Higher Engineering Mathematics
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Lecture 04
Predicates and Quantifires-I

Hello friends welcome to my lecture on Predicates and Quantifier. There will be two lectures on this topic. This is first lecture on this topic. We have so far discussed the simple statements and the logical techniques to combine simple statements into compound statements. Now these cannot, the technique cannot be applied, we cannot apply these techniques to convert the following sentence into a mathematical statements just by using propositional logic.

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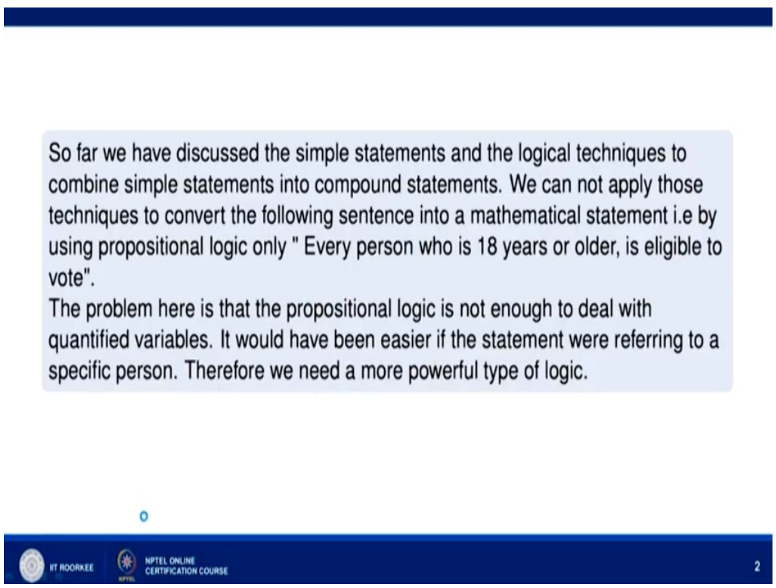

Every person who is 18 years or older is eligible to vote. The problem here is that the propositional logic is not enough to deal with quantified variables. It would have been easier if the statement were referring to a specific person, therefore, we need a more powerful type of logic.

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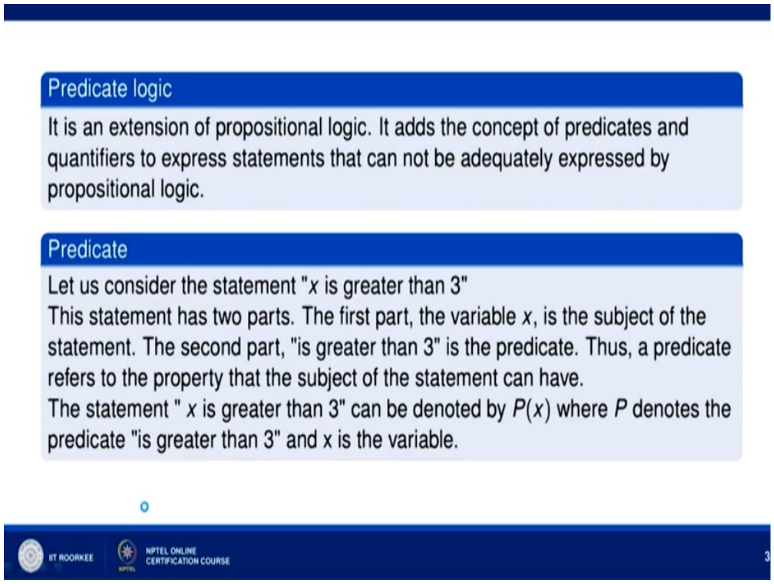

We discuss now predicate logic. It is an extension of propositional logic, it adds the concept of predicates and quantifier to express statements that cannot be adequately expressed by propositional logic. Now let us see what is a predicate, let us consider the statement x is greater than 3, this statement has two parts, the first part is the variable x which is called the subject of the x statement, the second part is greater than 3 is the predicate. Thus, a predicate refers to the property that the subject of the statement can have. The statement x is greater than 3 can be denoted by P(x), where P denotes the predicate, is greater than 3 and x is the variable.

