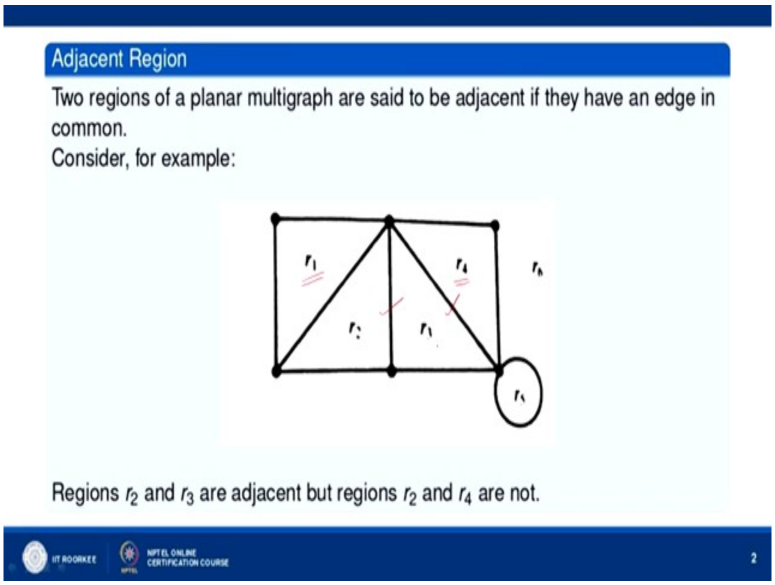
Higher Engineering Mathematics
Professor P. N. Agrawal
Department of Mathematics
Indian Institute of Technology Roorkee
Lecture 36:
Dual of a Graph

Hello friends, welcome to my lecture on Dual of a Graph, let us first define adjacent region.

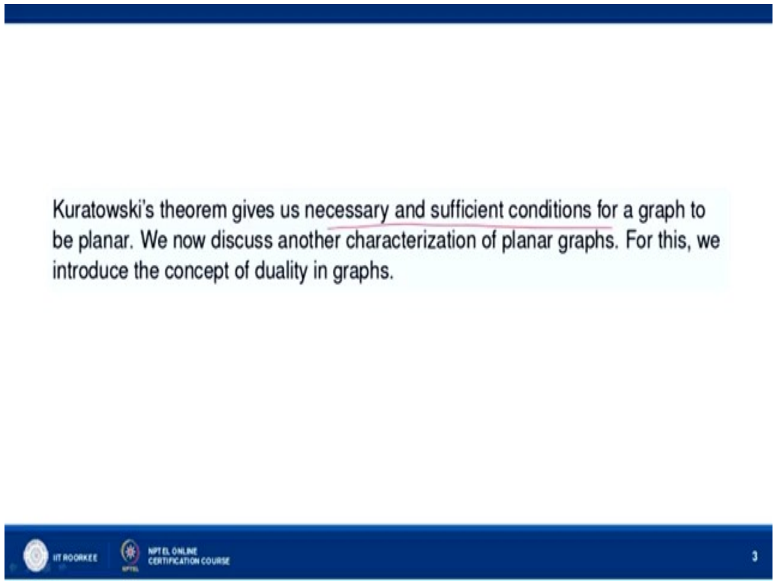
(Refer Slide Time: 00:33)



Two regions of a planar multigraph are called adjacent if they have an edge in common, say for example the region r_1 and r_2 are adjacent because they have this edge in common, okay and then r_2 and r_3 are adjacent regions because they have this edge in common, this one, okay and then r_3 and r_4 are adjacent because they have this edge in common, so r_1, r_2 are adjacent regions,r_2, r_3 are adjacent regions, r_3, r_4 are adjacent regions.

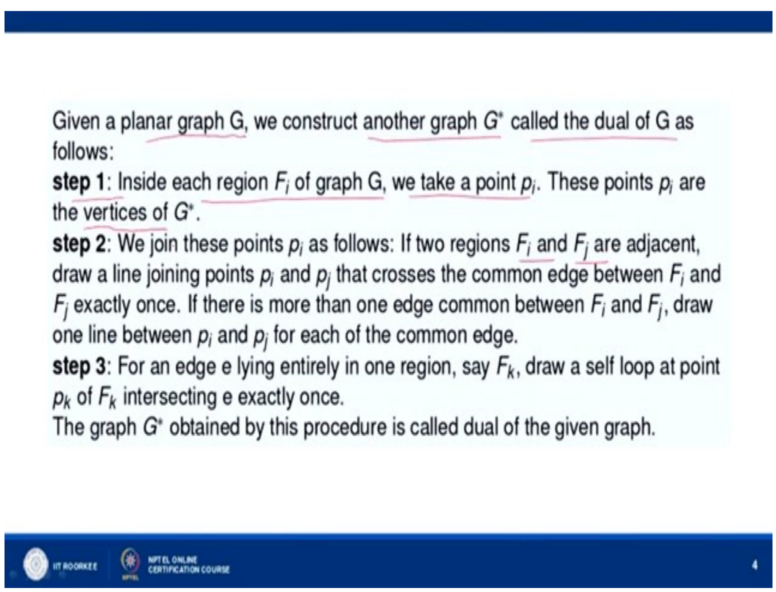
Now, if you will take say for example r_1 and r_4 okay, r_1 r_4 do not have any edge in common, so r_1 and r_4 are not adjacent regions. Similarly, r_2 and r_4 are not adjacent regions because they do not have any edges common, r_6, r_4, r_6,r_3, okay they are also non adjacent regions okay, r_5 is not adjacent to r_4 or r_3.

(Refer Slide Time: 01:40)



So Kuratowski’s theorem gives us necessary and sufficient conditions for a graph to be a planar, we have seen that, Kuratowski's theorem says that a graph is planar if and only if it does not have any sub graph which is homeomorphic to k_{33} or k_5. So it gives us necessary and sufficient conditions for a graph to be a planar graph.

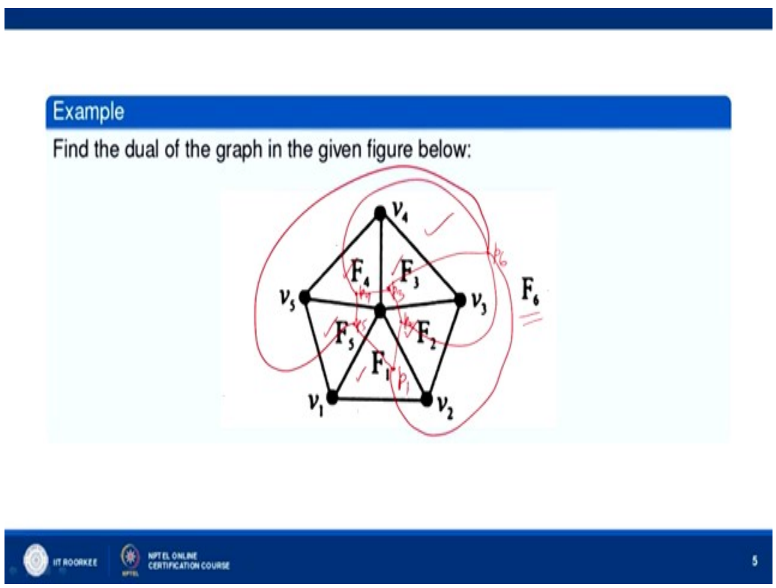
(Refer Slide Time: 02:03)



Now, let us see how we get the dual of a given planar graph, suppose we have a planer graph G, then we can construct another graph G star which is called the dual of G by following these steps, okay so inside each region F_i of graph G, we take a point p_i okay, these points p_i are called the vertices of, are the vertices of the dual graph G star, we join these points p_i as follows, if two regions F_i and F_j are adjacent, draw a line joining points p_i and p_j that crosses the common edge between F_i and F_j exactly once.

