Higher Engineering Mathematics
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Lecture - 50
Duality - I

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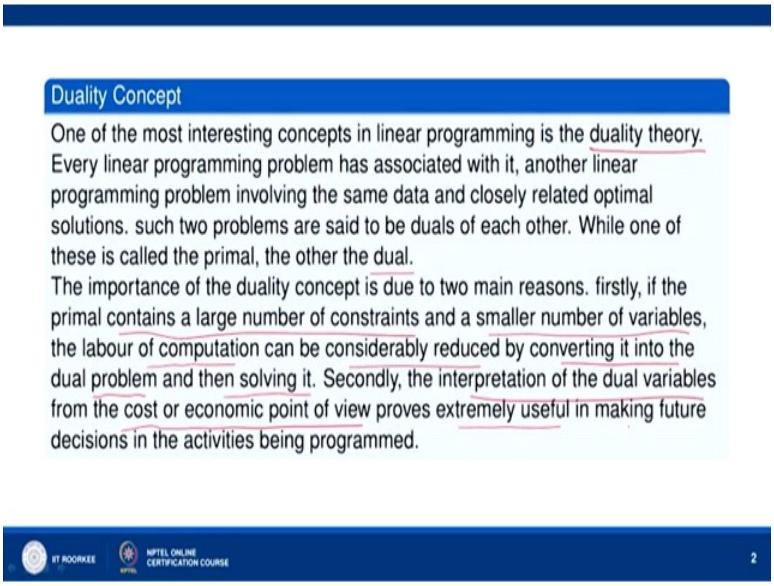

Hello friends, welcome to my lecture on The Concept of Duality, one of the most important and interesting concepts in linear programming is the Duality. Every linear programming problem has associated with it, another linear programming problem which involves the same data and is closely related to the optimal solutions, such two problems are said to be duals of each other.

While one of these problems is called the primal problem, the other one is called the dual problem. The importance of the duality concept is due to the two main reasons. Firstly, if the primal contains a large number of constraints and a smaller number of variables then the labour of computation can be considerably reduced by converting it into the dual problem and then solving it, because if there are say m constraints and n variables and m is greater than n then when we will consider the corresponding dual problem there the number of constraints will become n and number of variables will become m, ok.

So, number of constraints will reduce when we consider the dual problem and therefore it will be easier to solve the get the solution of the dual problem. So labour of computation can be reduced by converting the problem into the corresponding dual problem and then solving it. Secondly, the interpretation of the dual variables from the cost or economic point of view proves extremely useful in making future decisions in the activities being programmed.

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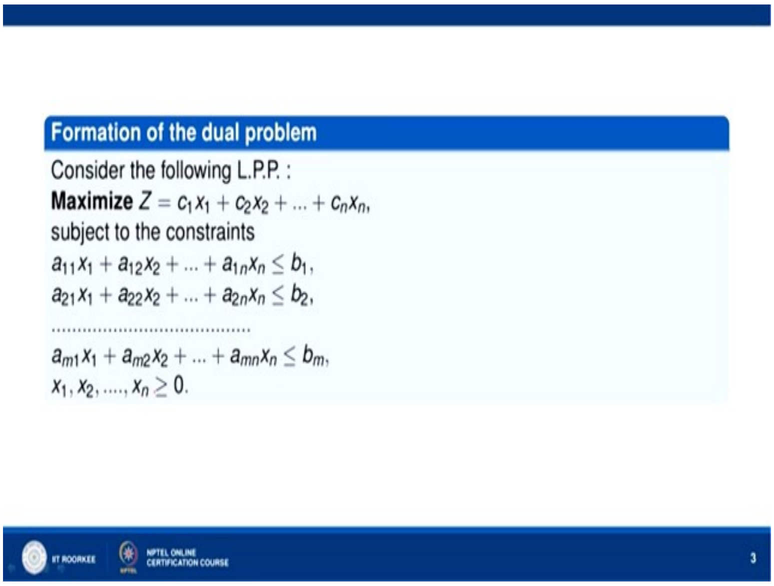

Now, let us consider the following L.P.P. maximize Z = $c_1\ x_1 + c_2 x_2$ and so on $c_n\ x_n$ subject to these m constraints $a_{11}\ x_1 + a_{12}\ x_2$ and so on $a_{1n}\ x_n \leq b_1$; $a_{21}\ x_1 + a_{22}\ x_2$ and so on $a_{2n}\ x_n \leq b_2$ and then the m at constraint $a_{m1}\ x_1 + a_{m2}\ x_2$ and so on $a_{mn}\ x_n \leq b_m$, x_1, x_2 $x_n$are all non-negative.