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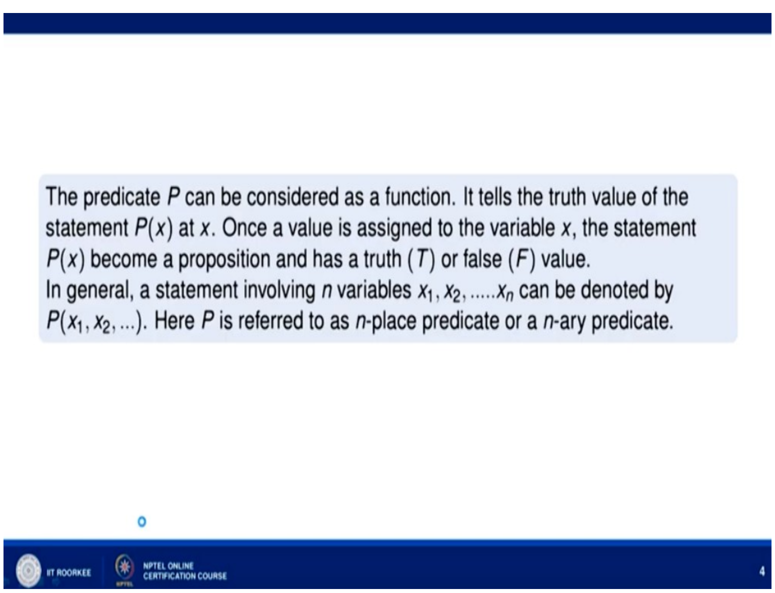

The predicate P can be considered as a function, it tells the truth value of the statement P(x) at x. Once a value is assigned to the variable x the statement P(x) becomes a proposition and has a truth value denoted by T or a false value denoted by F. P(x) in general statement involving n variables x_1, $x_2,\ldots,x_n$ can be denoted by P($x_1, x_2,\ldots$) here P is referred to as n-place predicate or an n-ary predicate.

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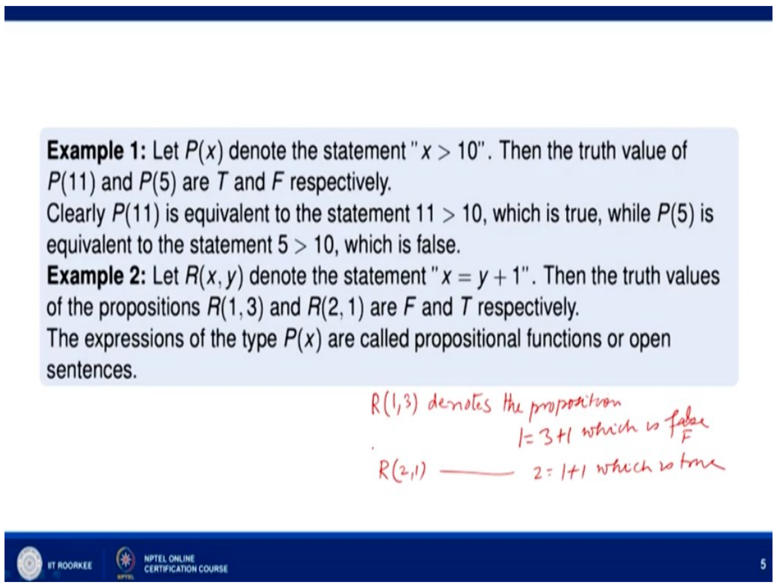

Let us look at the example one, in example 1 let P(x) statement denote the x>10, then the truth value of P(11) and P(5) are T and F respectively because P(11) is equivalent to the statement 11>10, which is true, while P(5) is equivalent to the statement 5>10 which is false. In example 2 let R(x,y) denote the statement "x=y+1". Then the truth values of the proposition R(1, 3) and R(2, 1) are F and T respectively.

R(1, 3) denotes the statement denote the propositions 1=3+1 which is false, while R(2, 1) denotes the proposition 2=1+1 which is true, so the value of the propositions R(1,3) is F and the value of R(2,1) the proposition R(2,1) is T. The expressions of the type P(x) are called propositional functions or open sentences.

