Higher Engineering Mathematics

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Lecture 05

Predicates and Quantifiers-II

Hello friends. Welcome to my second lecture on Predicates and Quantifiers. We are discussing now negation of a quantifier. We have seen that there two quantifiers existential quantifier and universal quantifier, so let us first discuss the negation of universal quantifier.

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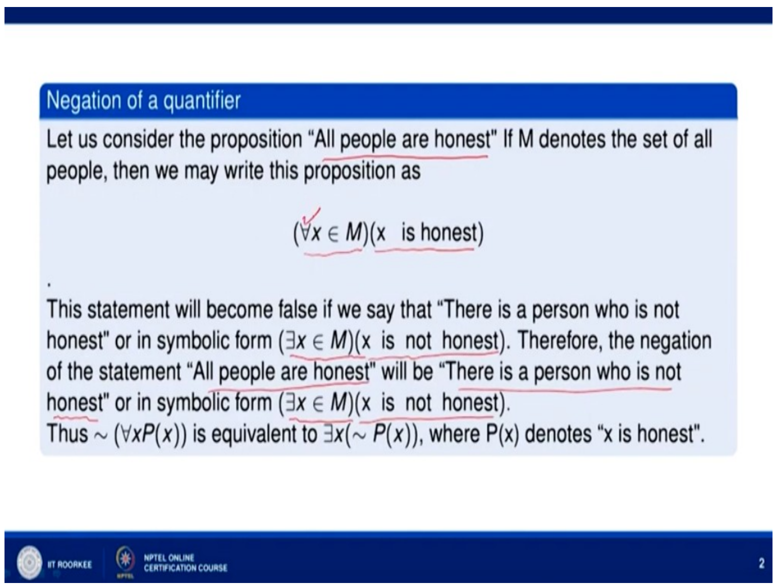

Let us consider the proposition 'All people are honest', if M denotes the set of all people, then we can write this proposition as ($\forall$ x$\in$M) (x is honest). This statement will become false if we say that 'There is a person who is not honest', in symbolic form we can write this statement as ($\exists$ x$\in$M) such that (x is not honest), therefore, the negation of the statement 'All people are honest' will be 'There is a person who is not honest' or in symbolic form the negation of the statement ($\forall$ x$\in$M) (x is honest) will be ($\exists$ x$\in$M) (x is not honest). Thus ~($\forall$ x P(x)) is equivalent to $\exists$ x(~P(x)), where P(x) denotes x is honest.

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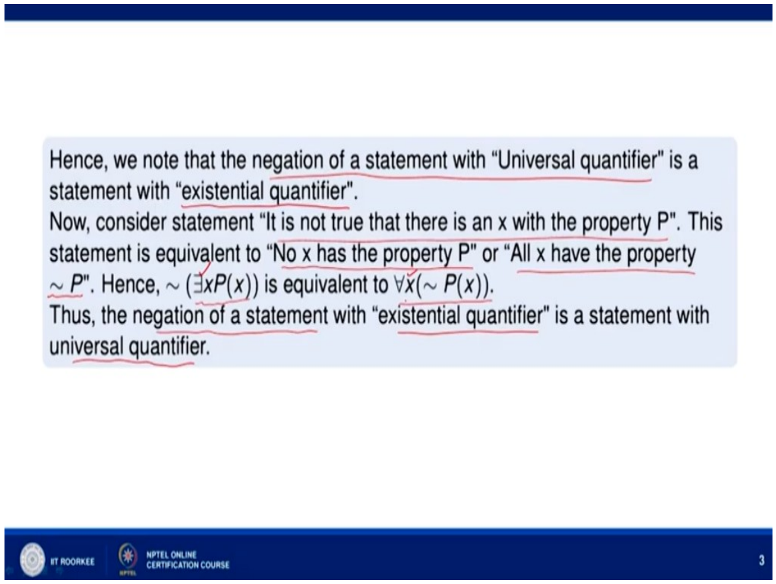

So we can say that the negation of a statement with a universal quantifier, you can see that here it is universal quantifier, this is universal quantifier so negation of the a statement the universal quantifier is a statement with existential quantifier, this is existential quantifier, this is existential quantifier, so negation of a statement with the universal quantifier is a statement with existential quantifier. Now consider this statement it is not true that there is an x with the property P, this statement is equivalent to saying no x has the property P or all x have the property ~P, hence $\sim(\exists xP(x)) \equiv \forall x(\sim P(x))$, so the negation of a statement with existential quantifier, this existential quantifier, this one this existential quantifier, so negation of a statement with existential quantifier, is a statement with universal quantifier, you can see this is universal quantifier. So, when we consider a negation of a statement with existential quantifier what we get is a statement with universal quantifier.

