Higher Engineering Mathematics
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Lecture 43
General Linear Programming Problem

Hello friends welcome to my lecture on General Linear Programming Problem. In our previous 2 lectures on linear programming we solved a linear programming problem involving exactly 2 variables by graphical method but if the number of variables in the linear programming problem are 3 or more than 3 then the graphical method cannot be used because even for the case where a linear programming problem involves exactly 3 variables, it will be difficult to solve it by the graphical method.

And so what we will do is, we will consider a generally, General Linear Programming Problem involving say n variables where n is greater than or equal to 2, okay. And then they will solve it by simplex method which is very well-known method to solve a linear programming problem involving n variables.

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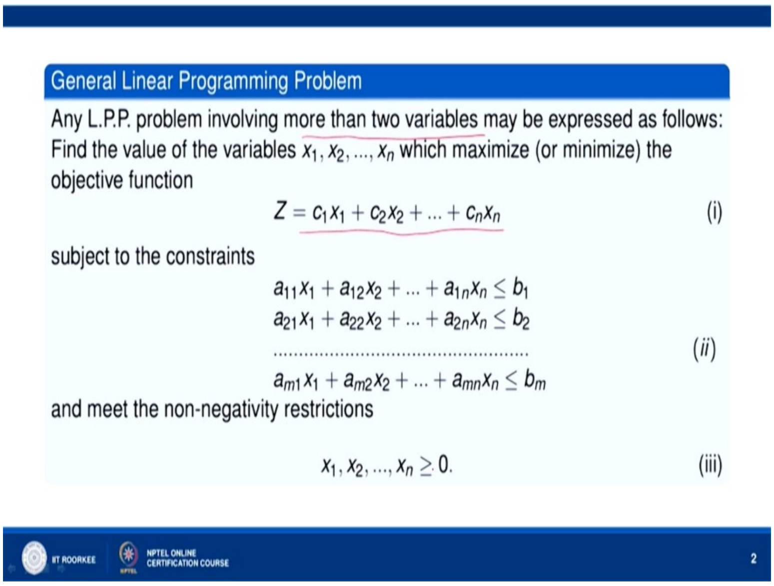

So let us discuss what is a General Linear Programming Problem? Any LPP, any linear programming problem involving more than 2 variables may be expressed as

$Z=c_1x_1+c_2x_2+...+c_nx_n$, which is our objective function and we want to maximize or minimize this objective function, subject to the constraints $a_{11}x_1+a_{12}x_2+...+a_{1n}x_n \leq b_1$, $a_{21}x_1+a_{22}x_2+...+a_{2n}x_n \leq b_2$

and so on. $a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n\leq b_m$ and the non-negativity restrictions on variables $x_1, x_2, \ldots, x_n \geq 0$.

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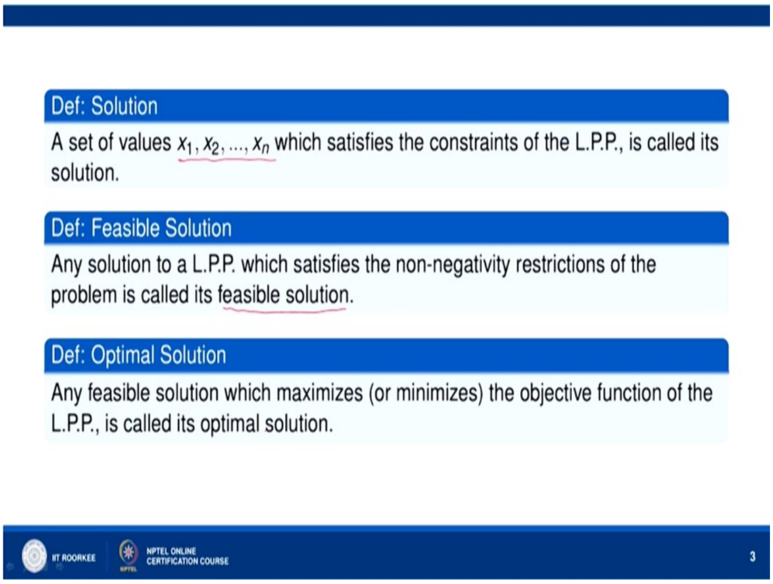

Now, let us define a solution. A set of values $x_1, x_2, \ldots, x_n$ which satisfies the constraints of the LPP. Constraints of the LPP means these inequalities, set of values of $x_1, x_2, \ldots, x_n \geq 0$ which satisfies these m constraints, okay is called a solution of LPP. Any solution to LPP which satisfies the non-negativity restrictions. Now, suppose our solution of LPP is such that it also satisfies these non-negativity restrictions $x_1, x_2, \ldots, x_n \geq 0$ then we call the solution to be a feasible solution. Feasible solution of LPP.