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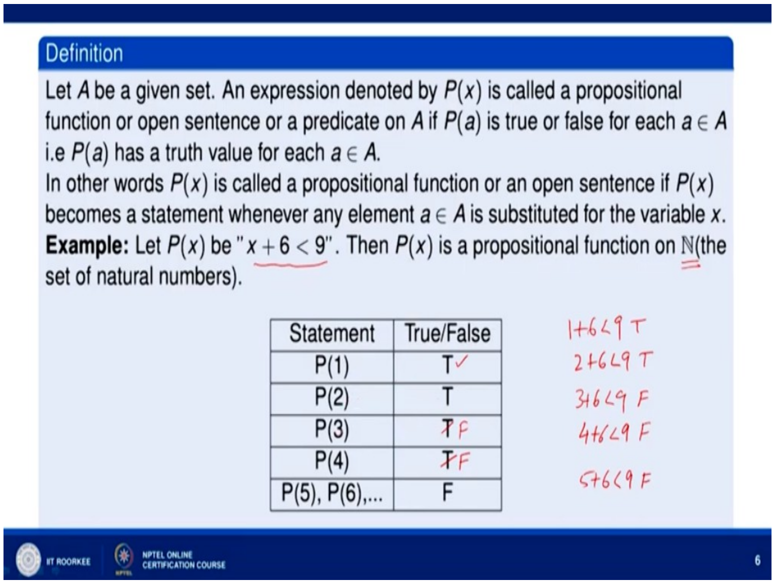

Let A be a given set, an expression denoted by P(x) is called a propositional function or open sentence or a predicate on A if P(a) is true or false for each a∈A, that is P(a) has a truth value for each a∈A, in other words P(x) is called a propositional function or an open sentence on A, if P(x) becomes a statement whenever any element a∈A is substituted for the variable x. Let P(x) be, for example, let P(x) be x+6<9, then P(x) is a propositional function on the set of all natural number.

Let us see that how it is propositional function, when you take the natural number 1 than P(1), P(1) =1+6 <9 which is true, so the value of P(1) is T here, P(2)=2+6<9 which is again true, value is T, 3+6 < 9 is not true, 9<9 is not true, so we have the value false here, 3+6=9, 9<9 is not true, so the value is false here and P(4) 4+6 < nine, is again untrue so the value is false here and P(5), P(6).

There are all false because 5+6<9 is not true then 6+6<9 is also not true, so P(5), P(6) they are all having truth value F, P(4) this is F and P(3) is also F, so the truth value of P(1) is T, the truth value of P(2) is T, the truth value of P(3), P(4), P(5), P(6) and so on they are all F.

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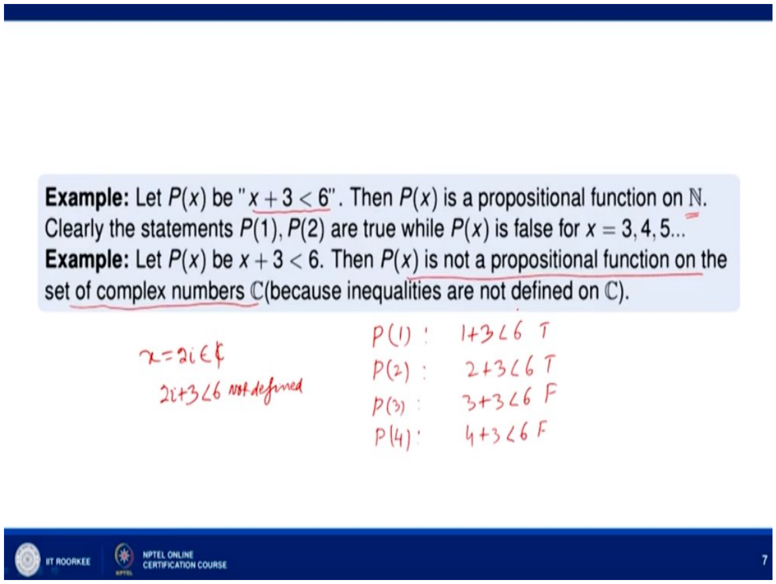

Now, let us take another example let P(x) be x+3<6 then P(x) is a propositional function on the set of all natural number clearly the statement P(1), P(2) are true, P(1) is P(1) is 1+3<6 which is true, so P(2) is 2+3<6 which is again true, now when x takes value three, P(3) then 3+3<6 is false, so we have the truth value of P(3) as F and truth value of P(4) is also F because 4+3<6 is false.

So the truth value is of P(3), P(4), P(5) and so on they are all F, thus P(x) has a truth value for each x∈N and therefore P(x) is a propositional function on the set of all natural number. Now let us take another example let P(x) be x+3<6 again we are taking P(x) to be the same inequality x+3<6 than P(x) is not a propositional function on the set of complex numbers C.

Set of complex number denoted by C because if you take x equal to say 2i then 2i∈C but 2i+3<6 is not defined, so since inequalities are not defined on C therefore, P(x) for are a given x ∈C, does not have a truth value T or F and therefore, P(x) is not a propositional function.