If there is more than one edge which are common between F_i and F_j then draw one line between p_i and p_j for each of the common edge. Let us first see some examples on this before we go to step number 3 okay, because in step number 3 we have the case where an edge e is lying entirely in one region so that will come later, let us first consider the case where we use step 1 and step 2 to get the dual of a graph.

(Refer Slide Time: 03:19)



Given a planar graph G, we construct another graph G^* called the dual of G as follows:

step 1: Inside each region F_i of graph G, we take a point p_i. These points p_i are the vertices of G^*.

step 2: We join these points p_i as follows: If two regions F_i and F_j are adjacent, draw a line joining points p_i and p_j that crosses the common edge between F_i and F_j exactly once. If there is more than one edge common between F_i and F_j, draw one line between p_i and p_j for each of the common edge.

step 3: For an edge e lying entirely in one region, say F_k, draw a self loop at point p_k of F_k intersecting e exactly once.

The graph G^* obtained by this procedure is called dual of the given graph.

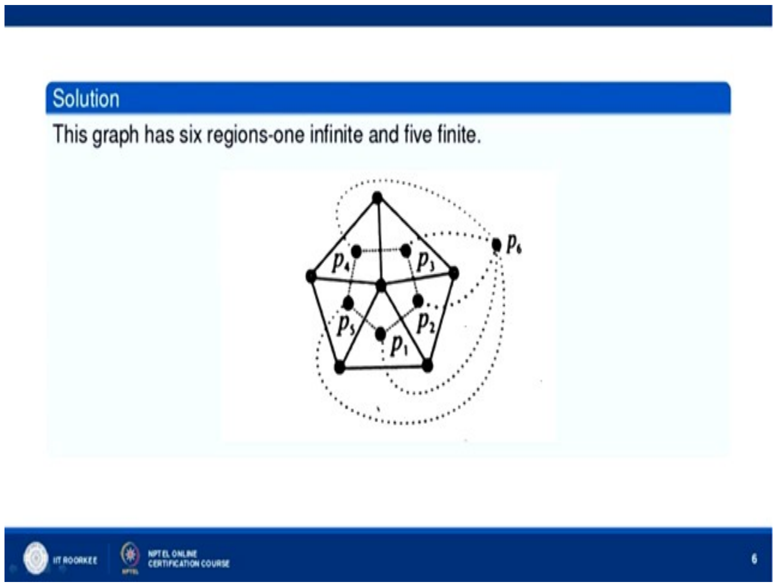
Say for example, let us look at this graph, okay, let us find the dual of this graph, so here we have in this graph we see that there are six regions okay F_1, F_2, F_3, F_4, F_5 and the region outside this, that is shown as F_6 okay, now what we do to obtain the dual of this graph, okay let us take a vertex in each of these regions okay.

Say, inside each region F_i of graph G take a point pi okay, so we can take a point $(p, a)\, p_1$ here, a point p_2 here, a point p_3 here, a point p_4 here, a point p_5 here, okay and a point p_6 here in the region F_6 and then the step 2 says join these points p_i as follows, if two regions F_i and F_j are adjacent okay, draw a line joining the points p_i and p_j, that crosses the common edge between F_i and F_j exactly once.

So, F_4 and F_3 are adjacent regions, so we join p_3 and p_4 by an edge which cross this common edge exactly once okay then F_3 and F_2 they are adjacent regions so we join p_3 to p_2 by crossing the edge which is common to F_3 and F_2 exactly once and then F_2 and F_1 are adjacent so we join p_2 to p_1, okay crossing the edge which is common to F_1 and F_2 exactly once and similarly we join p_1 to p_5 and then p_5 to p_4 because they are regions adjacent to this one, I mean this p_1 and p_5 are joined by this edge because F_1 and F_5 are adjacent regions and F_5 and F_4 are adjacent regions so we join p_5 to p_4.

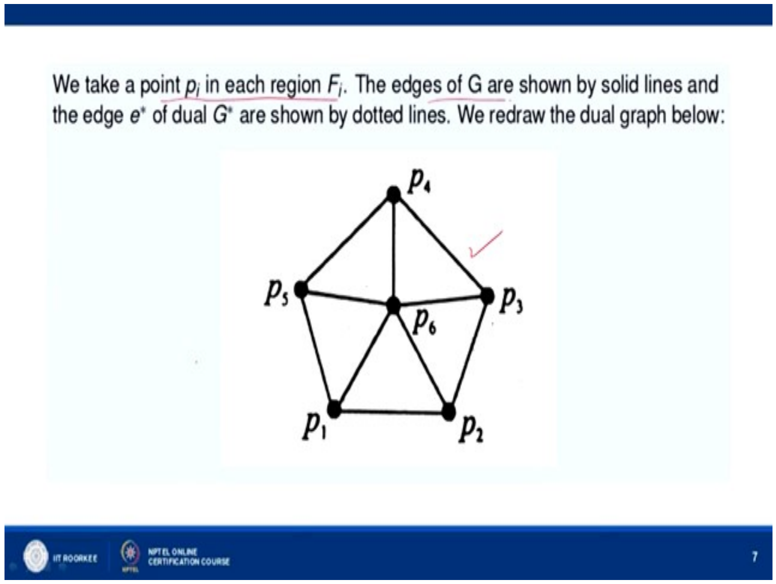
Now, p_6, okay p_6 is adjacent to p_3, so we join p_6 to p_3, we join p_6 to p_4 okay, we join p_6 to p_5 okay and then we join p_6 to p_2,p_6 to p_1 okay, so because this F_6 and F_3 are adjacent regions, they are, so we join p_6 to p_3 and F_6 and F_4 are adjacent, so we join p_6 to p_4 okay and similarly for p_6 and p_5, p_6 and p_1 and p_6 and p_2.

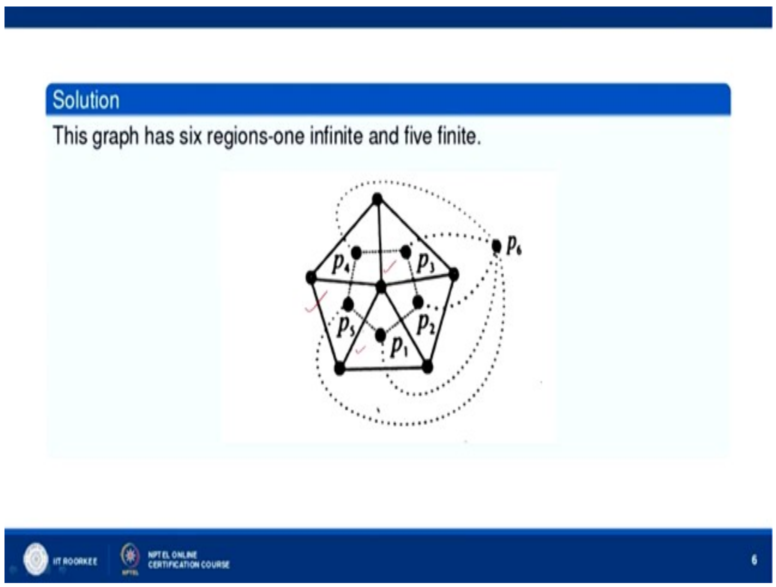
(Refer Slide Time: 06:15)



So, we get this graph, this graph we can see p_6 is joined to p_3 , p_6 is joined p_4 , p_6 is joined to p_5, p_6 is joined to p_1, p_6 is joined to p_6 and p_1, p_2, p_3, p_4, p_5 are joined by these edges okay, now this graph, so this graph shown by dots okay is the dual of the given graph.