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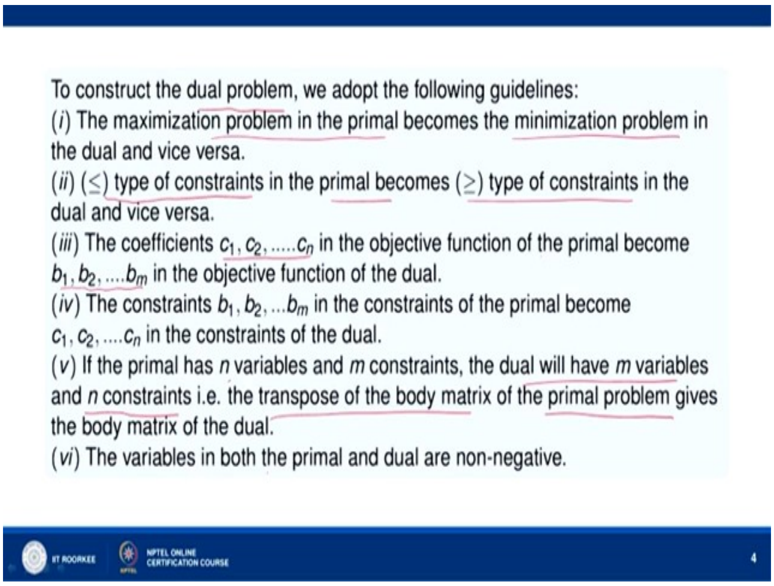

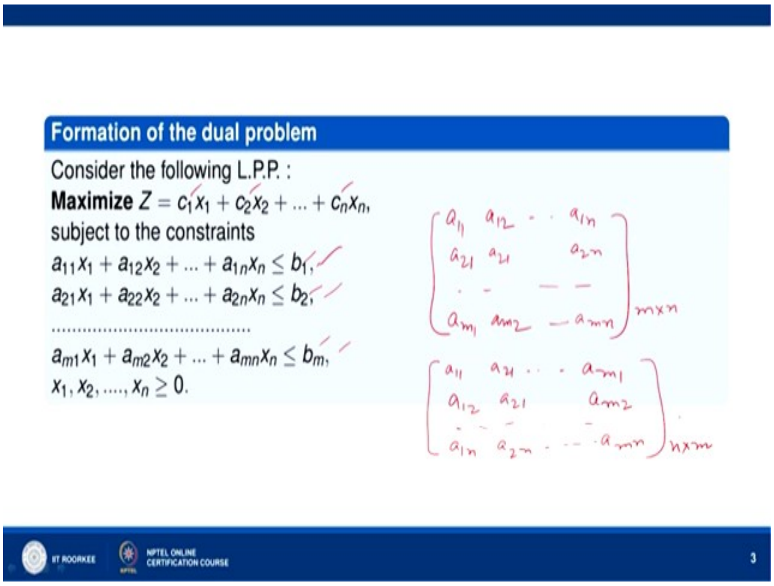

Now, in order to construct the dual problem we adopt the following guidelines, the maximization problem in the primal becomes the minimization problem in the dual and vice versa. $\leq$ type of constraints in the primal become $\geq$ type of constraints in the dual and vice versa.

The coefficients c_1, c_2,… c_n in the objective function, the coefficients c_1, c_2, …c_n in the objective function of the primal become b_1, b_2,…. b_m, ok so c_1, c_2,… c_n will be replaced by b_1, b_2,…. b_m, ok these b_1, b_2, ….b_m and x_1, x_2, ….x_n will be replaced by new variables y_1, y_2, …. y_m, ok.

So, the dual problem in the dual problem maximization will be replaced by minimization and we will have minimization of $Z^{¿}$ where we will have $Z^{¿} = b_1\, y_1 + b_2\, y_2$ and so on $b_m\, y_m$, ok.

If the primal has n variables and m constants like here we have n variables x_1, x_2, ….$x_n$and m constraints given by these m an equalities, ok so then the dual will have m variables and n constraints that is the transpose of the body matrix of the primal problem gives the body matrix of the dual.

