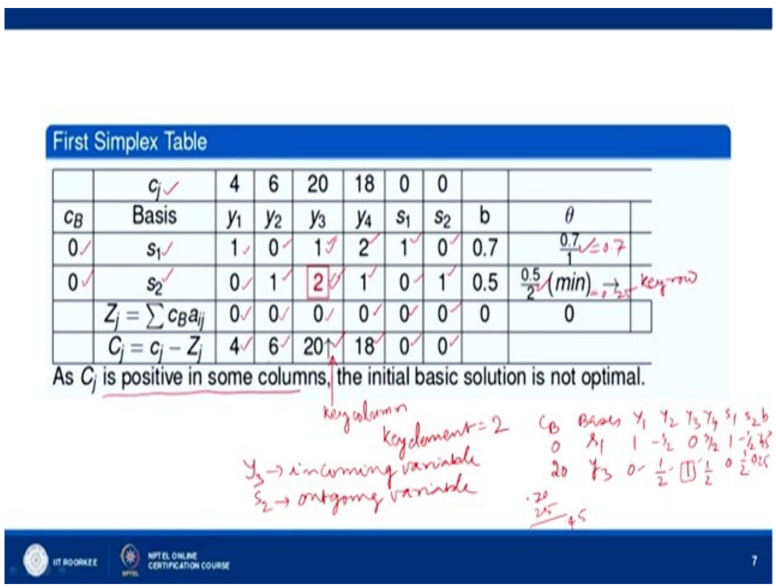

So, let us see the next simplex table 1 minus half 0, so 1 minus half 0, then we have this will be 3 by 2, 2 minus half 3 by 2 and here we have 0, so this will become 1 and then s_2, s_2 will become minus half, so we have first row is 1 minus half 0 3 by 2, 1 minus half 0 3 by 2, 1 minus half 40.45, 540.545 is 9 by 20, and the second row is 0 half, 1 half, 0 half 1 half, 0 half 0.25, 0 half 0.25, so this is our new rows.

Now, we will find the value Z_j, so 0 into 1 is 0, 20 into 0 is 0, so we get 0, 4 minus 0 is 4, than we get 0 into minus half 0, 20 into half is 10, so we get total 10 and then 0 into is 0, 20 into 1 is 20, so we get 20 here, then 0 into 3 by 2 is 0, 20 into half is 10, we get here 10 here, (())(14:54) 0 into 20, so 0 here, then 0 into minus half 20 into half, so we get 10 here and 0 into 9 by 20 is 0, 20 into (1 by 55), 1 by 4 so that is (1) 5 we get.

Now, this is 4, this is 6 minus 10 is minus 4, 20 minus 20 is 0, 18 minus 10 is 8, then 0 minus 0 is 0, 0 minus 10 is minus 10, so there are again C_j is positive in some columns, in this column C_j if 4, in this column C_j is eight, so we take maximum value, so eight is the maximum of the two, 4 and 8 and therefore this our key column, so this is key column.

Now, divide the values in the b column by the values in the key column, so 9 by 20 divided by 3 by 2 so that means 2 by 3, so we get 3 by 10 this value and then 1 by 4 divided by 1 by 2, so we get 1 by 2.

Now, out of the two positive values of theta, 3 by 10 and half, this is 0.5, this is 0.3, so 0.3 is the minimum and therefore this is our key row. So at the intersection of key row and key column we have 3 by 2, so key element is 3 by 2.

Now, this y_4, y_4 is incoming variable s_1 is outgoing variable, so y_4 is incoming variable and s_1 is outgoing variable, so in the next simplex table we shall have in the C_B column we shall have here the coefficient of y_4, coefficient of y_4 is 18, then we have 20 and in basis we will have s_1 is replaced by y_4, so y_4 here, and here we have y_3 and then the this this row, key row, elements are divided by key element that is 3 by 2 so we will have 2 by 3, 3 by 2, 1 is divided by 3 by 2 gives 2 by 3, and then minus half into 2 by 3, so 2 2 will cancel we get minus 1 by 3, then 0 divided by 3 by 2 is 0, 3 by 2 divided by 3 by 2 is 1 and then 1 divided by 3 by 2 is 2 by 3, minus half divided by 3 by 2, so minus half into 2 by 3, so minus 1 by 3, and then 9 by 20 divided by 3 by 2, so we get 9 by 20 into 2 by 3 that is 3 by 10, so we get here 3 by 10, so this is y_1,y_2, y_3, y_4 and then s_1 s_2 and b and now with the help of this one we make this entry 0 in the key column.