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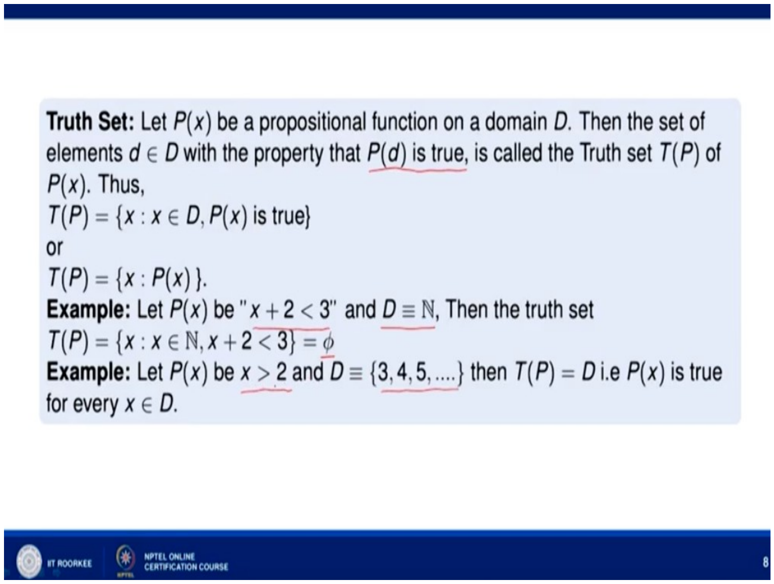

Let us define a truth set, let P(x) be a propositional function on a domain D then the set of elements $d \in D$ to the property that P(d) is true is called the truth set T(P) of P(x), so for all those $d \in D$ for which P(x) has the truth value T we say that those element form the set a truth set for of P, so the truth set T(P), we denote truth set by T(P) so $T(P)=\{x : x \in D, P(x) \text{ is true}\}$ or we can denote by $T(P)= \{x : P(x)\}$. Now example, let P(x) be $x+2<3$ and D is the set of all natural numbers than $T(P)=\{x: x \in N, x+2<3\}=\phi$.

Now, you can see if x takes the value 1 then $1+2<3$ is not true, similarly, when x takes values more than 1 that is 2, 3, 4 and so on, $x+2$ is not less than 3 therefore, T(P) is equal to ϕ because $x +2 <3$ is not true for any $x \in N$. Now let us take another example let P(x) be $x>2$ and $D=\{3, 4, 5, \ldots\}$ then $T(P)=D$ because for all $x \in D$, the inequality $x>2$ is true, so T(P) is the whole of T.

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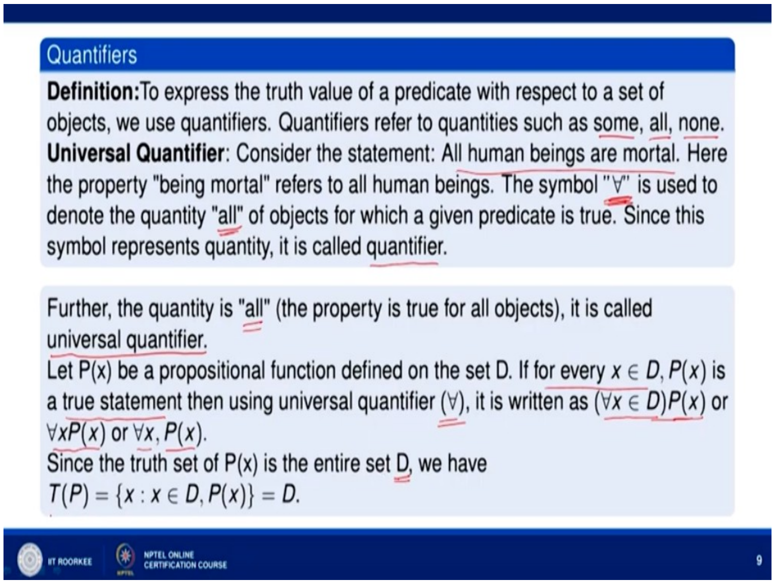

Let us now define quantifiers. To express the truth value of a predicate with respect to a set of objects, we use quantifiers, quantifiers refer to quantities such as some all none, some all and none, universal quantifier, consider the statement all human beings are mortal, here the property being mortal refers to all human beings. This symbol $\forall$ is used to denote the quantity all of objects for which a given predicate is true. Since this symbol represents a quantity we call a quantifier. Further, the quantity is all, the property that is the property is true all objects therefore, this symbol is called as universal quantifier.