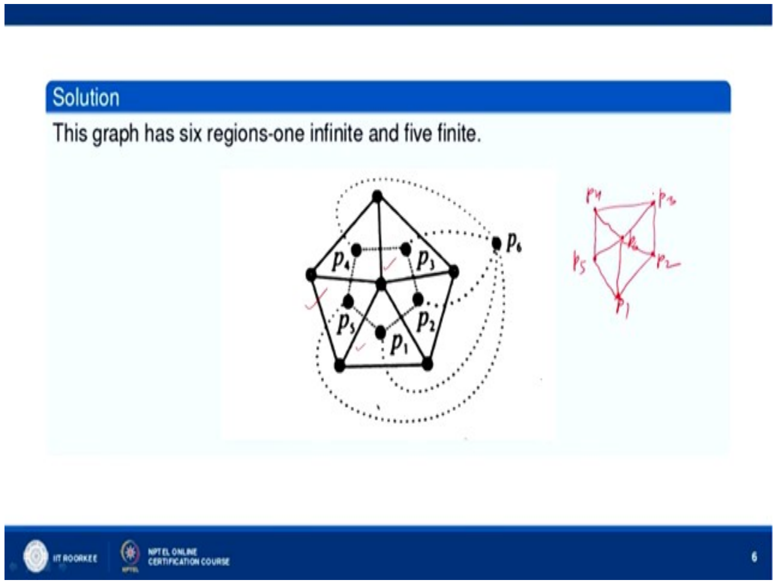
(Refer Slide Time: 06:44)

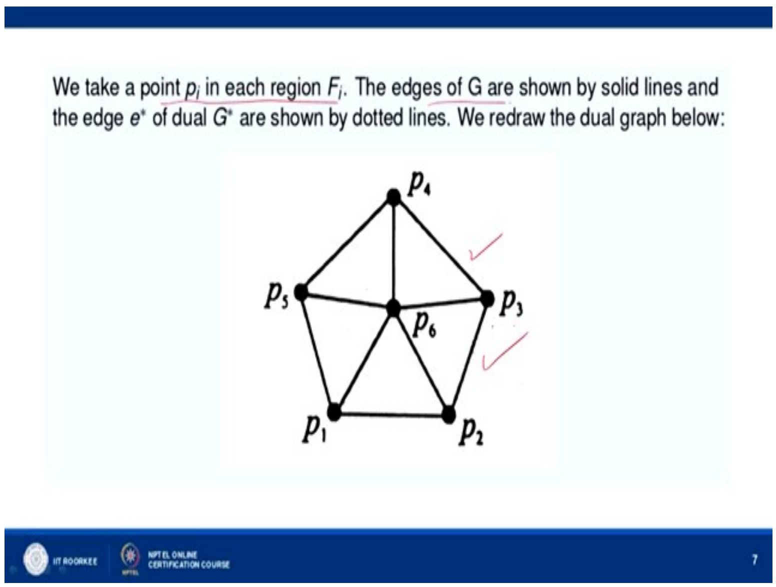




We can draw it like this, the dual graph can be drawn like this okay, we have drawn this graph by taking a point pi in each region Fi and the edges of G, here edges of G are shown by solid lines. The edges of G okay are shown by solid lines, while the edges of G star are shown by dots, dotted lines okay. Now, we draw the dual graph okay you can see we have taken this p_6 inside this and p_6 is then joined to p_3.

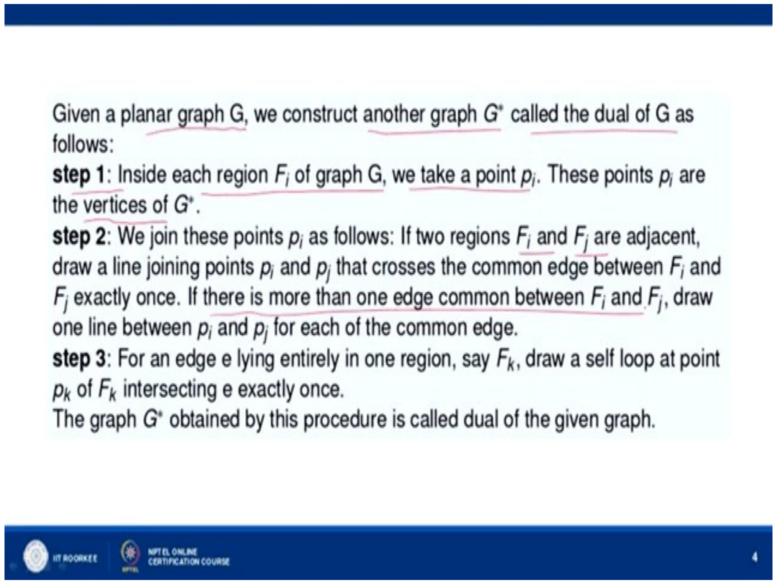
(Refer Slide Time: 7:19)





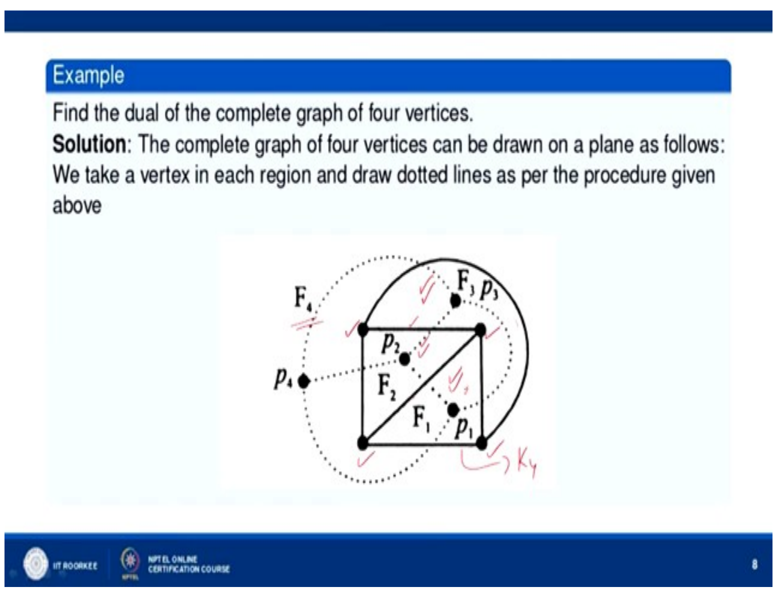
So, p_6 we have brought here okay so p_6 is joined inside this we have p_6, p_6 , p_5, p_2, p_1 okay so we have this okay and then we consider this p_6 here and we join p_6 to, this is p_4, p_3 okay, p_2 and we have p_1 here and we have p_5 here, so we join p_6 to each one of them, okay. So that is how we get the dual of the given graph, this is dual of the given graph.

(Refer Slide Time: 08:09)



Now, so here we consider the case where we did not have this situation that there is more than one edge common between F_i and ,F_j between F_i and F_j we have just one edge which was common. Now, let us go to more examples.