So, here the body matrix is this one, body matrix is a_{11}, a_{12} and so on a_{1n}, a_{21}, a_{22},.. a_{2n} and so on a_{m1}, a_{m2} and so on a_{mn} this is body matrix, ok of the primal problem.

In the dual problem this will be replaced with a_{11}, a_{21} and so on a_{m1}, a_{12}, a_{21}, a_{m2}, ok and then lastly a_{1n}, a_{2n} and so on a_{mn}, ok so here we have here we have m rows and n columns, so this is m by n matrix, here we have n rows and we can see we have m there are 1, 2 and so on m columns and n rows, so n by_m, ok.

So, here the body matrix, it is m by n size, here it is of the size n by m, transpose of the body matrix of the primal problem, ok. So transpose of the body matrix of the primal problem gives the body matrix of the dual.

The variables in both the primal and dual are non-negative, ok. So, now the dual problem will then be you see here, the dual problem will now be $Z^{¿}$ minimization of $Z^{¿} = b_1\ y_1 + b_2\ y_2$ and so on $b_m\ y_m$, subject to the constraints $a_{11}\ y_1 + a_{21}\ y_2$ and so on + $a_{m1} y_m \geq c_1$, ok $a_{12}\ y_1$, $a_{22}\ y_2$ …..+ $a_{m2}\ y_m \geq c_2$ and so on $a_{1n}\ y_1$+ $a_{2n}\ y_2$ and so on $a_{mn}\ y_m \geq c_n$, ok.

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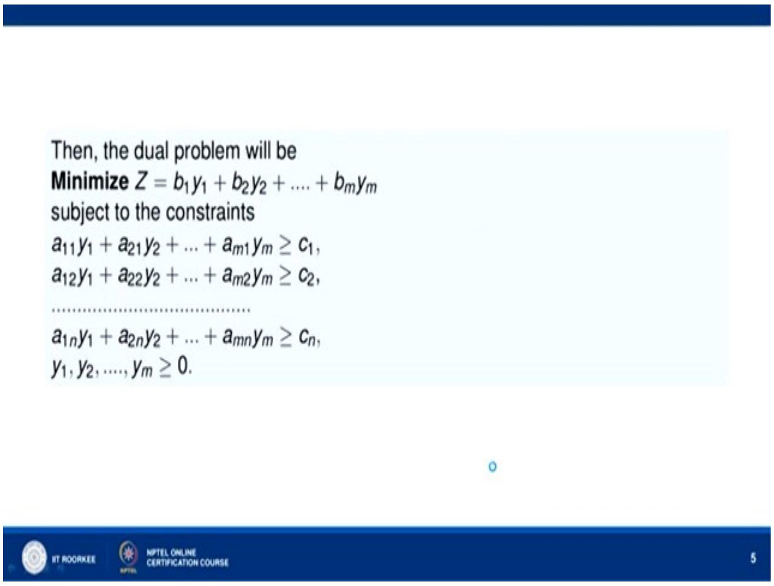

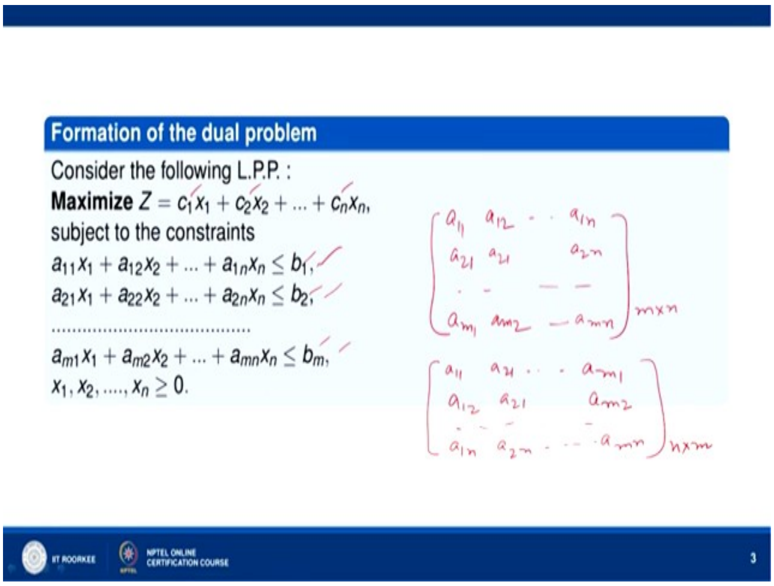

So, we will have these equations a 1 Z minimization of Z = b_1 y_1, b_2 y_2, b_m y_m subject to the constants a_{11} y_1 + a_{21} y_2,….+ a_{mn} y_m ≥yeah ≤will be replaced with ≥ ok in the dual. So, we will have ≥here, so ≥c_1, ≥c_2 and ≥c_n and y_1, y_2, y_m are non-negative variables, ok. ≤