Let PX be a propositional function defined on the set D if for every $x \in D$ P(x) is a true statement, then using universal quantifier, this universal quantifier $\forall$ be write it as $\forall x \in D$ P(x) or $\forall$ x P(x) or $\forall$ x, P(x), so this means that this notation means for every $x \in D$ P(x) is true, since the truth set of P(x) is the entire set D, the truth set of P(x), the truth set of P(x) is the because $\forall x \in D$ P(x) is true, so the truth set of P(x) will be entire set D and therefore T(P)=D.

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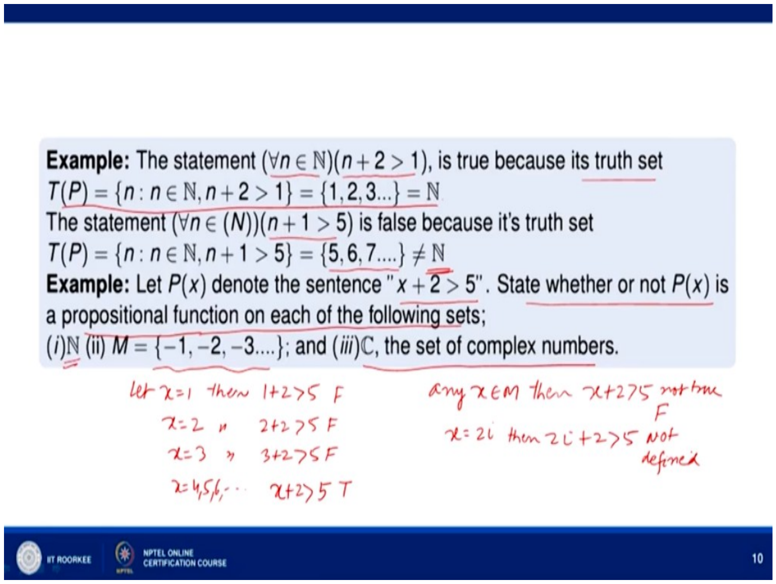

Now, let us check the example the statement (∀ n∈N), (n+2>1) you can see take any n∈N, say n={1, 2, 3,…} then n+2 is always greater than one, therefore it is truth set is the whole of N, T(P)={1,2,3,…}=N. The statement (∀ n∈N), (n+1>5) this is false because when n takes values 1, 2, 3, 4 then n+1>5 is not true, so truth, T(P)={5,6,7,…} when n takes values starting with 5 that is 5 6 7 and so on then only n+1 >5 is true.

So, T(P)={5,6,7,…} ≠ N and therefore this statement is false. So let P(x) denote the statement x+2>five, let P(x) denote the sentence x+2>5 state whether or not P(x) is a propositional function on the following sets, on each of the following sets, so let us take first the case N, in the case N you take any x∈N, so x is equal to one, let say that x=1 then 1+2 >5 is false, not true.

So we have truth value F, x=2, if we take then 2+2>5 is again not true so the truth value F and then we take x=3 then again 3+2>5 is not true, so truth value is 5 but when x takes value is 4, 5, 6,… then x+2>5 is always true, so the truth values is T, so for all x∈N, the sentence P(x) given by x+2>5 has a truth value T or F and therefore P(x) defines a propositional function on the set N.

While, now let us take the other case M={-1,-2,-3,…}, so here if you take x to be say minus 1 then -1+2=1, 1>5 is not true and you take x= -2, we get x+2=0, so 0>5 is not true when you take x= -3, again we get -1>5 which is not true, so for every x∈M, if you take then x+2>5 is not true, that is truth value is F and therefore of every $x \in M$, the sentence $x+2>5$ has a truth value which is F and therefore, P(x) defines propositional function on M.

Now, let us take the set of all complex numbers. If you take any $x \in C$, say x equal to if you take 2i, a complex number, a purely complex number if you take $x=2i$ then $2i+2>5$ is not defined because inequalities are not defined for complex numbers, so we cannot decide about the truth value whether it is T or F and therefore, P(x) does not define a propositional function on C.