(Refer Slide Time: 08:31)



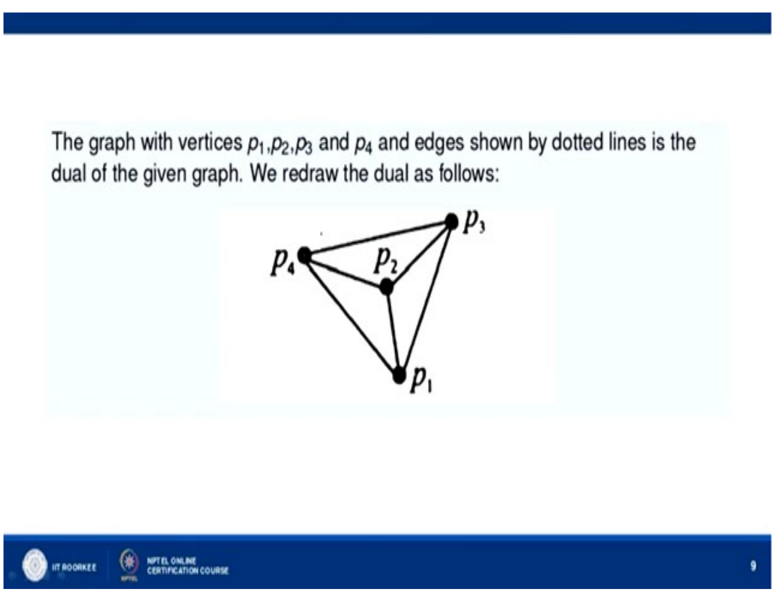
Okay, let us look at this, we have the complete graph of 4 vertices, you see we have 4 vertices, 1, 2, 3, 4, each vertex is joined to every other vertex and therefore it is complete graph k4 okay, it is complete graph k4. Now, the complete graph of 4 vertices can be drawn on a plane like this so what we have done, because it is a planar graph, so we have drawn it so that no edges of this graph cross each other.

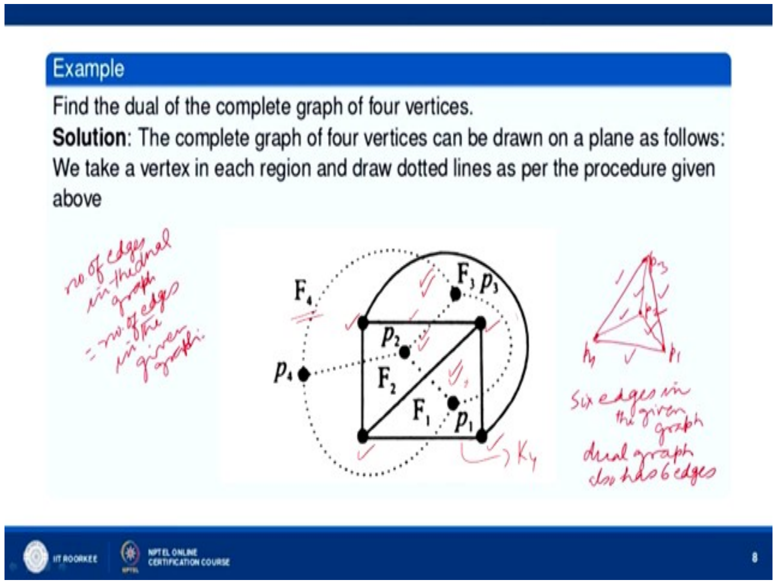
Now, we take a vertex, now we see there are how many regions? One region is this one, one region is this one, one region is here and one region outside okay which is shown by F_4 okay. So we have, we take a point p_1 in region F_1, we take a point p_2 in region F_2, we take a point p_3 in F_3 and we take a point p_4 in region F_4. Now, F_1 and F_2 are adjacent regions so p_1 and p_2 are joined by an edge which crosses the common edge between F_1 F_2 just exactly once and then, now you see this region F_3, okay.

F_3 has an edge common with F_2 okay, this edge common with F_2 okay and F_3 also has an edge common with F_1 okay, so F_3 has more than one edge common, no F_3 has an edge common with F_2, so we joined p_3 to p_2 and F_3 has one edge common with the this edge, common with F_1 so be p_3 join to p_1 then F_4 okay, $F_4$the region outside okay which is shown by F_4, we have p_4 here.

So, p_4 is then joined to, because this edge okay is common between F_4 and F_3 okay so p_4 is joined to p_3 and then p_4 is joined to p_2 because this edge is common between F_4 and F_2 and then this edge, this edge is common between F_4 and F_1 so we join p_4 to p_1 okay right so the solid lines graph is the given graph and the dotted lines graph is the dual graph, we can read around this graph as follows.

(Refer Slide Time: 11:08)

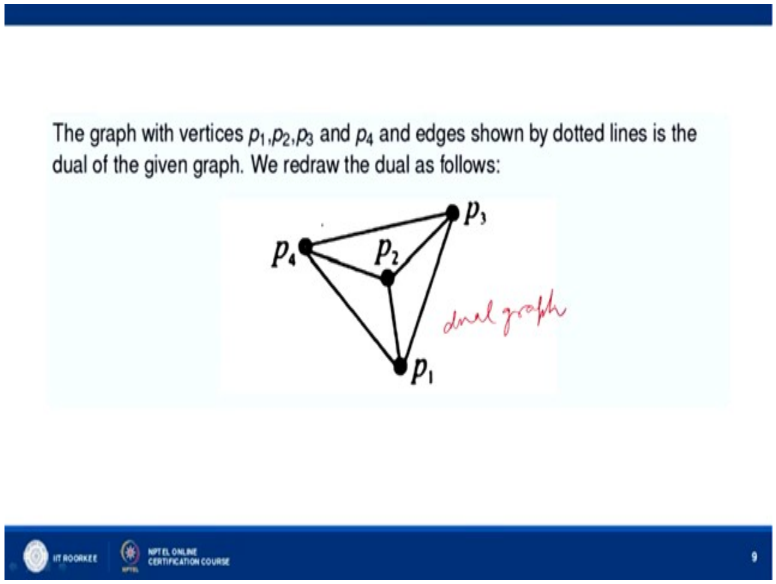


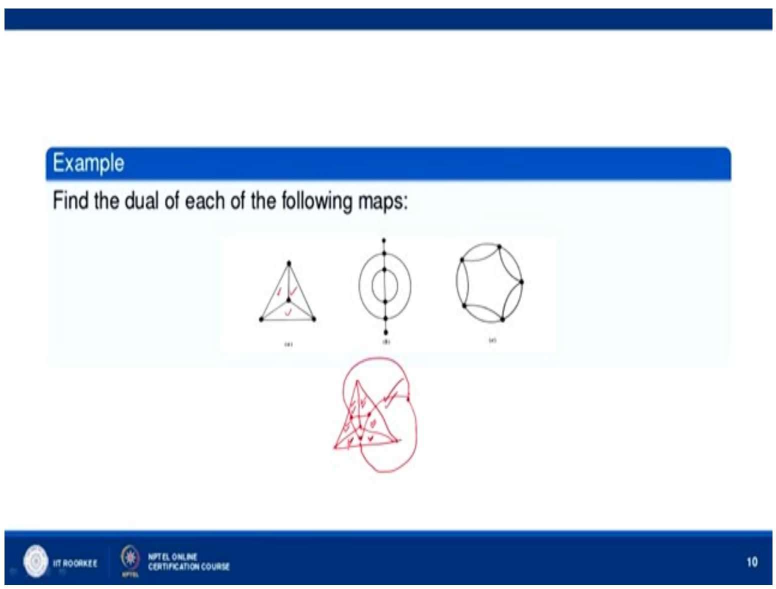


This one, okay this is the dual graph, you see p_3, p_2, p_4, p_1 you see p_4 is this one p_4 okay, so we have p_4 here and then we have p_3, we have p_2, p_1 okay so this is p_1 and p_2 we have here, okay then p_3, is joined to p_4 okay so this p_4 okay p_3 is joined to p_1, so we have this, okay then p_3 is joined to p_2 okay so we have this and then p_1 and p_4 okay, p_1 is joined to p_4 okay. So, we have this and then p_2, p_2 is joined to p_4, p_2 is joined to p_1, okay.