Now, any feasible solution which maximizes or minimizes the objective function of the LPP is called an optimal solution, okay. So if any feasible solution we have which maximizes or minimizes the objective function $Z=c_1x_1+c_2x_2+\ldots cx_n$ then it will be called as its optimal solution.

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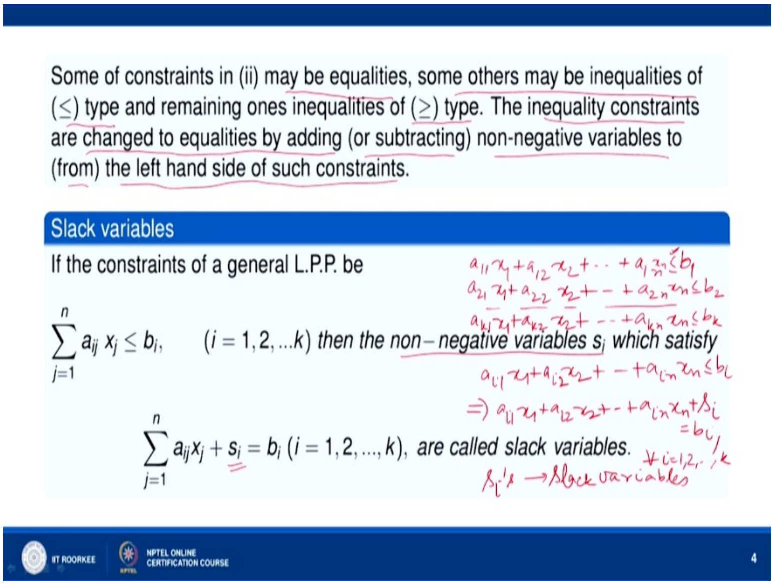

Some of the constraints in equations 2, some the constraints here equation 2 maybe equalities some others may be inequalities of less than or equal to type and remaining ones inequalities of greater than or equal to type. The inequality constraints are changed into equalities by adding or subtracting nonnegative variables to, from the left side of such constraints, okay.

If the inequality is less than or equal to type then we will add non-negative variables, okay to the left side. And if the inequalities of greater than or equal to type then we will subtract nonnegative variables from the left side of such constraints. So, let us now define slack variables. If the constraints of a general LPP be say $\sum_{j=1}^{n} a_{ij} x_j \le b_i, i=1,2,\dots,k.$

So you can see there are k equations because i when i is equal to one you get the first inequality as $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_1 \le b_1$, okay. The second one is $a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_1 \le b_2$, and so on. $a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_k \le b_k$. So suppose there are k inequalities which are of type less than or equal to, okay given by these k equations, okay.

$\sum_{j=1}^{n} a_{ij} x_j \le b_i, i=1,2,\dots,k.$ than the nonnegative variables s_i because these are less than or equal to type and when the inequalities are of less than or equal to type we add nonnegative variables to the left side of such constraints to make them equalities. So add s_i, okay to the ith equation here which is $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{¿}x_n \le b_i$, we add nonnegative variable s_i, okay.

So s_i is added to the i-th equation to convert it into n equality i-th equation is

$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{¿}x_n \le b_i$ To convert it into n equality we add s_i to this, nonnegative variable s_i to this, okay. So this gives us $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{¿}x_n + s_i = b_i, i = 1,2,\ldots,k$.

So we get the equality $\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i, i = 1,2,\ldots,k$. And these nonnegative variables s_i are called Slack variables. So slack variables are those which are added to inequality of type less than or equal to, to convert it into a equality.