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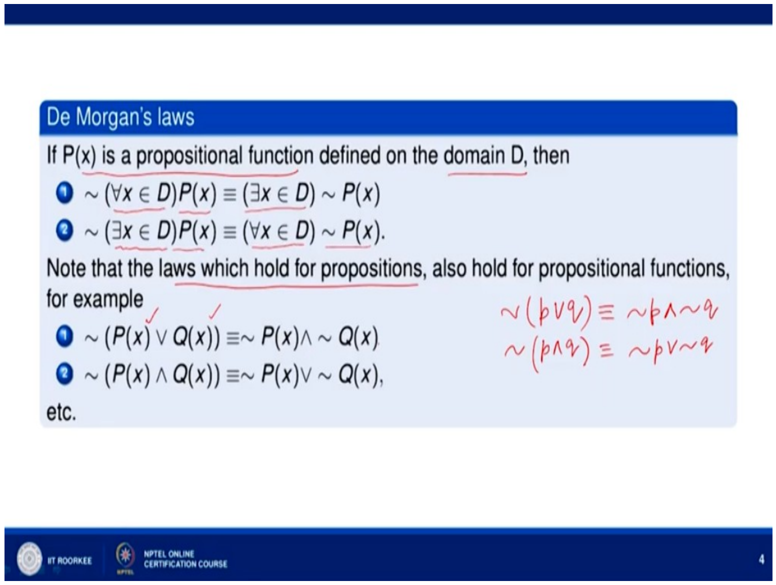

Now, we discuss De Morgan's laws. If P(x) is a propositional functional defined on the domain D then $\sim(\forall x \in D)P(x) \equiv (\exists x \in D)\sim P(x)$ and for $\sim(\exists x \in D)P(x) \equiv (\forall x \in D)\sim P(x)$, so when you consider negation of universal quantifier you get existential quantifier, when you get negation of when you have negation existential quantifier you get universal quantifier.

Now, the laws which hold for propositions, we have seen that $\sim(p \vee q)$ if p and q are two propositions then $\sim(p \vee q) \equiv \sim p \wedge \sim q$ and $\sim(p \wedge q) \equiv \sim p \vee \sim q$, so these laws hold for the propositions p and q, they also continue to hold here for propositional functions P(x) and Q(x), so P(x) and Q(x) are two propositional functions, then $\sim(P(x) \vee Q(x)) \equiv \sim P(x) \wedge \sim Q(x)$ and $\sim(P(x) \wedge Q(x)) \equiv \sim P(x) \vee \sim Q(x)$.

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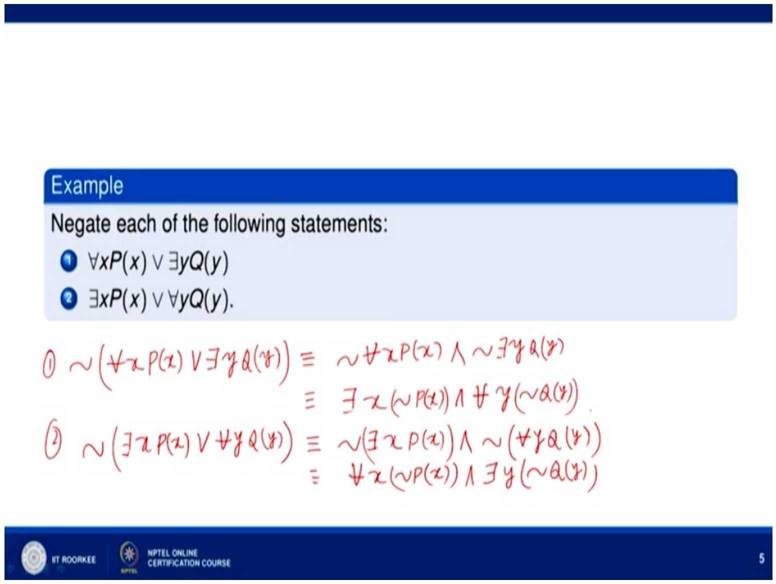

Let us discuss an example, so let us determine the negation of each of the statements here, let us take the first statement, so $\sim(\forall xP(x) \vee \exists yQ(y))$, so negation of this will be, negation of $\forall xP(x)$and $\vee$ becomes $\wedge$, and we get negation of $\exists yQ(y)$, so negation of this gives us $\exists x(\sim P(x))$, and then this gives us, $\exists$ becomes $\forall y(\sim Q(y))$, so$\sim(\forall xP(x) \vee \exists yQ(y)) \equiv \exists x(\sim P(x)) \wedge \forall y(\sim Q(y))$.