Let us note that number of edges in the given complete graph and the number of edges in the dual graph, they are equal okay here you can see the number of edges in the given graph, they are 1, 2, 3, 4, 5 and 6, there are 6 edges in the given graph and how many edges are there in the dual graph? 1, 2, 3, 4, 5, 6, so dual also has 6 edges. So number of edges in the dual graph is equal to number of edges in the given planar graph, they are equal.

(Refer Slide Time: 13:32)

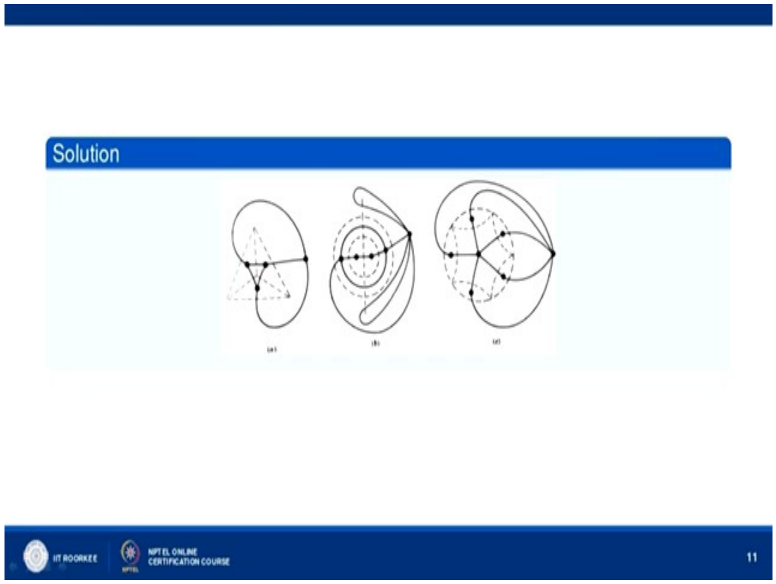


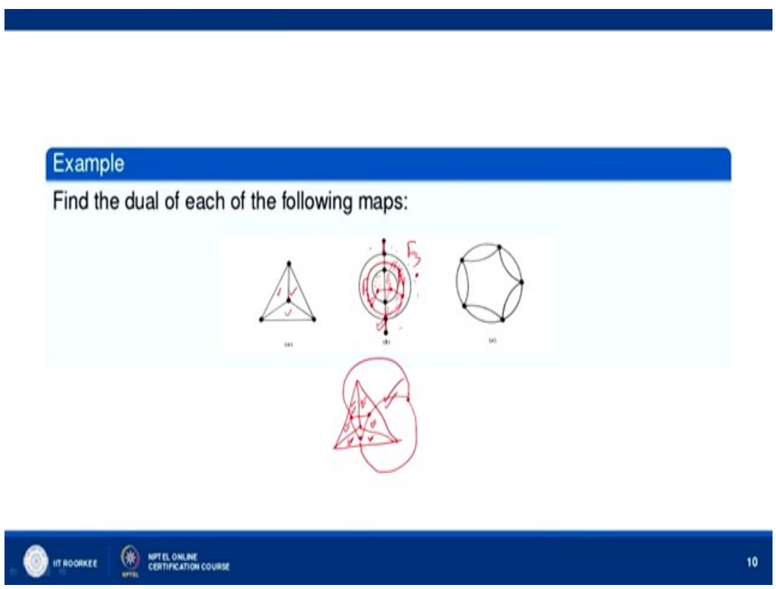


Now, let us go to, so this is the figure of the dual graph. Now, let us go to dual of each of the following maps okay, so we can, so there are one region here, one here, one here and one outside okay, so what we do, we can take a point for each of these regions so one here, one here, one here and one outside, okay. Now, this region and this region they are adjacent so we can join this by this line, now this region and this region are adjacent, so we can join them by this edge and then this region and this region they are adjacent so we can join like this okay.

Now, then this region outside okay has a edge common between this region, so we can join this like this and then we can join this to this one okay we can join this two this one okay and we have the following graph.

(Refer Slide Time: 14:44)



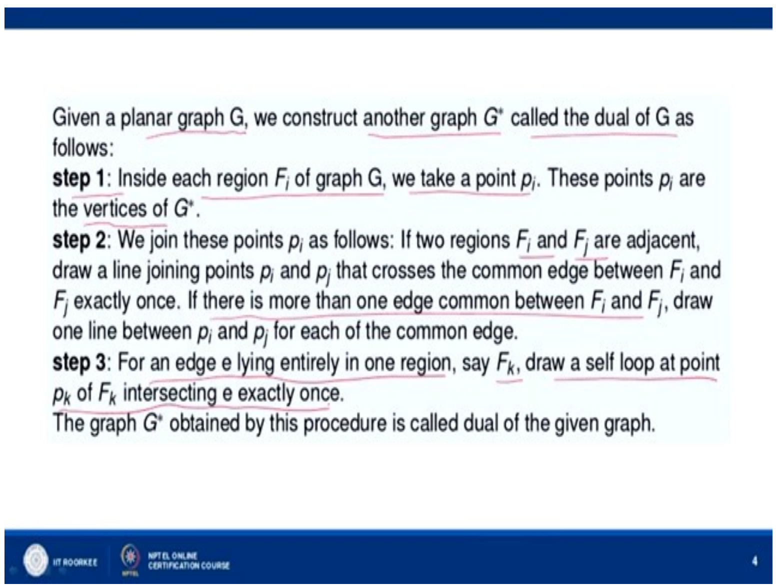


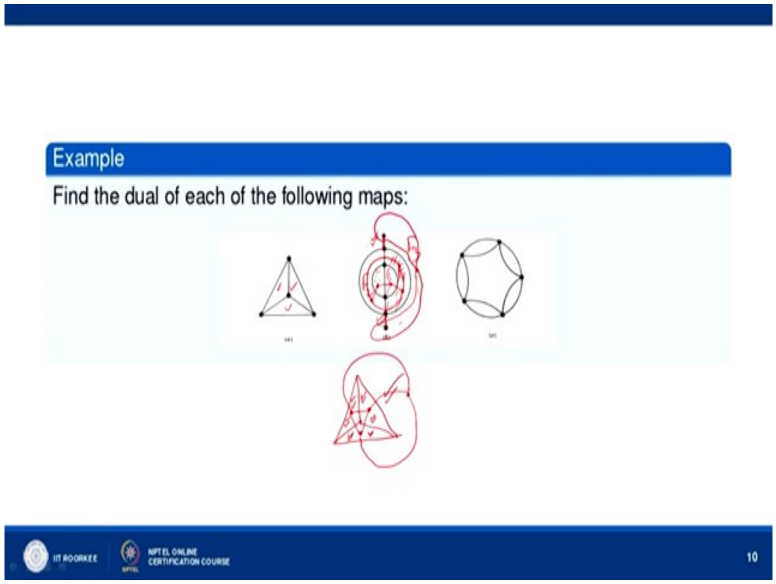
See this one, we have this okay and then join this to this, this to this and then this to this okay. Now, let us consider this one, in this graph how many regions are there, one region here F_1 okay this one, this region and one region here F_2 and then one region here F_3, F_3 is this one, this region okay and one region here F_4 this one, no we have F_3 only, there are not two regions, there is one region.