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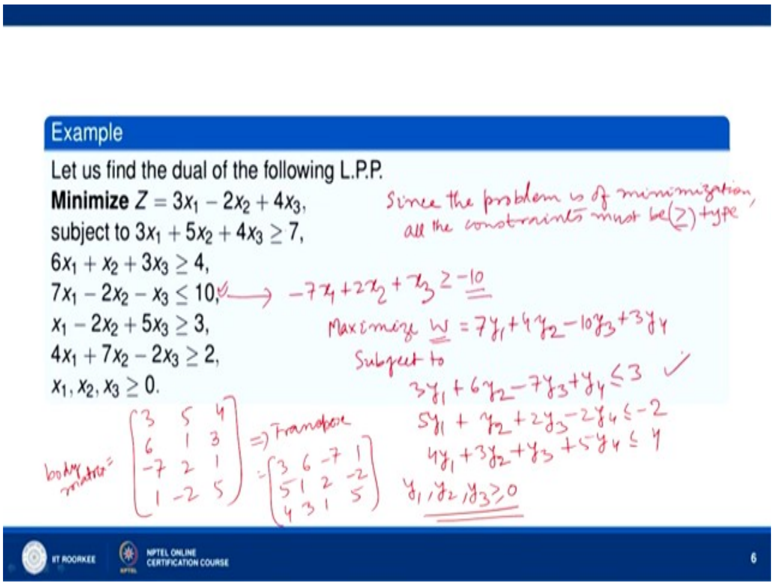

Let us find the dual of the following L.P.P. So, we have Z = 3x_1 - 2x_2 + 4x_3 subject to 3x_1 + 5 x_2 + 4x_3 ≥ 7, since the problem is of minimization you can see here, since the problem is of minimization all the constraints must be of ≥ type, all the constraints must be ≥ type, ok

so ≥ type.

So here what is happening in this constraint, this is ≤ type, so we have to convert it to ≥ type, so when you convert it to ≥ type it will become - 7x_1 + 2x_2, ok + x_3 ≥ - 10, ok - 7x_1 + 2x_2 + x_3 ≥ - 10, ok.

Now, so let us write the dual problem, so maximization of maximize Z will be replaced by some W, W = now this is c_1, c_2, c_3 W replaced by with b_1, b_2,.. b_m, so 7 y_1 + 4 y_2 - 10 y_3, ok and then 3 y_4, ok and then we will have so this is our objective function in the dual problem and (corres subs) subject to the constraints, we will have now the (bod) the matrix here is body this body matrix is 3, 5, 4, ok 6, 1, 3, ok then we have - 7, 2, 1, ok we have 1 - 2, 5 this is body matrix in the primal problem.

So, in the dual problem we have to take transpose of this, so we will have subject to constants 3y_1 + 6y_2 - 7y_3 + y_4 and ≤ we will have ≤c_1, c_1 is 3, ok then we have 5y_1 + y_2 + 2y_3 and we get - $2y_4 \leq$ - 2, ok and then $4y_1 + 3y_2 + y_3 + 5y_4 \leq 4$, ok so c_1, c_2, c_3, ok and y_1, y_2, $y_3 \geq 0$, so this is our dual problem, we replaced (mak) minimization by maximization, the variable Z will be replaced by some other variable,

some other variable, so we can write it as W.

Then the c_1, c_2, c_3 are replaced with b_1, b_2, b_3, b_4, ok so b_1 is 7, b_2 is 4, b_3 is - 10, b_4 is 3, ok and we use new variables y_1, y_2, y_3, y_4, so $7y_1 + 4y_2 - 10y_3 + 3y_4$, the body matrix here is 3, 5, 4, 6, 1, 3 - 7, 2, 1 and then 1 - 2, 5 we take transpose of this and then it becomes transpose of this becomes 3, 6, - 7, 1, 5, 1, 2, - 2, 4, 3, 1, 5, ok.