Let us take the second one, now so $\sim(\exists xP(x) \vee \forall yQ(y))$, so this is equivalent to $\sim(\exists xP(x))$, $\vee$ becomes $\wedge$, for $\sim(\forall yQ(y))$ and this gives us negation of existential becomes universal, so $\forall x(\sim P(x))$ and here we get negation of for every becomes existential, so $\exists y(\sim Q(y))$, so is what we get when we consider the statement of negation of true.

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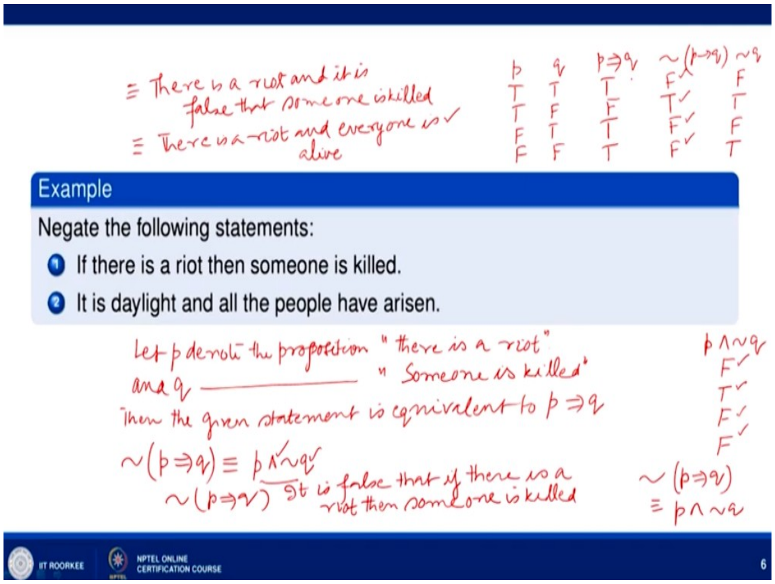

Now, negate the following statements, if there is a riot, if there is a riot then someone is killed. Let us say, let p denote the proposition 'there is a riot', and q denotes the proposition 'someone is killed', 'someone is killed' then the given statement, then the given statement implies p gives is equivalent to $p\Rightarrow q$ if p then q, so negation of, we are taking about $\sim(p\Rightarrow q)$, and $\sim(p\Rightarrow q)\equiv p\wedge\sim q$. We can verify this by the truth table supposed this is p and here is q, the truth values are TT TF FT FF then $p\Rightarrow q$ has truth values T TF when P is T Q is F then it is F.

When p is F, q is T it is T and when p is F, q is F it is T again, so $\sim(p\Rightarrow q)$, negation of, $\sim(p\Rightarrow q)$ is for T it becomes F, F it becomes T, here and this FT becomes F, and this T becomes F. Now $p\wedge\sim q$ let us see $\sim q$ has truth values F, T F T so p and q, we can find the truth values of $p\wedge\sim q$, p has truth value T, q has, $\sim q$ has truth value F so $p\wedge\sim q$ has truth value F, and then p has truth value T, $\sim q$ has truth value T, so $p\wedge\sim q$ has will have truth value T and then p has truth value F, $\sim q$ has truth value F, so $p\wedge\sim q$ has truth value F and then p has truth value F and the $\sim q$ has truth value T, so $p\wedge\sim q$ will have truth value F.

Now, let us see whether the is of $p\wedge\sim q$ or same as the truth value is of $\sim(p\Rightarrow q)$, so here you can see this is F T F F and here also we have F T F F, so $\sim(p\Rightarrow q)\equiv p\wedge\sim q$, now so from this now, let us write it is $\sim(p\Rightarrow q)$ means $\sim(p\Rightarrow q)$ means it is false that if there is a riot, if there is a riot then someone is killed $\sim(p\Rightarrow q)$ say it is false that if there is a riot then someone is killed which is equivalent to there is a riot and it is false that someone is killed.