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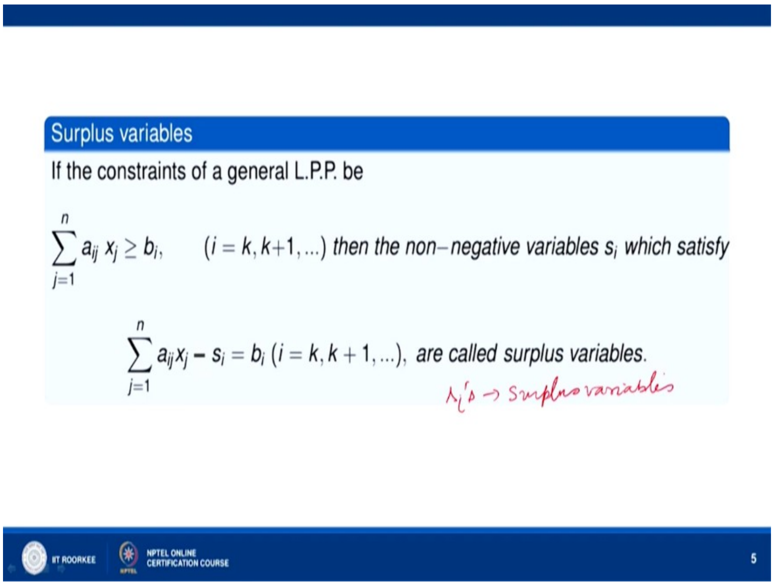

Now, let us go to next possibility, if the constraints of a general LPP be of the type

$\sum_{j=1}^{n} a_{ij}x_j \ge b_i, i = 1,2,\ldots,k$. okay. Where i takes values k, k plus 1, k plus and so on, okay. Then what we will do? We will subtract nonnegative variables from the left side, okay of this constraint to convert into a equality. So $\sum_{j=1}^{n} a_{ij}x_j - s_i = b_i$.

$\sum_{j=1}^{n} a_{ij}x_j - s_i = b_i$ where with this s_i nonnegative variables s_i are called surplus variables. So this we do for every i. These s_i's are called surplus variables.

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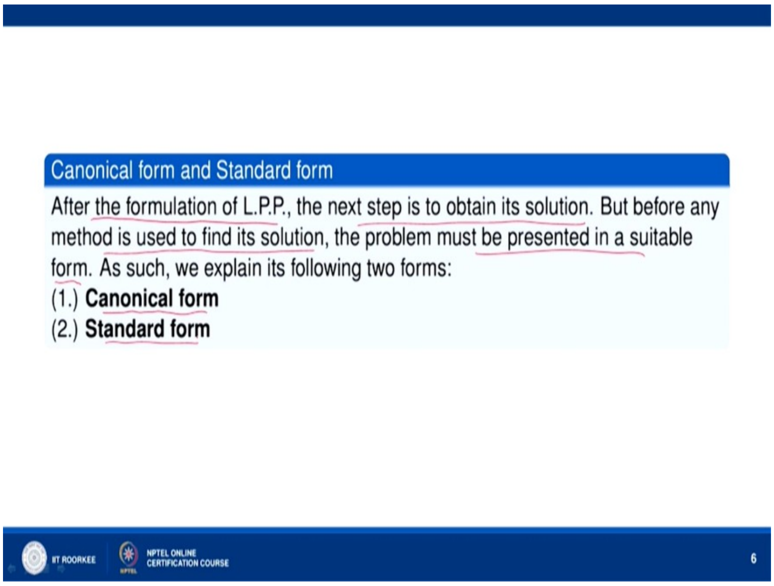

Now let us discuss the 2 forms of LPP they are canonical form and standard form. After the formulation of LPP, the next step is to obtain its solution. But before any method is used to find its solution, the problem must be presented in a suitable form. Now what are those suitable forms? One form is canonical form, the other for me standard form. Let us define what do we mean by canonical form and a standard form?