So, we have this, not F_4, F_3 only, in F_3 region we have two edges, this edge, this edge and this edge in the F_3 region we have two edges. Now, you see F_1 okay this region F_1, this is adjacent to this region okay so what we do, we take a vertex here and a vertex here and join them by an edge okay, then this region and this region, they are adjacent so we take a vertex here and join like this and then this region, this one and this one F_2 they are adjacent so we take a point here okay and join like this okay.

Now, this region is adjacent to this region okay, this region is adjacent to this region and they have two common edges okay, two common edges so we can join this by this like this, and we can join this to this okay, yeah by two edges because this region and this region, F_1 and F_2 have two common edges, okay. Now, so we take an edge which crosses here and we take another edge which crosses here and then F_3, F_3 is here you take point here. Now, in F_3 there are two common edges, this one, it has two edges okay, so what we will do, we take a self-loop.

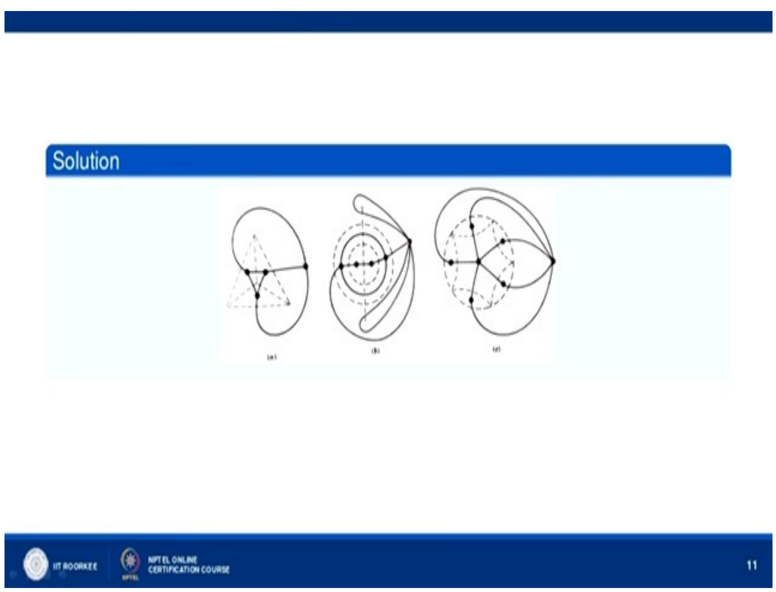
(Refer Slide Time: 17:03)

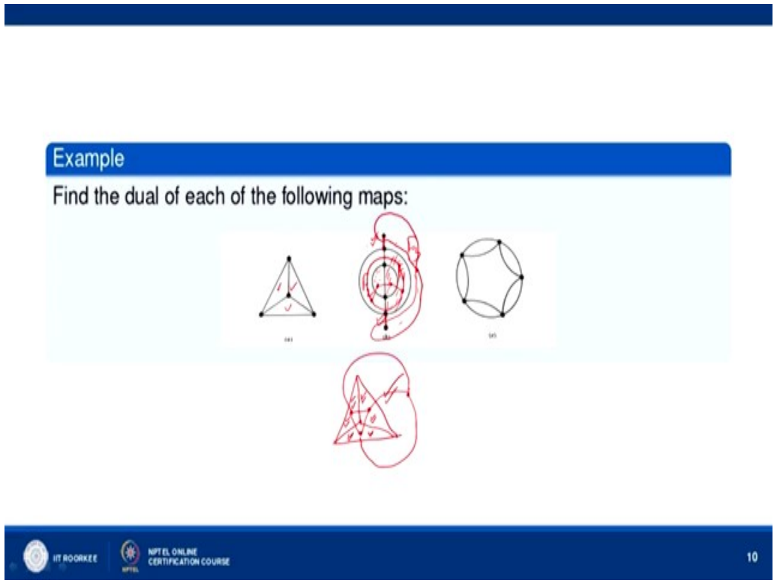




Here we see, for an edge lying entirely in one region okay say F_k, draw a self-loop at point p_k of F_k intersecting e exactly once okay, so we have to draw a self-loop at point pk of F_k intersecting e exactly once. So, we can draw a loop here like this okay exactly and exactly intersecting this edge exactly once and then we can draw another loop like this okay, so this edge, this loop also crosses this edge exactly once, okay.

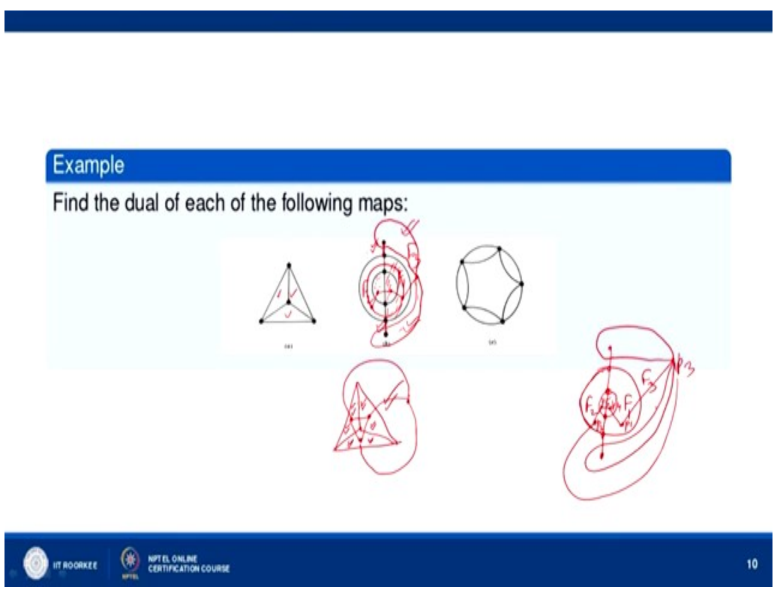
(Refer Slide Time: 17:44)

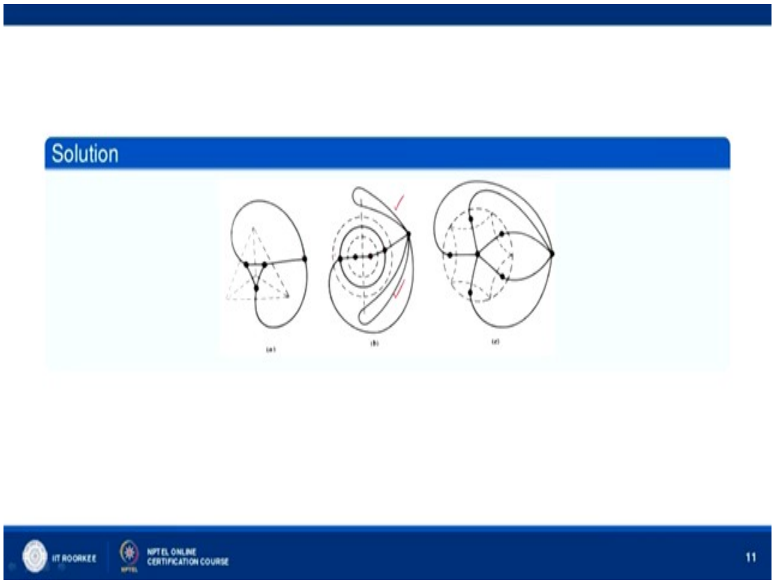




Now, so this is what we have, this graph okay, you can see here this loop crosses this edge exactly once okay and this loop crosses this edge exactly once alright and then we have joined this to this one okay and we have joined okay this one to this one, this we have joined, this region F_3 outside has two edges okay lying in F_3, so we have drawn two loops okay crossing each edge exactly once, one loop is this one, another loop is this one okay.