So, we multiply by half and subtract from this row, so we multiply by half, half when you multiply becomes 1 by 3, 1 by 3 we subtract from 0 so we get minus 1 by 3 and then we are here we are multiplying by 1 by 2, 1 by 2 when we multiply here we get minus 1 by 6 and after that we subtract, so 1 by 2 + 1 by 6 we have, then here we have 0, 0 is multiplied by half and then subtracted from 1, so it will remain 1 than this will become 0 and this will become we are multiplying this by 1 by 2 so this will become 1 by 3, 1 by 3 we subtract from 0 so we get minus 1 by 3 we are multiplying by half and then subtracting.

So, this is minus 1 by 6, we are subtracting it, so what we get s2 column? s2 column is 1 by 2, so 1 by 2 + 1 by 6 we have, and here we get 1 by 4 we are multiplying it by half and subtracting, so minus 3 by 10 into 1 by 2, so what we shall have 1 by 2 + 1 by 6 is equal to 3 + 1 divided by 6 so that is 4 by 6 this is 2 by 3 and so 1 by 2 + 1 by 6 2 by 3 this 2 by 3 and then we get 1 by 4 minus 3 by 20, so we get 20, 3 by 20,

so we get 20 here so 5 minus 3, so get 2 by 20, so we get 1 10, so this is 1 by 10.

So first row, let us see first row is 2 by 3 minus 1 by 3 0 2 by 3 minus 1 by 3 minus 0 than we have 1 2 by 3 minus 1 by 3 1 2 by 3 minus 1 by 3, 3 by 10, second row is minus 1 by 3, then 2 by 3, so minus 1 by 3, then 2 by 3, then 1 0, so we get 1 0 and then minus 1 by 3, minus 1 by 3 than we have this one is 2 by 3 so we get 2 by 3 here and this becomes 1 by 10, so we get 1 by 10 here.

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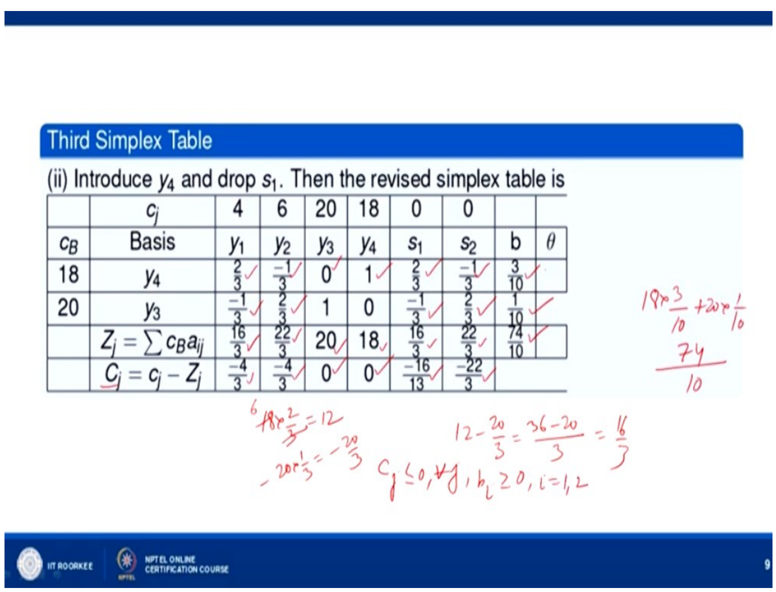

Now, let us find Z_j value, so Z_j value comes out to be 18 into 2 by 3 so that means 18 into 2 by 3 means 12 so this when you multiply 18 into 2 by 3 you get 12 and you multiply 20 into minus 1 by 3 is 20 into 1 by 3, so this is minus 20 by 3.

So, let us make a total of this 12 minus 20 by 3 is equal to minus 20 by 3 so this is 16 by 3, so we get 16 by 3 here. Similarly, here Z_j is 22 by 3, here 20, here 18, here 16 by 3, here 22 by 3, and here when we multiply 18 to 3 by 10 + 20 into 1 by 10, so we get 54 + 20 74 by 10, so, this is 74 by 10.

Now let us find capital C_j so 4 minus 16 is by 3 is minus 4 by 3 and 6 minus 22 by 3 is minus 4 by 3 by 20 minus 20 is 0, 18 minus 18 is 0, 0 minus 16 by 3 is minus 16 by 3 0 minus 22 by 3 is minus 22 by 3, so we can see that C_j is ≤ 0 for all J and so b_i is ≥ 0 $b_1 \geq 3$ by 10, $b_2 = 1$ by 10 so i = 1 ,2, so $b_i \geq 0$ for i = 1, 2.