So this is equivalent to there is riot and it is false that, someone is killed, because we have $\sim(p \Rightarrow q) \equiv p \wedge \sim q$, so from $p \wedge \sim q$ we can say that p denotes the proposition there is a riot, so there is riot this is and, this notation is and, and ~q is it is false that someone is killed because q is denoting the proposition someone killed. Now there is a equivalent to saying that there is a riot and everyone is alive, so negation of the statement there is riot then someone is killed is this statement there is riot and everyone is alive.

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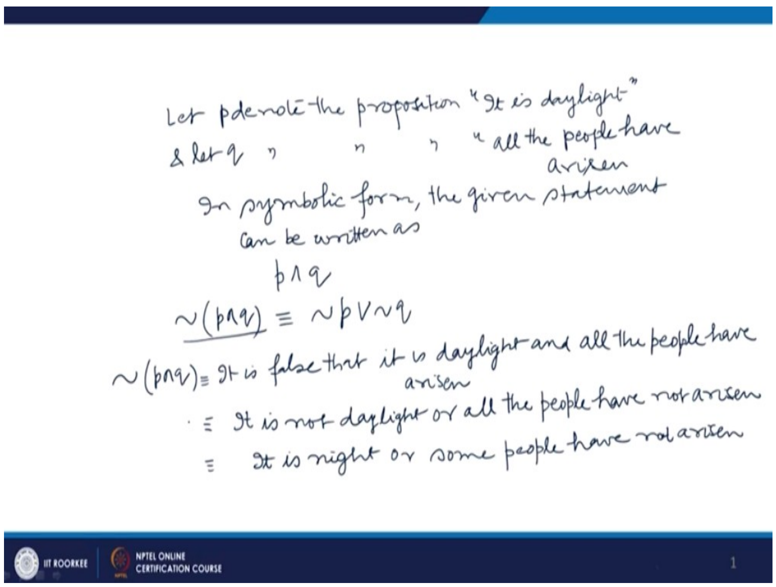

Now, look at the second statement it is daylight and all the people have arisen, so let us do it like this, it is daylight denote it is daylight let p denote the proposition it is daylight all the people has arisen, let q denote this statement 'All the people have arisen' then the given statement symbolically in the symbolic form we can write as in symbolic form the given statement can be written as $p \wedge q$ it is daylight and all the people have arisen, so in symbolic form we can write the given statement as $p \wedge q$ now we want to find negation of the given statement that is $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

So De Morgan's law, so this is $\sim p \vee \sim q$, now so let us now write in the let us write the corresponding statements $\sim(p \wedge q)$ means it is false that it is daylight and all the people have arisen so $\sim(p \wedge q)$ is the statement it is false that it is daylight it is all the people has arisen now this equivalent to $\sim p \vee \sim q$ so it is false that or negation is of p is it is not daylight negation of it is daylight means it is not daylight or all the people ~q means all the people have not arisen.

So this is equivalent to sign this is night it is not daylight means it is night or some people have not arisen the we get negation of all is sum negation of universal quantifier give existential quantifier existential quantifier denotes there exist are some, so some people have not arisen, so ~(p^q) is the statement it is night or some people have arisen that is the negation of the given statement.

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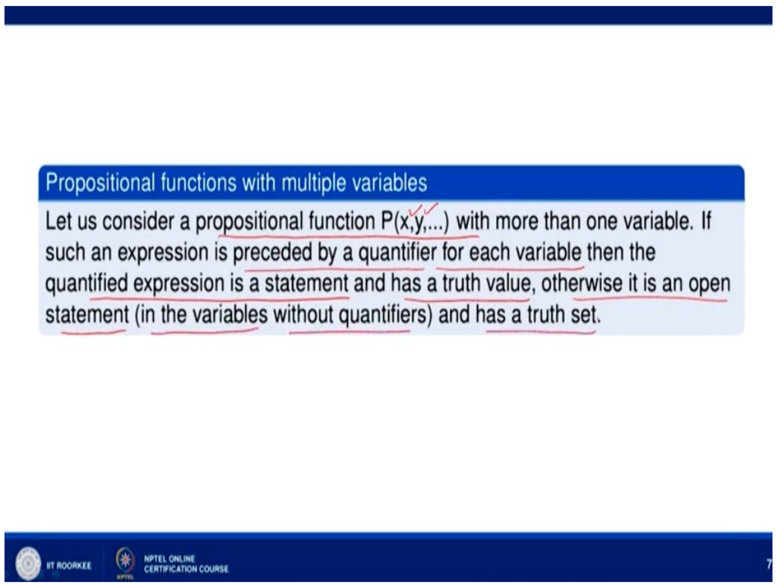