So, we get the constraints as $3y_1 + 6y_2 - 7y_3 + y_4 \leq c_1$, $5y_1 + y_2 + 2y_3 - 2y_4 \leq$ - 2, $4y_1 + 3y_2 + y_3 + 5y_4 \leq 4$, so this and where y_1, y_2, y_3 are non-negative, so this is the dual problem, ok.

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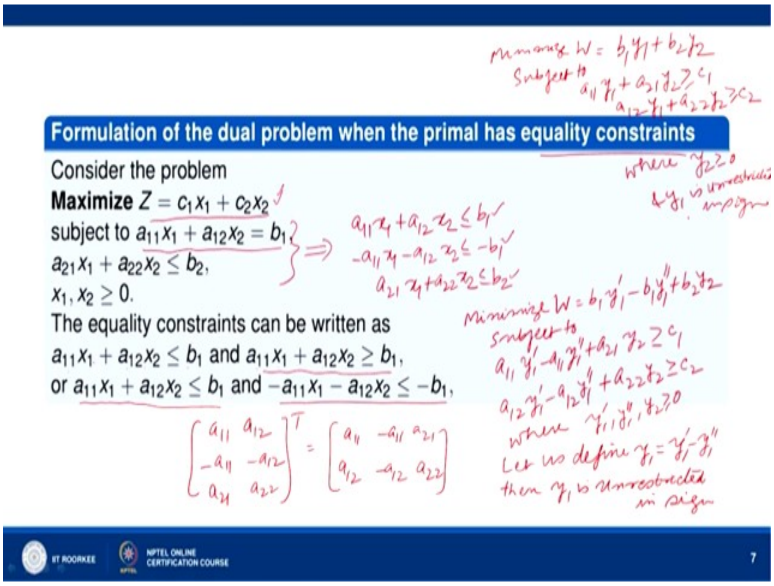

Now, let us consider the problem where we have to maximize Z = c_1 x_1 + c_2 x_2 and what we have we are have here the change here is that 1 constraint is an equality, ok there is a equality constraints a_{11} x_1 + a_{12} x_2 = b_1, so how we will (siles) find the dual of this as L.P.P. ok. So, what we will do is the equality constraint a_{11} x_1 + a_{12} x_2 = b_1 can be written as a_{11} x_1 + a_{12} x_2 $\leq b_1$ and a_{11} x_1 + a_{12} x_2 $\geq b_1$, so in the problem is of maximization, ok all the constraints should be of $\leq$ type, ok.

So, first constraint will be a_{11} x_1 + a_{12} x_2 $\leq b_1$, the second constraint this one will be converted into $\leq$ type and we will write it as - a_{11} x_1 - a_{12} x_2 $\leq$ - b_1. So, now the problems will this be replaced by a_{11} x_1 + a_{12} x_2 $\leq b_1$, - a_{11} x_1 - a_{12} x_2 $\leq$ - b_1, ok and a_{21} x_1, a_{22} x_2 $\leq b_2$, ok.

Now, so when we write the dual problem here we write maximize maximization we replaced by minimization minimize W = now here we have b_1, - b_1, b_2, ok so we write b_1, ok we can write b_1 y_1', ok b_1 c_1, c_2 will be replaced with b_1, - b_1 and b_2, so b_1 y_1'- b_1 y_1'', let us take the variables y_1', y_1'' and then b_2 y_2, ok.

So, this and the in the body matrix is now a_{11}, a_{12}, ok - a_{11}, - a_{12}, a_{21}, a_{22} this is 3 by_2 matrix, ok 3 rows and columns, so we take the transpose of this body matrix, so transpose of this will be giving you a_{11}, - a_{11}, a_{21} and then a_{12}, - a_{12} and then a_{22}, ok so subject to constraints a_{11} y_1' - a_{11} y_1''+ a_{21} y_2 $\geq$ we will have because we have $\leq$ here, so $\geq c_1$, ok then a_{12} y_1'- a we have here a_{12} here we have - a_{12}, ha - a_{12} y_1''+ a_{22} y_2 $\geq c_2$, so where y_1' $y_1''$and y_2 are all $\geq$0.