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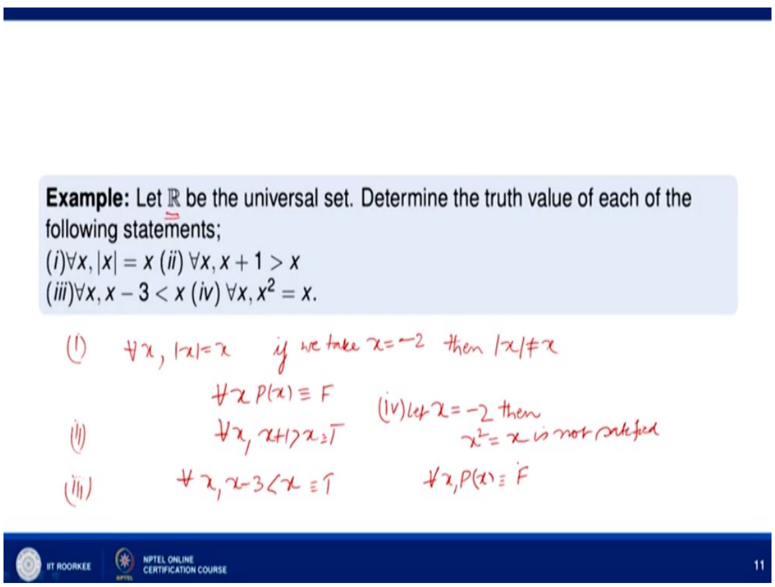

Let take the example, let R be the universal set R be the set of real numbers then determine the truth value of each of the following statement, first statement let us take $\forall$ x , $|x|=x$, now if you take x equal to if we take $x = -2$, then $|x| \neq x$ because $|x| = 2$ while $x = -2$, so $|x| \neq x$ and therefore, the truth value of the statement is F, so $\forall$ x P(x), hence truth value F, truth value is F, and then here second case $\forall$ x, $x+1>x$ if we take any $x \in R$ than $x+1>x$ is always satisfied so this statements is always true, so it is value is true, so it is value is true.

Now let us take the third case in this case $\forall$ x, $x-3<x$ now take any $x \in R$ then $x-3<x$ is always true, so it is truth value is T, and then forth case $\forall$ x, $x^2 = x$ or P(x) here is $x^2 = x$, now if we take x equal to say minus 2 let $x=-2$ then $x^2 = x$ is not satisfied and therefore truth value is F, x^2 $\forall$ x P(x), P(x) is $x^2 = x$, the truth value is F.

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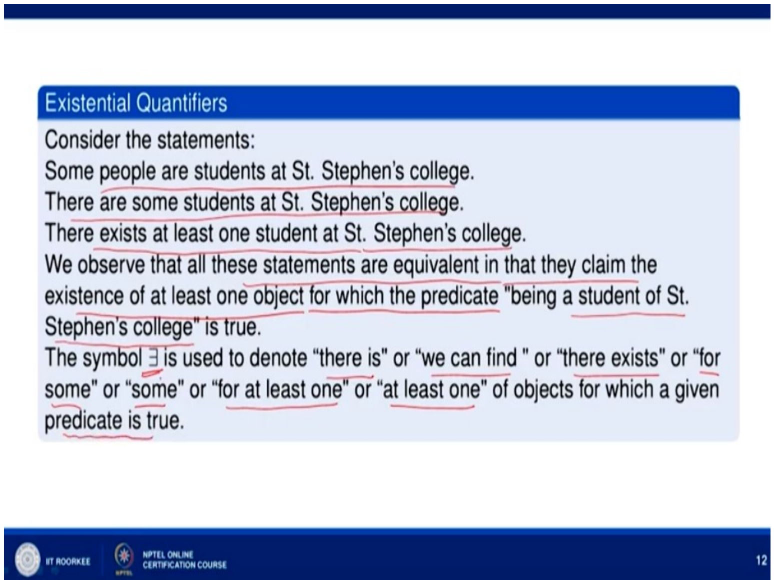

Now, let us discuss existential quantifiers Let us consider the statements, 'some people are students at St. Stephen's college'. The second statement 'there are some students at St. Stephen's college', third is 'there exist at least one student at St. Stephen's college', now notice that all these statement are equivalent in that they claim the existence of at least one object for which the predicate as 'student of St. Stephen's college' is true, the symbol $\exists$ is use to denote there is we can find there exist for some, some for at least one least one, so in all this cases of object for which a given predicate is true, we use this symbol.