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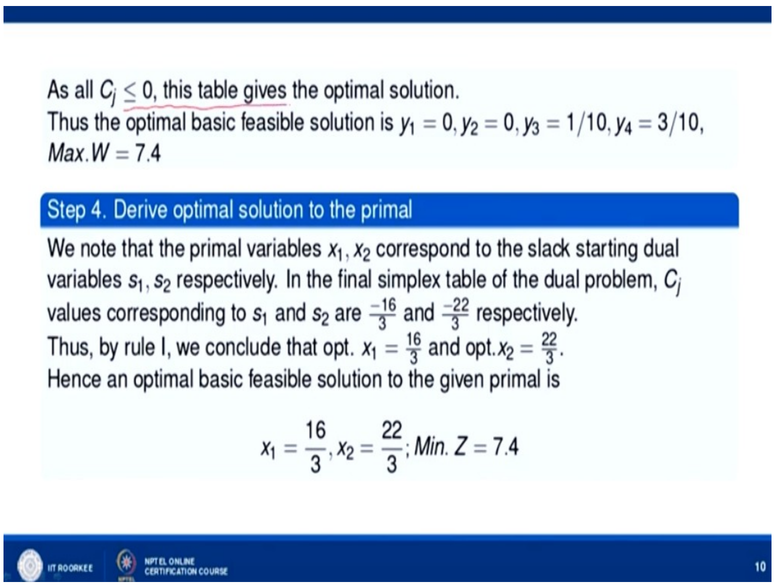

And therefore, since all CJ is ≤ 0 this gives the optimal solution and the optimal basic solution is than y1 equal to 0.

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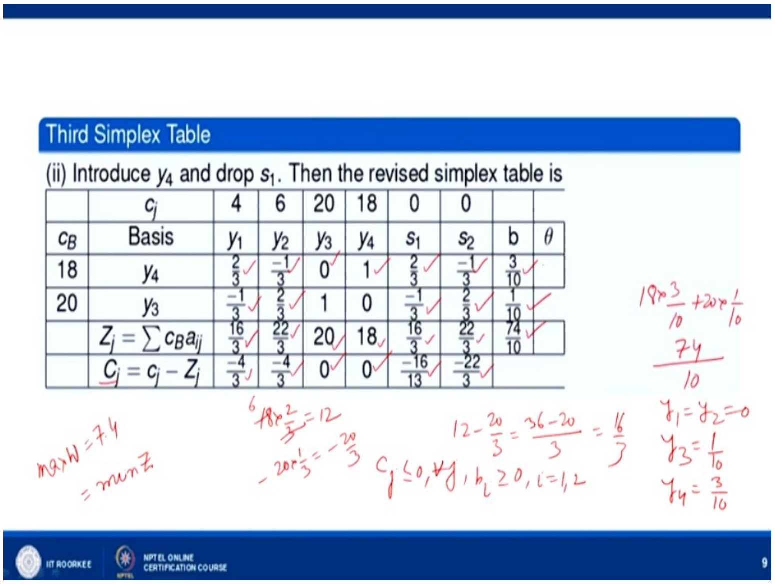

So, $y_1 = 0$, $y_2 = 0$, $y_3 = 3$ by 10, , y_3 we get as 3 by 10 and , y_4 we get as , y_3 we get as 1 by 10 , y_3 we get as 1 by 10 and y_4 we get as 3 by 10 and y_1 y_2 equal to 0 and maximum value of Z is 74 by 10 that is 7.4 which is minimum value of this maximum value of we are using W so maximum value of W is 7.4 which is equal to minimum value of a Z so minimum value of Z is 7.4.

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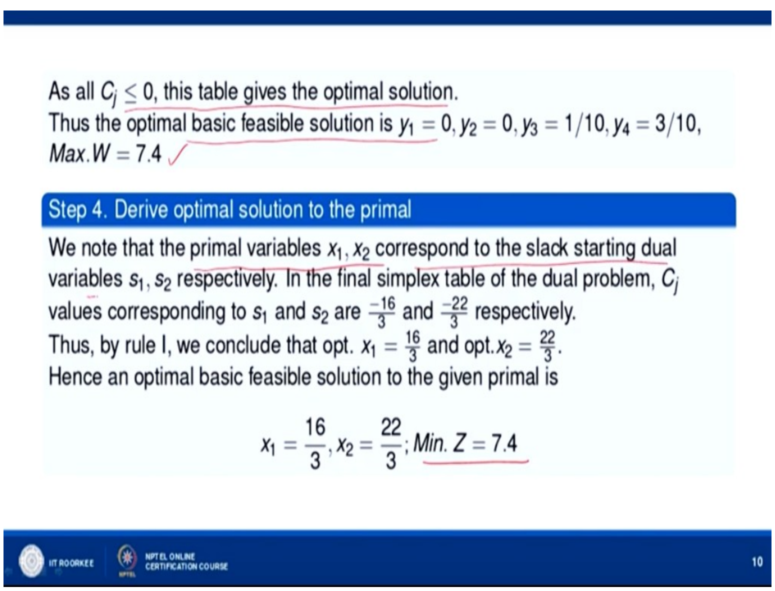