Now, let us define $y_1 = y_1' - y_1''$, since y_1' and y_1'' are greater than both non-negative y_1 is unrestricted in sign, ok it is not restricted in sign, so y_1 then y_1 is unrestricted in sign and we will get this problem as minimization of, so this will then change to minimization of W = b_1 $y_1' - y_1''$ will be replaced by y_1, so $b_1\ y_1 + b_2\ y_2$, ok subject to a_{11} into $y_1' - y_1''$.

So, $a_{11}\ y_1 + a_{21}\ y_2 \geq c_1$ and then a_{12} multiplied by $y_1' - y_1''$ that is y_1, so $a_{12}\ y_1 + a_{22}\ y_2 \geq c_2$ where $y_2 \geq 0$ and y_1 is unrestricted in sign, so y_1 can be negative as well as positive, so this is the dual of the given problem when the primal has equality constraints we will do this, ok.

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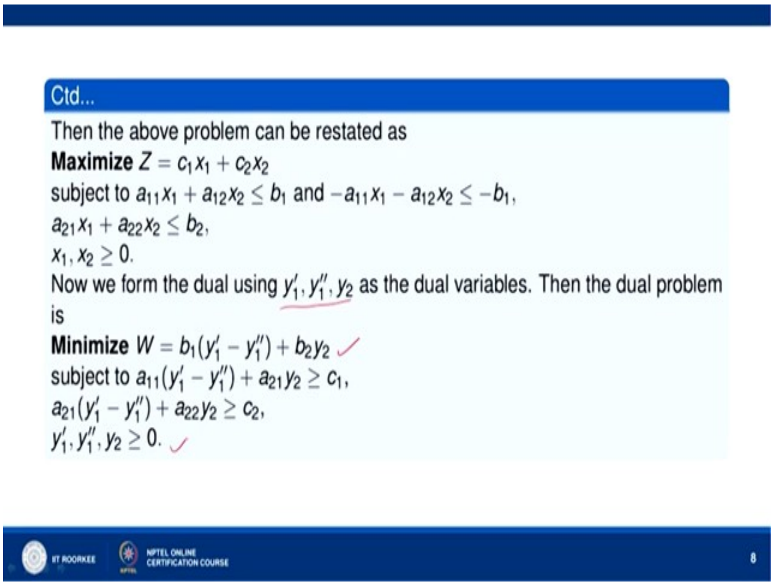

So, this what we have then, this is maximize Z = this, subject to this, ok this constraints x_1, x_2 ≥ 0 then we use the variables y_1', y_1'', y_2, ok and we get W, W = this, subject to this, ok and which gives us this problem, ok.

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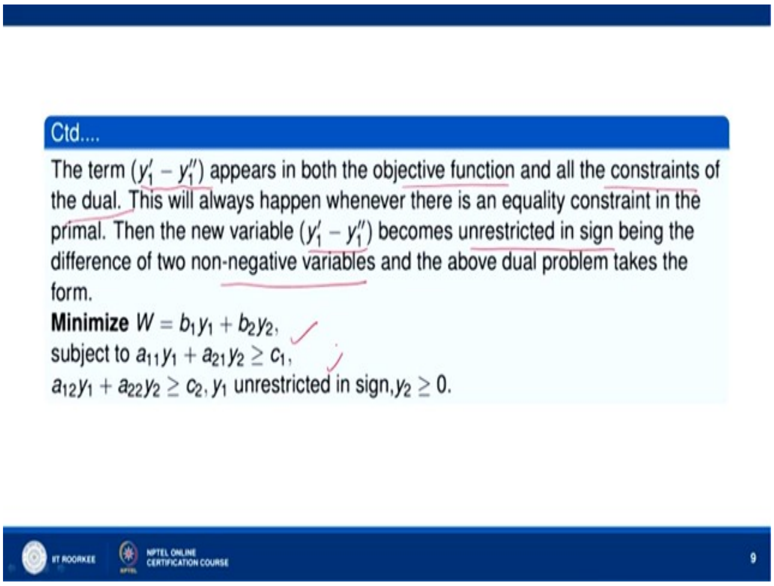

The term $y_1' - y_1''$ appears both in the objective function and all the constraints of the dual. This will always happen whenever there is an equality constraint in the primal. The new variable this becomes unrestricted in sign, ok because of the difference of two non-negative variables, so the above dual problem then takes this form, ok which we have just now seen, ok.