Now, let us go to propositional function with multi variable let us consider propositional functions P(x,y,…) with more than one variable x,y… on if such an expression is preceded by a quantifier for each variable for each variable x,y,… if it is preceded by a quantifier then the quantified expression is called a statement it is a statement and it has truth value T or F otherwise it is an open statement if the propositional function P(x,y,…) is not preceded by as many quantifier as the variable are there then it will be called and open sentence open variable without quantifiers it will have a truth set T(P).

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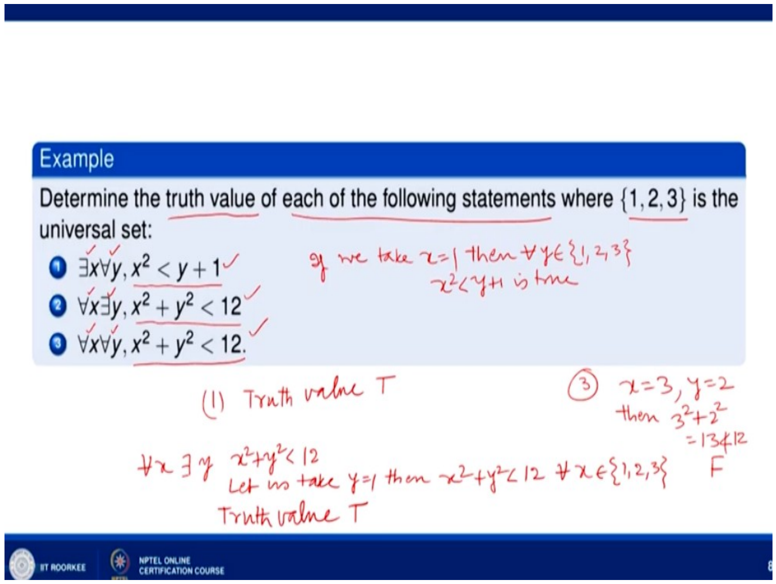

So let us discuss some example on this determine the truth value, we have to determine the truth of each of the following statement now each of this statement you can see that, we have two variables x and y and the propositional function $x^2 < y+1$ is preceded by two quantifier $\exists$ and $\forall$ here also, this propositional function is preceded is preceded by two quantifier universal and existential In this case this propositional function is preceded by two universal quantifier, so there are two variables in the propositional, propositional function is a function of two variables and it is preceded by two quantifier.

So every case and every example 1, 2, 3 is a statement, now let us find the truth value of the first statement we are given that {1,2,3} is the universal set, so can see we have $\exists$ x $\forall$ y, $x^2 < y+1$, if we take x=1, if we take x=1 then for every y$\in$ {1 2 3} the inequality $x^2 < y+1$ is true, because y can take value 1 if y takes value 1 then we are taking X equal to 1 so 1 will be less than 2 and if we take y equal to 2 than again 1 is less than 3 and when y is 3 than 1 is less than four.

So if we take x=1 then $\forall$ y$\in${1,2,3}, $x^2 < y+1$ is true, hence the first statement has truth value F T, so first statement truth value is T, now let us look at the second statement, $\forall$ x $\exists$ y, that $x^2 + y^2 < 12$, now x$\in$ {1,2,3}, so we need to find one y so that $x^2 + y^2 < 12$. Let us take y=1, let us take y=1 then $x^2 + y^2 < 12$ is 2, $\forall$ x$\in${1,2,3}, because if x takes the value 1 then 1+ y^2, y is one.

So 1+1=2, 2<12 when x takes value 2 then 4+1=5, 5<12 and when x takes the value 3, 9+1=10<12, so the truth value is here is T. Now let us look at the third statement ∀ x and ∀ y, $x^2 + y^2 < 12$, so third statement plus look at the third, so let us take x=3, y=2 then $(3)^2 + (2)^2 = 13$ which is not less than 12, it is greater than twelve.