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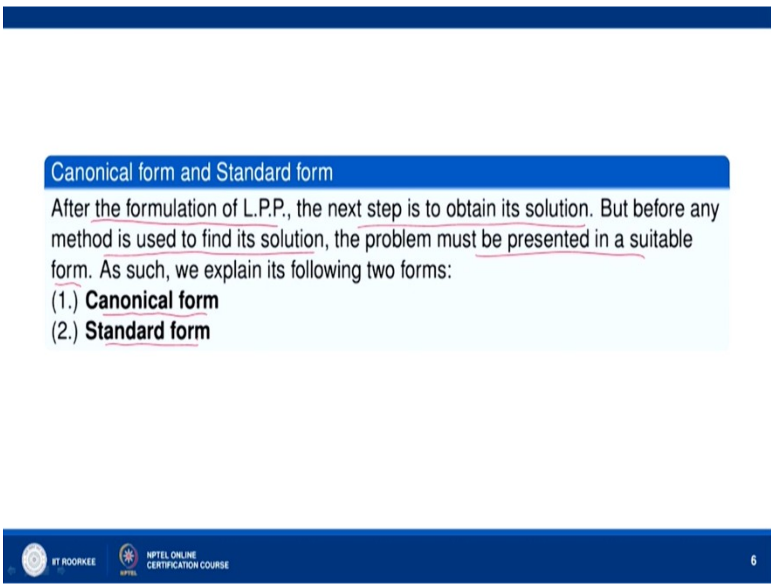

Now, let us first take the canonical form. The general linear programming problem can always be expressed in the following form maximize $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$.

Subject to the constraints $a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n \le b_i, i=1,2,\ldots,m$. So there are m constraints here, okay given by the m inequalities which are of less than or equal to type and

$x_1, x_2, \ldots, x_n$ are nonnegative variables. So non-negativity restriction is there on $x_1, x_2, \ldots, x_n$. Now, this General Linear Programming Problem can be expressed, now suppose a constant is of greater than or equal to type, okay. Greater than or equal to type instead of less than or equal to type then we multiply this constant by minus 1 and bring here instead of greater than or equal to less than or equal to. Say for example, say certain equation

$a_{k1}x_1+a_{k2}x_2+\ldots+a_{kn}x_n \ge b_k$ is there, okay in the LPP.

Then we will write it as $a_{k1}x_1-a_{k2}x_2-\ldots-a_{kn}x_n \le b_k$, so all the constraints, okay will have to be brought in the form less than or equal to. If there is anyone which is greater than or equal to that we multiply by minus 1, okay the whole inequality and bring it to form less than or equal to. And now here the objective function if they are given that you minimize Z.

$Z=c_1x_1+c_2x_2+\ldots+c_nx_n$ than equivalently we can write it as maximize, we convert it into the form maximize $Z'=-Z$, okay. So that is equal to $-c_1x_1-c_2x_2-\ldots-c_nx_n$, okay. So if there is a minimization problem, we have to minimize the objective function then for that we consider the corresponding maximization of $Z'=-Z$.

$Z'=-Z$, okay which is $-c_1x_1-c_2x_2-\ldots-c_nx_n$ and bring the LPP in the form maximization of $Z=c_1x_1+c_2x_2+\ldots+c_nx_n$ and all the constraints are brought in the form less than or equal to. $x_1, x_2, \ldots, x_n \ge 0$ is already given to us. Now that is what we say by making some elementary transformations. The LPP can always be brought into this form. This form of LPP is called as canonical form and it has the following characteristics.

What are the characteristics of this canonical form? First thing is that objective function is of maximization type, okay. Objective function is of maximization type, if it is not of maximization type in the LPP then we have to bring it to the form of maximization, right considering maximization of $Z'=-Z$. Now all constraints are of less than or equal to type.

All constraints must be of less than or equal to type. All variables xi are nonnegative. So all the variables xi are nonnegative. The canonical form is a format for a LPP which will find its use in the Duality theory. When we will consider duality theory that we will see that this form of LPP which is called canonical form of LPP, it is a format for the duality theory. In the case of linear programming problem it is a format in the case of Duality theory.

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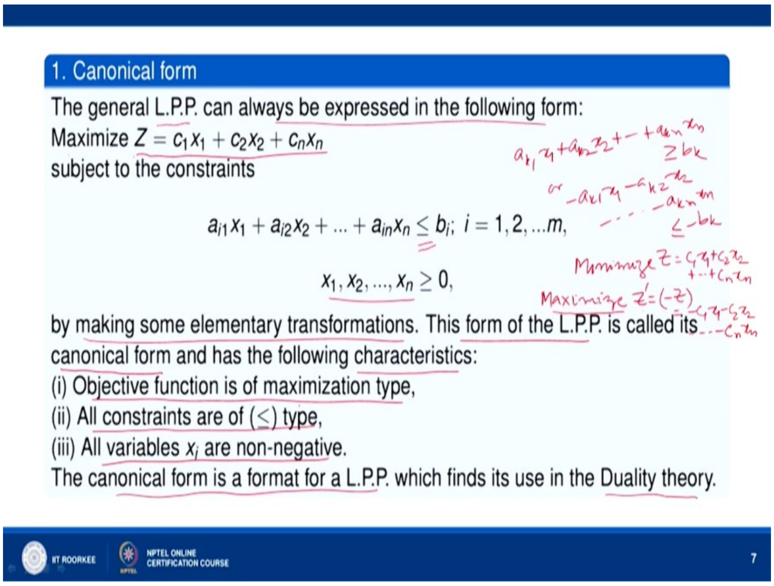