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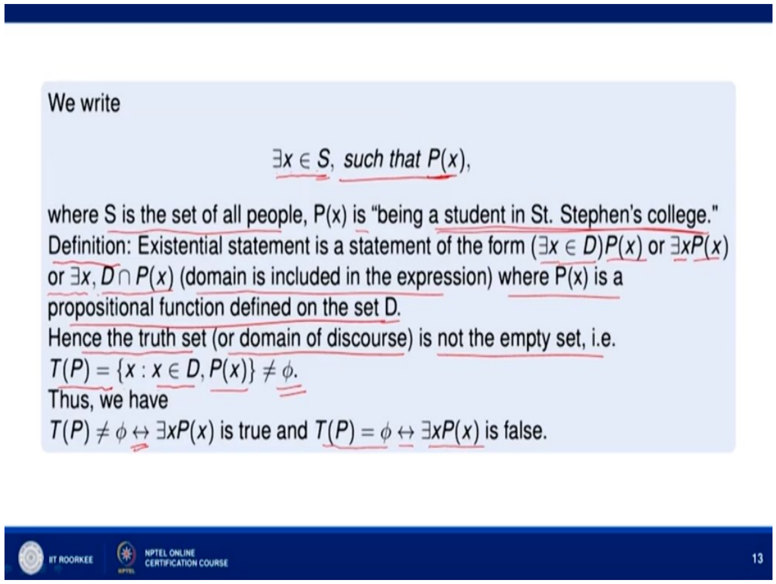

Now, we write $\exists x \in S$ this means there exist, there exist are for some for some are at least one, at least one and so on, so this symbol $\exists x \in S$ such that P(x), we write this where S is the set of all people, S is the set of people, P(x) is being a student in St. Stephen's College, so this P(x) is denote a student in St. Stephen's college and this denote a set of people. So we can write the sentence these sentences these statements, symbolically in the following form.

If our S denote the set of all people then we can write those three statements in the, symbolically in the following form $\exists x \in X$ such that where P(x) is the predicate a student in St. Stephen College. Now let us define existential statement, so existential statement is a statement of the form($\exists x \in D$)P(x) or $\exists x P(x)$ or $\exists x$, $D \cap P(x)$, here domain you can see domain D is included in the expression, so if we do not write D here then we include D in the in the predicate we write instead of predicate P(x) we write $D \cap P(x)$.

So here P(x) is a propositional function defined on the set D. P(x) is the propositional function defined on the set D, hence the truth set and now truth set is also defined as domain of discourse, so it is not an empty set because there always exist to some $x \in S$ for which P(x) is true, so the truth set is not the empty set is here, that is $T(P)=\{x: x \in D, P(x)\} \neq \phi$, so we can say the following that $T(P) \neq \phi \leftrightarrow \exists x P(x)$ is true and $T(P)=\phi \leftrightarrow \exists x P(x)$ is false, because if there does not exist any $x \in S$ for which are D for which PX is true then $T(P)=\phi$, so the truth set of P is equal to phi if and only if $\exists x$:P(x) is false.

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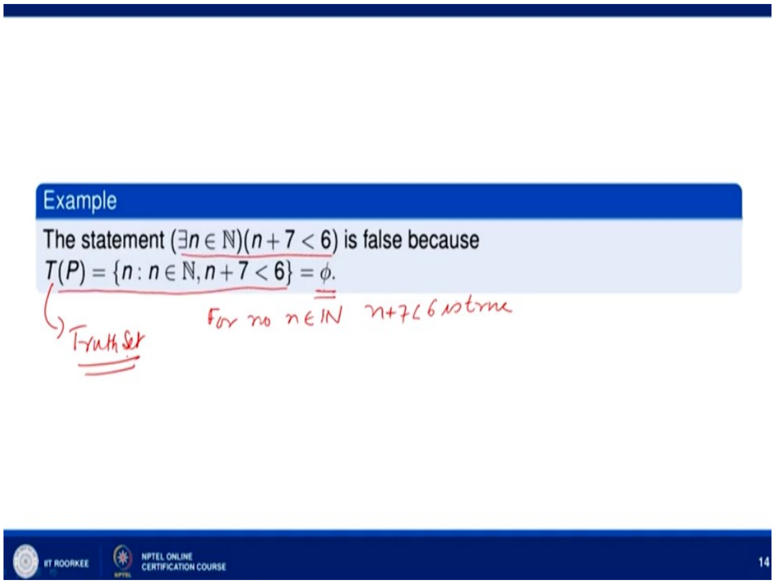

Now, let us look at this statement, ∃ n∈N, n+7 <6, you can see if you take any n, n=1 or 2 or 3 than n+7<6 is always false, so for no n∈N, n+7<6 is true, therefore T(P), T(P)={n:n∈N, n+7<6}=ϕ , so the truth value of this statement is, truth set of this statement, this is truth set, truth set of this statement is empty.