So, we get this maximum value of W is 7.4, so minimum value of W is 7.4 and we let us go to the rule, rule says that the primal variable x_1, x_2, here correspond to the slack starting dual variable s_1, s_2, respectively you can see here s_1, s_2, , so x_1, x_2, corresponds to s_1, s_2, here and therefore and the final simplex table of the dual problem C_j value is corresponds to s_1, s_2, are C_j values corresponding s_1, s_2, are minus 15 by 3 minus 22 by 3.

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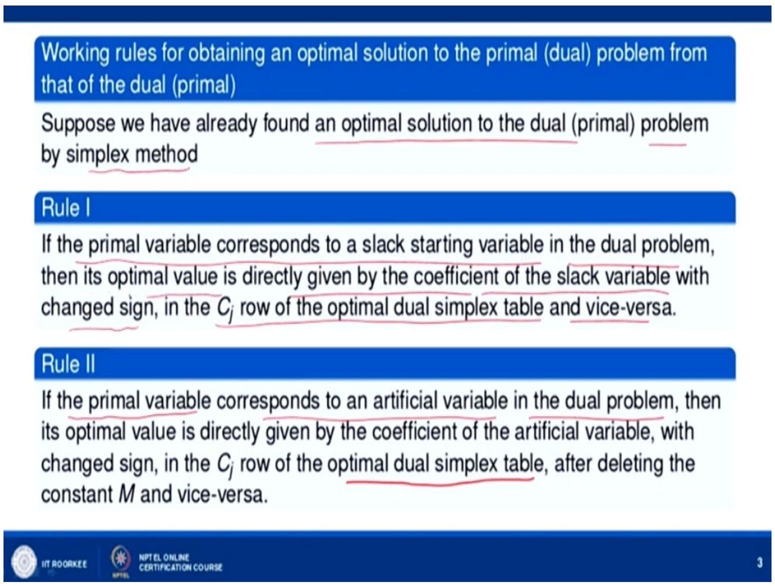

So, by rule one this is the rule, this is the rule, by rule one if the primal variable corresponds to slack starting variable in the dual problem than its optimal value is directly given by the coefficient of the slack variable with changed sign in the C_j row of the optimal dual simplex table and vice-versa.

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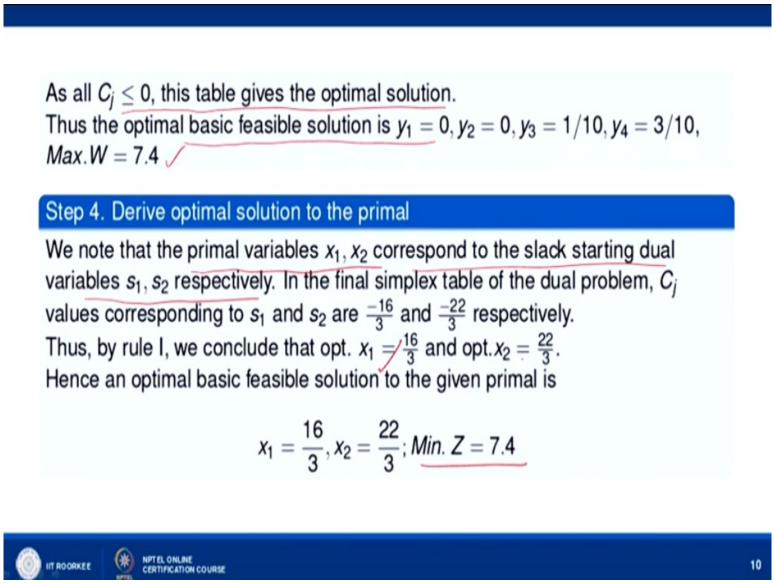

So, we will change their sign to get the corresponding value for x_1, x_2, so here this is C_j is minus 16 by 3 for s_1 and for s_2 it is minus 22 by 3, so corresponding to them the values of x_1, x_2, are 16 by 3 and 22 by 3, so x_1, is 16 by 3, x_2 is 22 by 3 minimum value of Z is 7.4, so this is the optimal basic feasible solution to the given primal in this case.

So we have discussed the, how to solve the given LPP by converting it into the dual problem and we solve it there and then determine the solution of the given problem.

So, we have seen that by writing dual of the given problem and solving it by the simplex method, we can determine the solution of the given primal problem. So that is all in this lecture, thank you very much for your attention.