Now, the second one is Standard form. The General Linear Programming Problem can also be put in the form maximization of $Z=c_1x_1+c_2x_2+\ldots+c_nx_n$. Subject to the constraints $a_{i1}x_1+a_{i2}x_2+\ldots+a_{¿}x_n=b_i, i=1,2,\ldots,m$ and $x_1,x_2,\ldots,x_n\geq 0$. So here the constraints are all of equality type, okay. We have $a_{i1}x_1+a_{i2}x_2+\ldots+a_{¿}x_n=b_i$, so we are given m equations here.

There must be m equations and the objective function must be a maximization type. So this form of the LPP is called standard form and it has the following characteristics. Now standard form of LPP we will discuss when we use simplex method to solve a General Linear Programming Problem. In the simplex method first thing that we do is we bring the LPP into the standard form, so by bringing that into the standard form means if the function, if the objective function Z is to be minimized we will consider the corresponding problem where maximization of Z dash. $Z'=-Z$ we will consider, so maximization of minus Z we will find.

So first thing is that objective function must be maximized and moreover that the m equations are there, okay. Equality type, okay. So if the constraints in the general linear programming problem are less than or equal to or greater than or equal to then by adding, in the case of less than or equal to the Slack variables and in the case of your greater than or equal to subtracting surplus variables from the left side you will be converting them to equality into equations with equality sign. So there are m equations with equality here, okay. So all constraints are expressed as equations, okay.

And now this is very important right-hand side of each constraint must be nonnegative that is bi has to be nonnegative, okay. So bi if it is nonnegative we multiply the equation by minus 1 and bring it to the form where the right-hand side is nonnegative. And all variables must be nonnegative. So when we have here constraint of less than or equal to type, we are adding Slack variables s_1, s_2 and so on which are greater than or equal to 0 and when we have greater than or equal to type here in the constraint then we are subtracting surplus variable s_1, s_2 and so on which are again nonnegative, okay.

So first thing is that objective function must be of maximization type. All constraints must be equations. Third thing is right-hand side must be nonnegative of each constraint and forth is all variables must be nonnegative.

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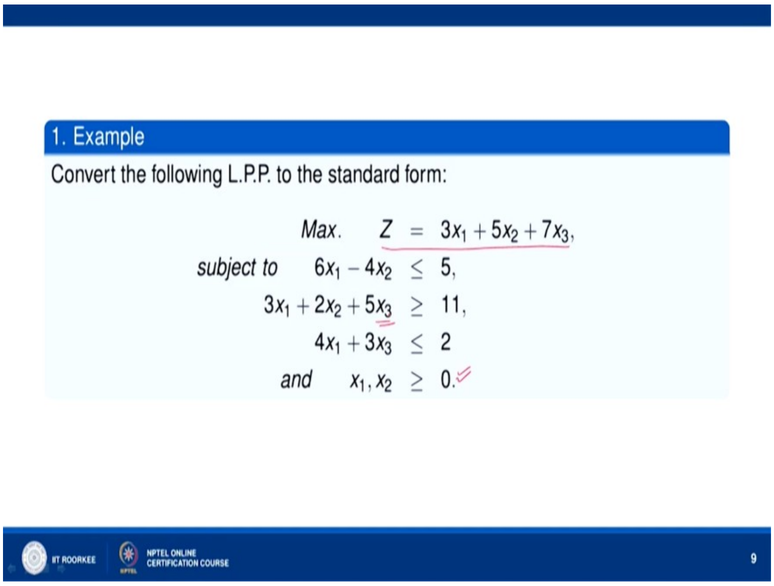

Now for example let us convert the following LPP to the standard form. We have here the objective function $Z=3x_1+5x_2+7x_3$ and we want to maximize this and the constraints are $6x_1-4x_2 \leq 5, 3x_1+2x_2+5x_3 \geq 11, 4x_1+3x_3 \leq 2, x_1, x_2 \geq 0$.