(Refer Slide Time: 18:19)



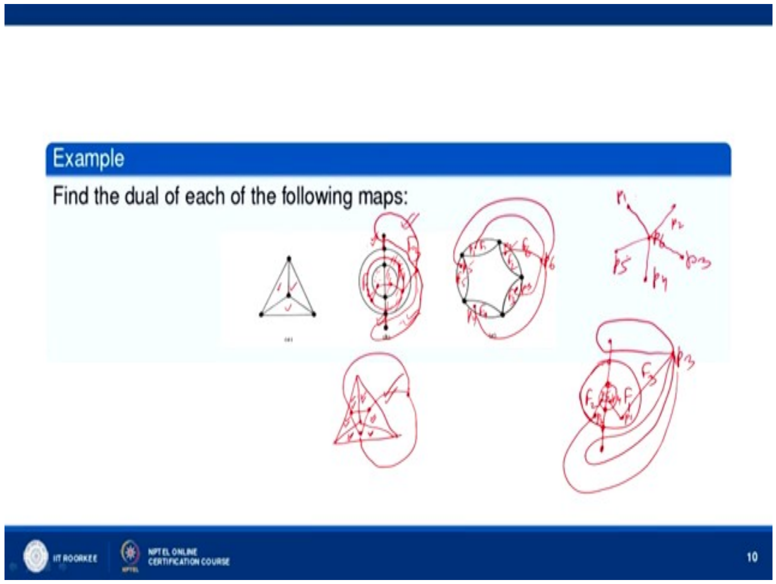


Now, we also have to join this region with this one because this region okay is (isomorphic,) I mean is adjacent to this one okay we have this graph, we have this situation, okay so this is F_1 we have drawn it as F_1, this is F_2 and we have F_3 and we have taken a point here okay, this point and a point here, this point okay and a point here and a point here and then join them like this okay, one point here we have taken.

Now, we have two self-loops here crossing this edge exactly once, okay so we had this, one this and then another one like this, okay. Now, we join this to this because F_1 and F_3 are adjacent regions, there is a one common edge here and then F_3 is also adjacent to F_2, there is a common edge here, this common edge, this one, okay so F_3 can be joined to this p_3, this you can call as p_1 and then this we can call as p_2, we have p_3, p_4 and here we have p_5 okay so p_3 we have joined p_1, p_3 also has to be joined to p_2 like this, okay.

So that is the dual here, we can see this is the dual here, we have joined this two points then we joined this to this and joined this to this, okay and then we have this loop which cross the common edge exactly once and then we have joined this to this okay so that is the dual now let us see this case okay, here we can see, here.

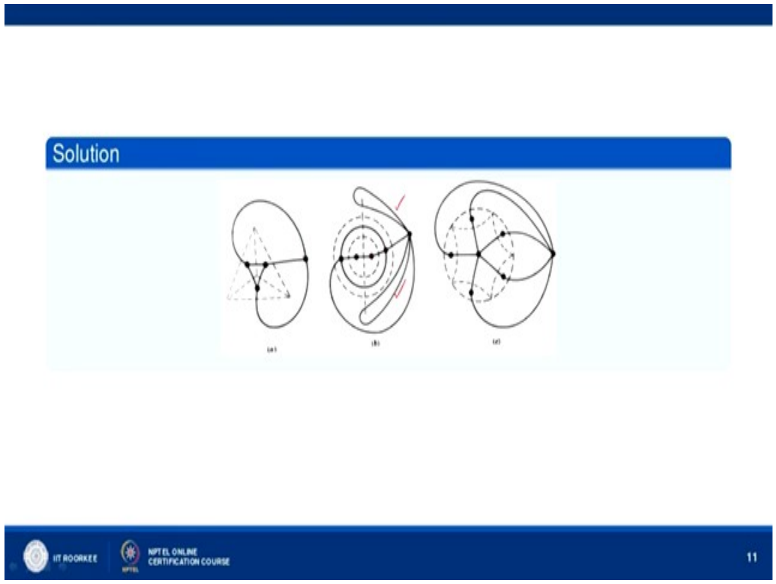
(Refer Slide Time: 20:55)

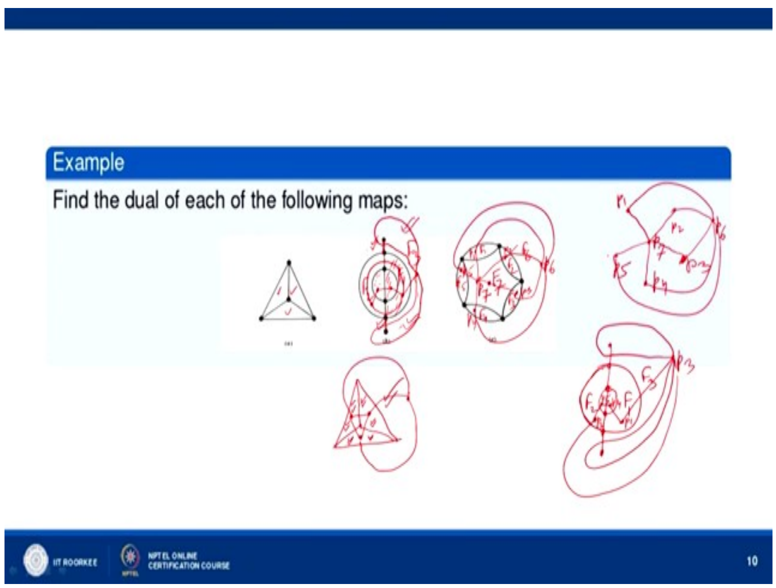


So there are how many regions, F_1, F_2, F_3,F_4, F_5 and we have F_6 here, okay. So we take a point p_1 here, p_2 here, p_3 here, p_4 here, p_5 here and p_6 here okay then since F_1 and F_2 are not adjacent regions okay we cannot join them by an edge crossing their common edge exactly once, but F_6 has a common edge with F_2 so we can join like this, F_6 has a common edge with F_3 so we can join like this and then similarly you can join this way and then this way okay and then we can join like this okay.

So that is the graph in this case and we can read write like this, so you see we have let us say this is p_1, we have p_2 here, p_3 here, p_4 here, p_5 here and p_6 here okay so we have p_1,p_2, p_3, p_5 and p_6, $p_6$is joined to each one of them, p_6 is joined to p_1, it is joined to p_2, it is joined to p_3, oh no, it is joined to, p_6 is joined to p_3, p_6 is joined to p_4, p_6 is joined to p_5, p_1, p_2 are not joined to each other because between the regions F_1, F_2 there is no common edge, they are not adjacent to each other.