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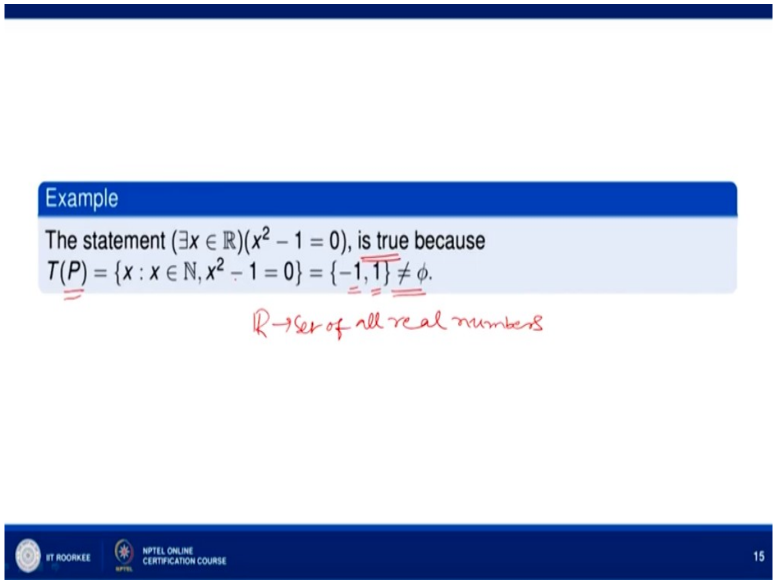

Let us take another example the statement ($\exists$ x$\in$R), R is the set of all real numbers, so the statement ($\exists$ x$\in$R)($x^2 - 1 = 0$), this statement is true because T(P) $\neq$ ϕ, T(P)={-1,1}, because for x=-1 and x=1, $x^2 - 1 = 0$ is satisfied, so when T(P) $\neq \phi$ we have seen here, when T(P) $\neq \phi$ is the statement existential statement, this is existential statement, the existential statement is true.

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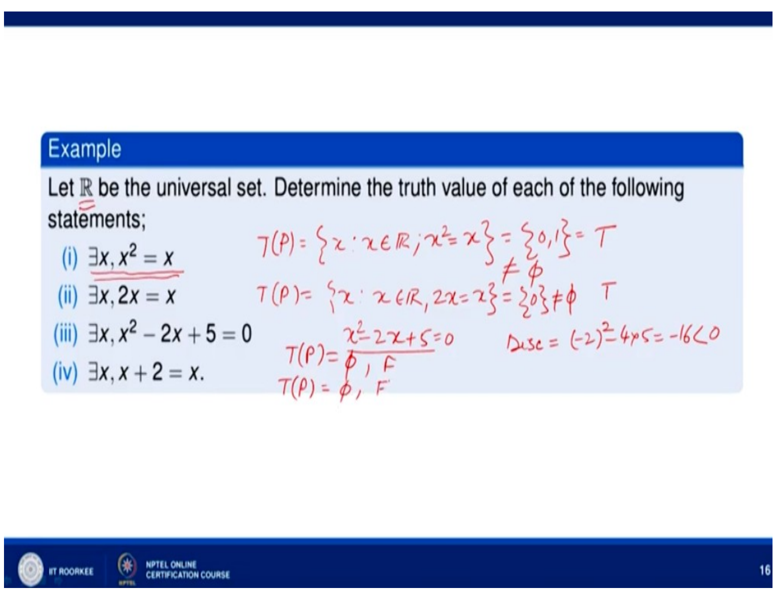

So let us say let the set of real number R be the universal set, then let us consider this statement $\exists x, x^2 = x$. Now here T(P)={x: x$\in$R, $x^2 = x$}={0,1}=T$\neq \phi$ and here is this case T(P)={x: x $\in$R, 2x=x}={0}$\neq \phi$, so again it is true and then $\exists$ x, $x^2 - 2x + 5 = 0$.

Now, $x^2 - 2x + 5 = 0$ if we consider this equation then you can see discriminant here, discriminant $= B^2 - 4AC = (-2)^2 - 4 \times 5 = -16$ so discriminant negative and therefore, its roots are complex, so for no real value of x, $x^2 - 2x + 5$=0 and therefore, T(P)=ϕ , T(P)=ϕ , so truth value of this statement is F, and then . Now for no x$\in$R, x+2=x so this here T(P)=ϕ again, so the truth value of this statement is F. With that I would like to end my lecture, thank you very much for your attention.