Now we are given these restrictions on x_1, x_2 okay. x_1, x_2 are given to be nonnegative but the third variable is unrestricted, there is no restriction on the third variable x_3. So we will convert it into this LPP into the standard form.

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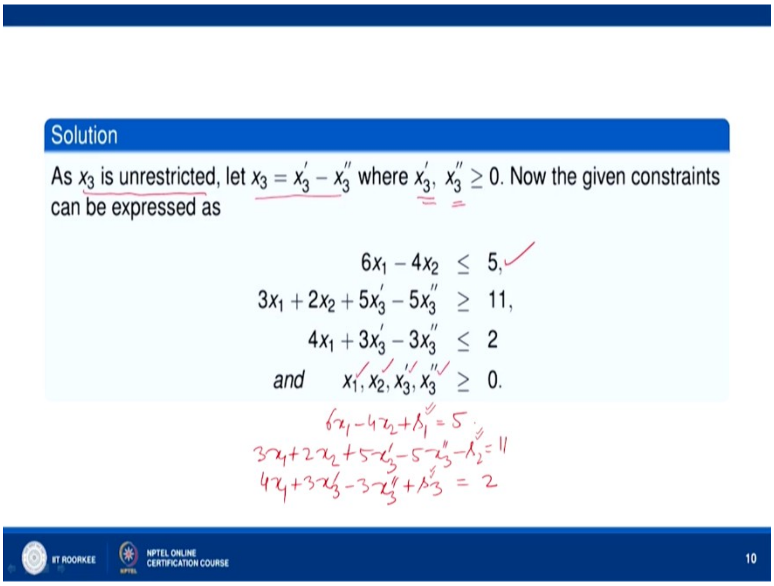

What we will do? Since x_3 is unrestricted, let us put $x_3 = x_3' - x_3''$. We can always write x3 in the form $x_3 = x_3' - x_3''$ where $x_3', x_3'' \geq 0$. Now the given constraint can be expressed as, so with $x_3 = x_3' - x_3''$, what we will have?

The new constraints will be of the type $6x_1$ this is an affected because there is no x_3 here, So

$6x_1 - 4x_2 \leq 5$ that is the first inequality constraint. Second constraint is $3x_1 + 2x_2 + 5x_3 \geq 11$, so $x_3 = x_3' - x_3''$, and we get $3x_1 + 2x_2 + 5(x_3' - x_3'') \geq 11$.

And the third constraint is $4x_1 + 3(x_3' - x_3'') \leq 2$. So here we get $4x_1 + 3x_3' - 3x_3'' \leq 2$ and here all the variables $x_1, x_2, x_3, x_3', x_3'' \geq 0$. So now we can see this problem LPP the original problem, General Linear Programming Problem we have brought into the form where we have objective function is to be maximized.

Second thing all the constraints we have to express them into equations, okay. So these have to be expressed into equations. It is not still into the standard form, we just have observe that x_3 is unrestricted, so x_3 can be written as x_3' or as x_3'', we have written the constraints in the form of variables $x_1, x_2, x_3, x_3', x_3''$ where they are all nonnegative.

Now, let us convert them into equations. So since this is less than or equal to 5 we will have to add a Slack variable there. So $6x_1 + 4x_2 + s_1 = 5$, okay where s_1 is a Slack variable. Now here we have greater than or equal to, so we will subtract a surplus variable, so $3x_1 + 2x_2 + 5x_3 = 5$, okay. Now here again it is less than or equal to type, so $4x_1 + 3x_3$ there

$-3x_3^{''}$ and we add a Slack variable $s_3 = 2$ okay. So by the help of 2 Slack variables s_1, s_3 and a surplus variable s_2 we are able to convert the given constraints into equations, so we have this.