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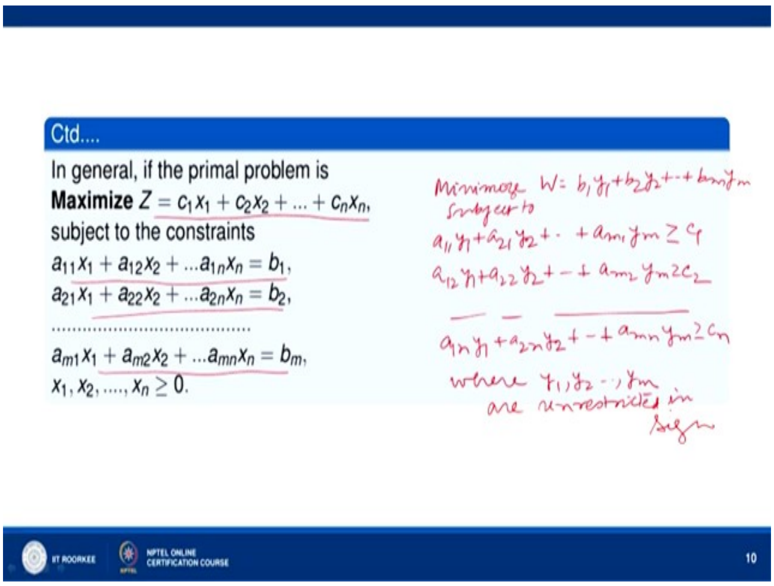

Now, let us consider in general if the primal problem is Z = c_1 x_1 + c_2 x_2 and so on c_n x_n subject to this constraints, this is the = b_1, this is the = b_2, this is the = b_m, ok then it will be replaced with the it is dual of this will be minimize W, W = $b_1\ y_1 + b_2\ y_2$ and so on $b_m\ y_m$, ok $b_m\ y_m$ subject to the constraints a_{11}, yeah $a_{11}\ y_1 + a_{21}\ y_2$ and so on $a_{m1}\ y_m$, ok.

We will have because it is maximization problems, so we will have $\leq$ here we will have $\geq c_1$, ok then $a_{12}\ y_1 + a_{22}\ y_2$ and so on $a_{m2}\ y_m \geq c_2$ and so on, we shall have $a_{1n}\ y_1$, $a_{2n}\ y_2$ and so on $a_{mn}\ \ y_m \geq c_m$, ok c_n sorry c_n where y_1, y_2,.. y_m are all unrestricted in sign, so this is the dual, ok.

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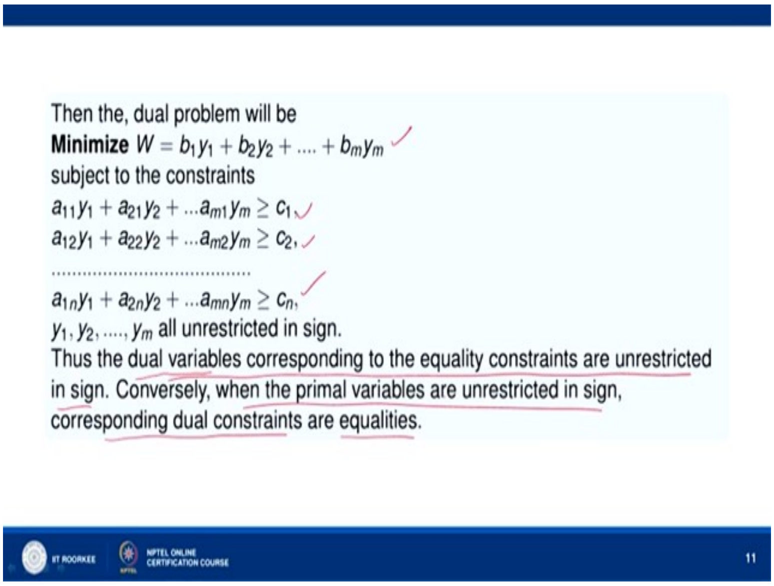

So, the dual will be W = $b_1\ y_1 + b_2\ y_2$ and so on $b_m\ y_m$ subject to this constraints, ok where these are all unrestricted in sign, thus the dual variables corresponding to the equality constraints are unrestricted in sign. Conversely, when the primal variables are unrestricted in sign corresponding to dual constraints are equalities, ok so if so when the when we have equality constraints the dual variables are unrestricted in sign and if the primal variables unrestricted in sign then the corresponding dual constraints are equalities, ok.