So $x^2 + y^2 < 12$ is not true when x=3, y=2, so ∀ x ∀ y, $x^2 + y^2 < 12$ is not true therefore, the truth value is F, so they are three statements, the first two statements have truth value T and the third statement has truth value F.

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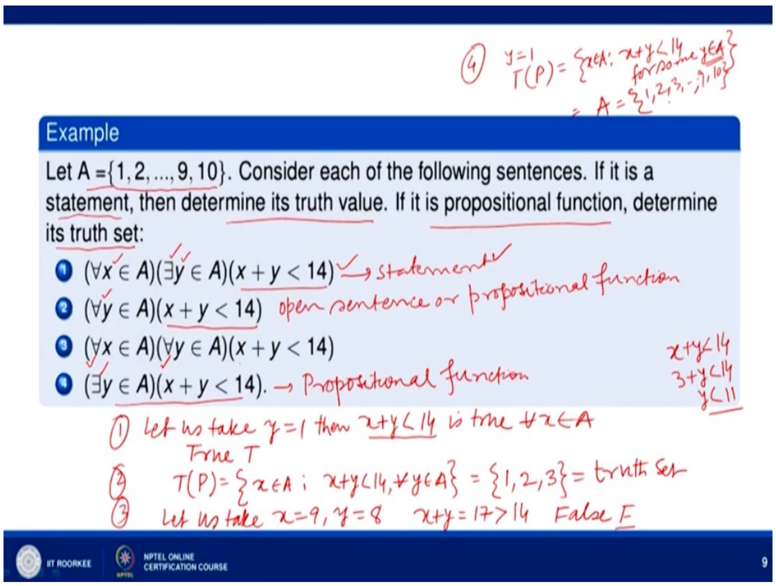

Now, let us take another example, let us take A={1, 2, 3,…,10}, consider each of the following sentences, so let us consider these four sentences, if it is a statement than determine the truth value if it is a propositional function determine the truth set, truth set means T(P). Now let us see here we have two variables x and y, and the propositional function is x+y<14, it is preceded by two quantifiers ∃ and ∀ the universal quantifier and the existential quantifier.

So it is a statement because there are two variables and this is preceded by P(x), this is P(x), P(x) is preceded by two quantifiers and here we have again two variables but there is one quantifier which is the universal quantifier, so number of quantifiers are not two, and therefore it is an open sentence, or you can say it is a propositional function. In the example 3 we have again two variables x and y and we have two quantifiers both of them are universal quantifiers or in the four case also, we have two variables x and y and there is only one quantifier which is existential quantifier so this is also propositional function.

Now, here we can see, first let us look at this (∀ x∈A)(∃ y∈A), such that x+y<14, let us find it truth value because it is a statement, so it will have a truth value. x∈{1,2,3,…,10}, let us take y=1, then x+y<14 is true for every x∈A because A={1,2,3,…,10}, so even if x takes the value 10, 10 +1=11<14.

So ∀ x∈A if y=1 then x+y<14, and so the first statement is true, its truth value is T. Let us go to the second, it is an open sentence are propositional function, so x+y<14, ∀ y∈A, x+y<14. Now we have to find the set of value of x, so T(P)={ x∈A: x+y<14, ∀ y∈A}, so y∈A and we want those value of x for which x+y<14 ∀ y∈A, so if you take x=1 then x+y<14, ∀ y∈ A.

So one value of x could be taken as 1 when x takes the value 2 then, 2+y<14 is true for all belonging to A, so another value of x could be 2, when x takes the value 3, then again x+y<14 we want, x+y<14 we want we are taking x=3, so 3+y<14 are y<11. Now y can take any values less than 11, so the values that y∈{1,2,..,10}, so this inequality is valid. So for x=3 also x+y<14, ∀ y∈A. Now if you go one more step that is take x=4 then x+y<14 will not be possible because ∀ y∈A because when y will take value 10 then 4+10=14, 14<14 is false.

So T(P) be truth set here is the set containing the three element {1 2 3}. Now let us go to the third statement, ∀ x∈A, ∀ y∈A, x+y<14, so you can see here if I take x=9 and y=8 then x+y<14 is not true, so in the third statement let us take x=9, y=8 then x,y∈A, but x+y=17>14.