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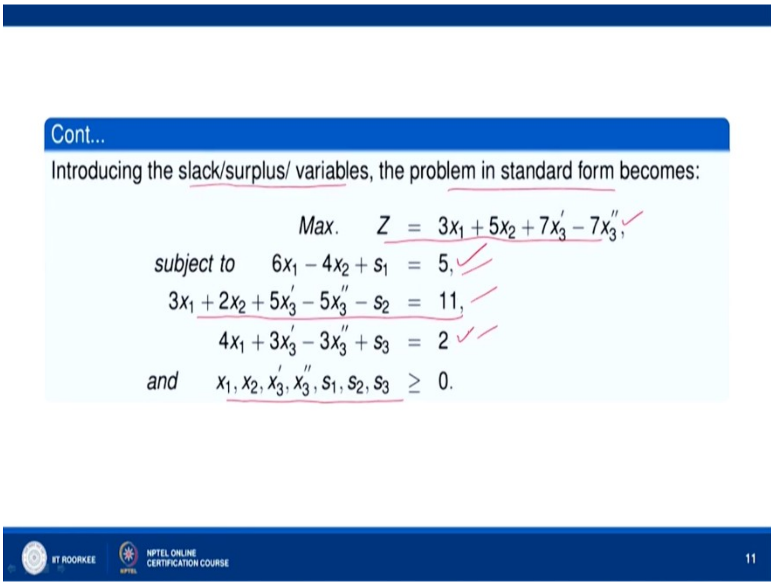

Now we introducing Slack surplus variables, the problem in standard form becomes maximization of Z equal to, no $Z = 3x_1 + 5x_2 + 7x_3$, Means $7x_3 = 7x_3' - 7x_3^{''}$ because $x_3 = x_3' - x_3^{''}$ and subject to the equations $6x_1 - 4x_2 + s_1 = 5$, this is this equation. Then $3x_1 + 2x_2 + 5(x_3' - x_3^{''}) + s_2 = 11$. So this is this equation, okay.

And then the third equation $4x_1 + 3(x_3' - x_3^{''}) + s_3 = 2$, that is the third equation, okay and we have $x_1, x_2, s_1, s_2, s_3, x_3', x_3^{''} \geq 0$, so they can see all the characteristics of the standard form of the LPP are there. Our objective function is of maximization type, this objective function is of maximization type, okay.

All constraints are expressed as equations. All constraints are expressed as equations, okay. Right-hand side of each constraint must be nonnegative. Now this is 5, this is 11 and this is 2, so right-hand side of each constraint is nonnegative. All variables are nonnegative, so all variables $x_1, x_2, s_1, s_2, s_3, x_3', x_3^{''} \geq 0$. So the LPP, okay this LPP it has been brought into the standard form, so this is a standard form of the LPP.

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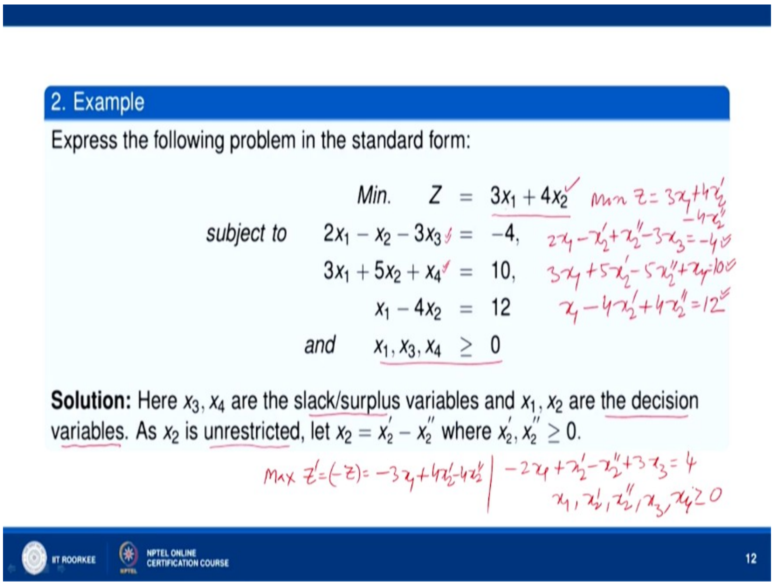

Now let's consider another example where we have to minimize the objective function

$Z=3x_1+4x_2$ and we are given the constraints which are given in the form of equations

$2x_1-x_2-3x_3=-4, 3x_1+5x_2+x_4=10, x_1-4x_2=12$ and $x_1, x_3, x_4 \geq 0$. Now you can see there is no restriction on the variable x_2, okay so x_2 is unrestricted variable, okay.