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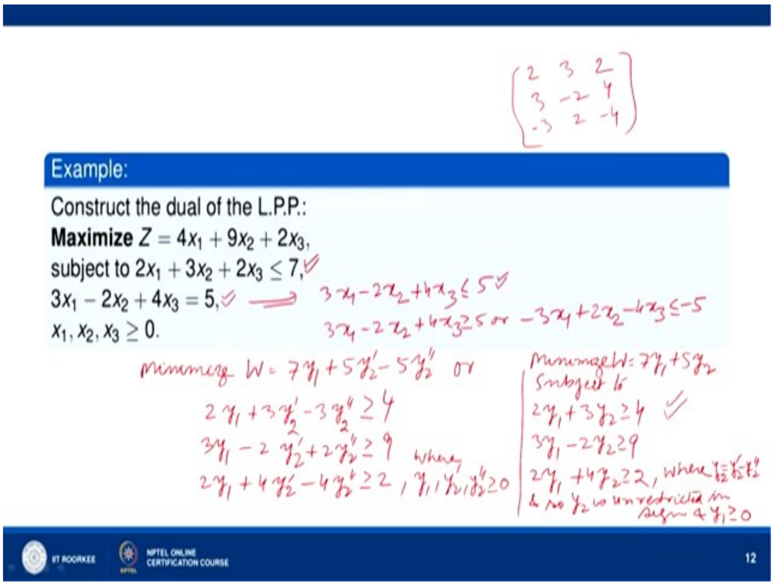

Now, let us construct the dual of the L.P.P. Z = (mik) $4x_1 + 9x_2 + 2x_3$ subject to $2x_1 + 3x_2 + 2x_3 \le 7$ and $3x_1$ - $2x_2$, so there is a 1 equality constraints here, so we have here we can write it as $3x_1 - 2x_2 + 4x_3$ or ≤ 5, ok and $3x_1 - 2x_2 + 4x_3 \ge 5$, we will have to convert to $\le$ because the problem is of maximization, so $- 3x_1 + 2x_2 - 4x_3 \le - 5$, ok.

So, we have the three constraints one is this one, ok $2x_1 + 3x_2 + 2x_3 \le 7$ and other one $3x_1 - 2x_2 + 4x_3 \le 5$ and third one is $- 3x_1 + 2x_2 - 4x_3 \le 5$, so $\le - 5$, so the dual problem will be minimization of W = $7y_1$, ok $7y_1 + 5y_2^{'} - 5y_2^{''}$, ok and then we shall have the body matrix is what here? 2, 3, 2, ok 3, - 2, 4, ok and then we have - 3, 2, - 4, so this is body matrix, ok we have to take the corresponding transpose for the body matrix of the dual problems.

So, we get 2, 3, - 3, so $2y_1 + 3y_1^{'} - 3y_1^{''}$, ok and we will have to put ≥ 4, ok and then we have $3y_1$ - 2, so this should be written as y_2 and $y_2^{''}$ not y sorry, this should be, ok so, ok. So, $3y_1 - 2y_2^{'} + 2y_2^{''} \ge 9$ and then we have $2y_1 + 4y_2^{'} - 4y_2^{''} \ge 2$, ok or we can write it as where y_1, $y_2^{'}$, $y_2^{''}$ they are all non-negative, ok.

Now, we can write the this problem also in the form minimize W = $7y_1 + 5y_2$, we will write $y_2 = y_2^{'} - y_2^{''}$, ok and then subject to $2y_1 + 3y_2 \ge 4$, $3y_1 - 2y_2 \ge 9$ and $2y_1 + 4y_2 \ge 2$ where $y_2 = y_2^{'} - y_2^{''}$, ok and so y_2 is unrestricted in sign, ok.

So, the dual problem is minimize W = $7y_1 + 5y_2$ subject to the constraints $2y_1 + 3y_2 \geq 4$, $3y_1 - 2y_2 \geq 9$, $2y_1 + 4y_2 \geq 2$, y_2 is unrestricted in sign, $y_1 \geq 0$. So, this is how we write the dual of this L.P.P. where 1 constraint is of equality type. So that is all in this lecture, thank you very much for your attention.