So x+y<14 is not true and therefore we truth value of the third statement is false. The third statement of truth value F. Now let us go to the 4 case, so in the case of the fourth statement ∃ y ∈A, x+y<14, so we have to consider those values of x for which x+y<14, for some y∈A. So you can see ∃ y∈ A y could be taken as one, y could be taken as 1 and then x can takes all values from 1 to 10, x+y<14 will be true.

So T(P) ={x: x+y<14 for some y∈A}, this is what we want x∈A, so we can take 2 that element y∈A as on if we take y∈A as 1 than ∀ x∈A, we will have x+y<14 so T(P) =A={ 1, 2, 3,…,10}, so TP in this case will be the universal set A.

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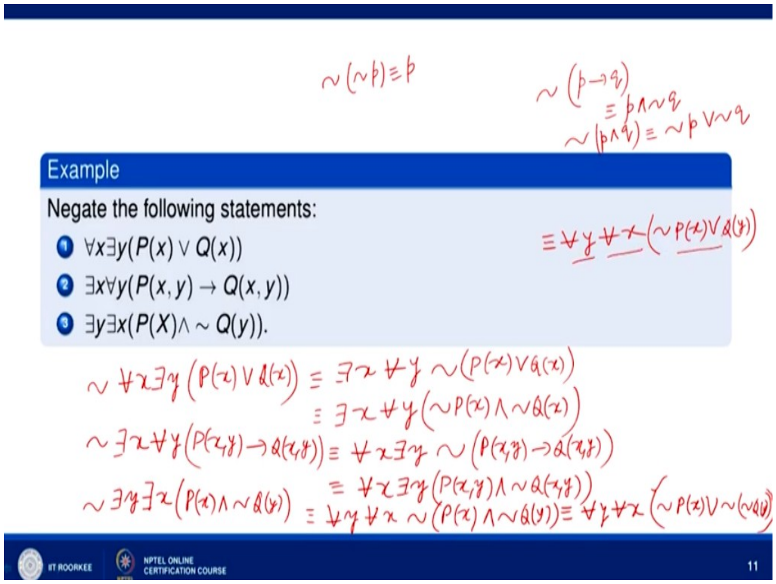

Now, let us go to the next example. Negate each of the following statements, so $\sim\exists x\,\forall y,\ P(x,y) \equiv \forall x\ \exists y, \sim P(x,y)$ existential quantifier becomes universal quantifier for $\sim\forall x\,\forall y\ P(x,y)$ is same as for every universal quantifier becomes existential quantifier, so $\exists x\,\exists y \sim P(x,y)$ then $\sim\exists y\,\exists x\,\forall z\ P(x,y,z)$.

So here negation becomes negation of there exist means $\forall y\,\forall x\,\exists z, \sim P(x,y,z)$, so this is how we can find the negation of each of this statements you can see here this are statement because here we have two variable x and y there are two quantifiers, here also there two variable and there two quantifier here there are three variables and there are three quantifiers, so they are all statement. Now negate the following statement.

Now, let us recall that negation of the propositional functions follows the same rule and negation of the formula for propositions, so $\sim\forall x\,\exists y(P(x)\vee Q(x)) \equiv \exists x\,\forall y \sim(P(x)\vee Q(x)) \equiv \exists x\,\forall y \sim P(x)\wedge\sim Q(x)$, Now let $\sim\ \exists x\,\forall y\ (P(x,y)\rightarrow Q(x,y))$

So there exist becomes universal quantifier, universal quantifier becomes existential quantifier and then $\sim(P(x,y)\rightarrow Q(x,y))$ and we have seen that in the case of proposition p implies $\sim(p\rightarrow q) \equiv p\wedge\sim q$, so we can write this as $\forall x\,\exists y \sim(P(x,y)\rightarrow Q(x,y))$, is $P(x,y)\wedge\sim Q(x,y)$, now let us go to the third, so $\sim\exists y\,\exists x(P(x)\wedge\sim Q(x))$, so $\sim\exists$ becomes $\forall$, then $\sim(P(x)\wedge Q(y))$.

Now, this is equivalent to $\forall y \forall x \sim(P(x) \wedge \sim Q(y)) \equiv \forall y \forall x(\sim P(x) \vee \sim(\sim Q(y)) \equiv \forall y \forall x(\sim P(x) \vee Q(y))$. So, with that I would like to end my lecture, thank you very much for your attention.