So here x_3 again moreover that we have in the objective function x_1, x_2, x_3, x_4 are not there in the objective function, so x_3, x_4 are Slack oblique surplus variables, okay. These variables x_1, x_2 which determine the minimum value of Z are the decision variables, okay. Now x_2 is unrestricted, so we can write $x_2 = x_2' - x_2''$ where $x_2', x_2'' \geq 0$ okay.

Now, with $x_2 = x_2' - x_2''$ what we will have? This problem will become minimum of Z ¿ $3x_1 + x_2, x_2' - x_2''$, so $4x_2' - 4x_2''$, okay. And these equations will become

$2x_1 - x_2 + x_2'' - 3x_3 = -3x_1 + 5x_2' - 5x_2'' + x_4 = 10$, okay. And then $x_1 - 4x_2' - 4x_2'' = 12$, okay.

Now right-hand side of all the constraints must be nonnegative, okay but we notice here that first constraint in the right-hand side of the first constraint we have negative constraint minus 4 is there. So we multiply this first constraint by minus 1 and bring it to the form $2x_1 - x_2 + x_2'' - 3x_3 = 4$ okay.

So this first equation is termed defined in such a way that right-hand side becomes plus 4, okay. The second equation and third equation are okay because right-hand side has is already nonnegative. The minimization of Z is given as the LPP, so we convert it to maximization problem, so maximization of $Z^{'}=-Z=3x_1+4x_2^{'}-4x_2^{''}$, okay. This is how we convert this $min\,imum\,Z=maxi\,mum\,Z^{'}$. So now we have maximization of minus

$3x_1+4x_2^{'}-4x_2^{''}$ subject to the constraints are $-2x_1+4x_2^{'}-4x_2^{''}=4$ that is one constraint and the other 2 are $3x_1+5x_2^{'}-5x_2^{''}+x_4=10$ and $x_1-4x_2^{'}-4x_2^{''}=12$.

And we have $x_1, x_2^{'}, x_2^{''}, x_3, x_4 \geq 0$, okay.

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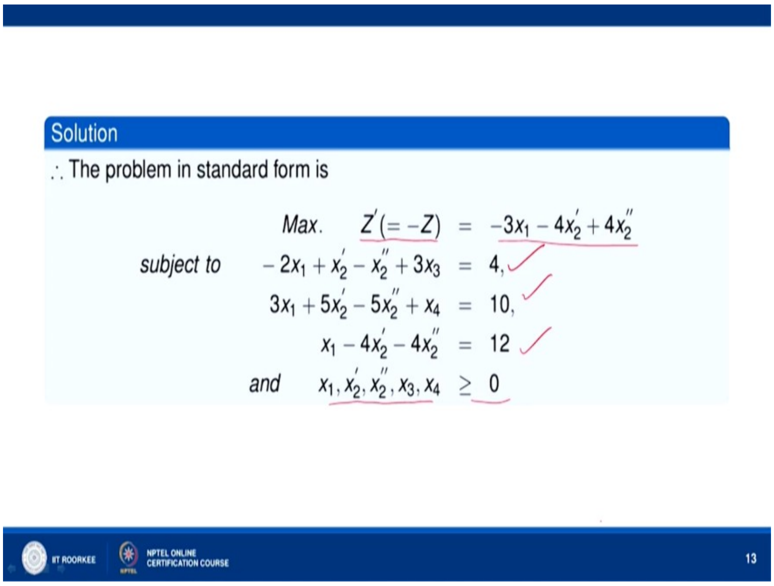

So the problem has now been converted to the standard form maximization of

$Z^{'}=-Z=-3x_1-4x_2^{'}-x_2^{''}$. The first constraint is $-2x_1+x_2^{'}-x_2^{''}+3x_3=4$, this one. And then the second one is $3x_1+5x_2^{'}-5x_2^{''}+x_4=10$, so this is the second one.

And then $x_1-4x_2^{'}-4x_2^{''}=12$ that is third, okay. And all these variables are nonnegative. So the given problem has been brought into the standard form and now we will proceed and solve it by using simplex method which we will discuss in the next lecture.

So we have brought a General Linear Programming Problem into the standard form and we have seen how we can do that. Now in the next lecture we shall discuss how to solve a General Linear Programming Problem. We will bring it to the standard form and then use simplex method to solve that problem. With that I would like to end this lecture, thank you very much for your attention.