(Refer Slide Time: 23:03)

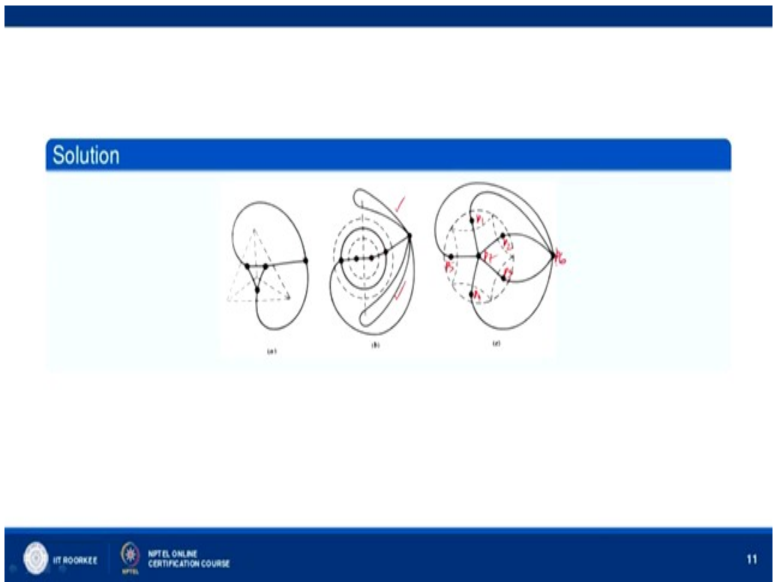




Okay, so we have this graph, this one, see we have, yeah so this is p_1,p_2,p_3, p_4, p_5 and this is p_6 okay and yes yes yes we have I missed one thing, so this is p_6 let us say okay, this is F_6, we have F_1,F_2,F_3, F_4, F_5, this is F_7 you can say, that one was F_6, this is F_7, okay. So, I can take this one as p_7 okay this is, here we have to take, so we can take that as this one okay, that one is joined to, p_7 is now p_7, this F_7 has a common edge with F_1,F_2,F_3, F_4, F_5 so p_7 will be joined to p_1, it will be joined to p_2, it will be joined to p_3, it will be joined to p_4, it will be joined to p_5, okay.

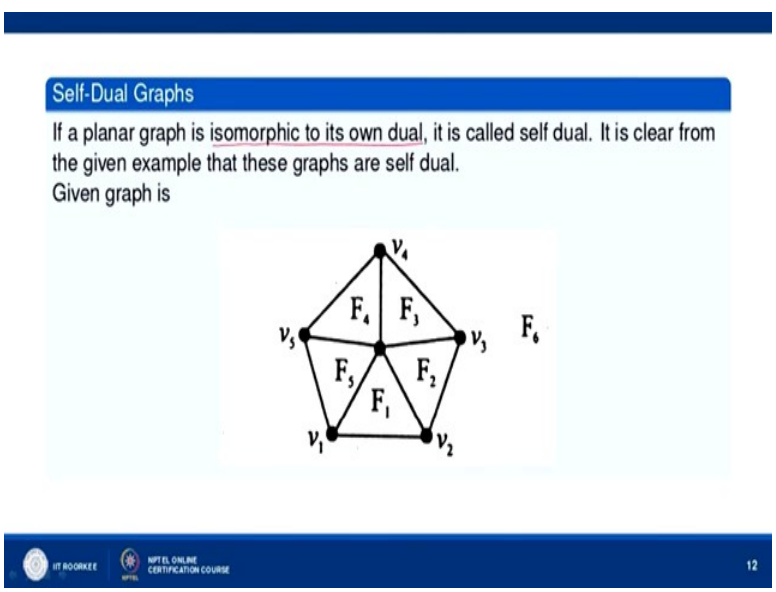
So, I should take this as, if I take this as p_7 then and join like this okay so p_7 is joined to p_1, p_7 is joined to p_2, p_7 is joined to p_3, p_7 is joined to p_4, p_7 is joined to p_5. Now, p_7 is not joined to p_6 because they do not have a common region okay, now p_6, so we have p_6 here, p_6 is joined to p_1, p_6 is joined to p_2, p_6 is joined to p_3, p_6 is joined to p_4, p_6 is joined to p_5 okay like this.

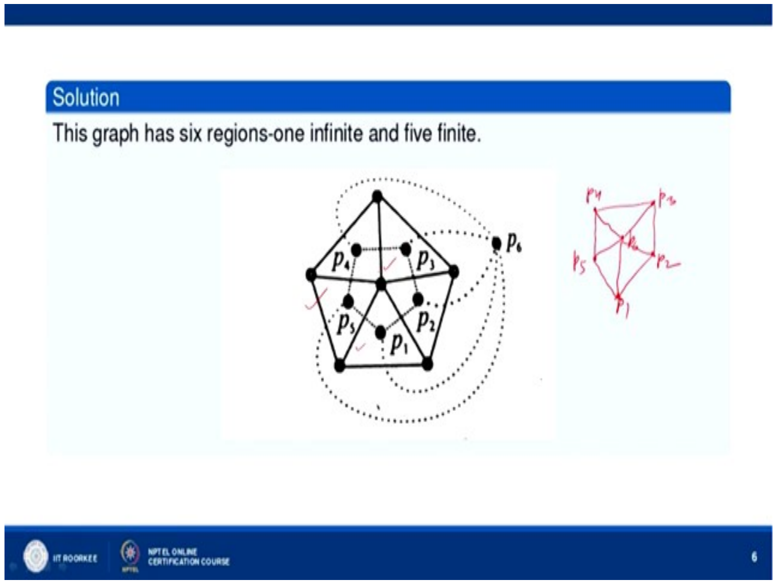
(Refer Slide Time: 25:03)

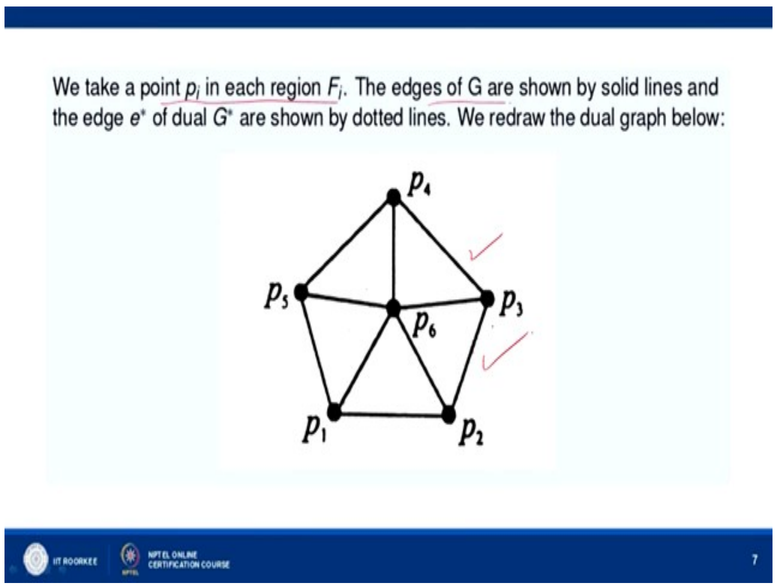


And we can draw this graph in this manner you see okay, this p_1, p_2, p_3, p_4, p_5, this is p_7 and this is p_6, okay so that is the graph which is dual of the given graph.

(Refer Slide Time: 25:24)

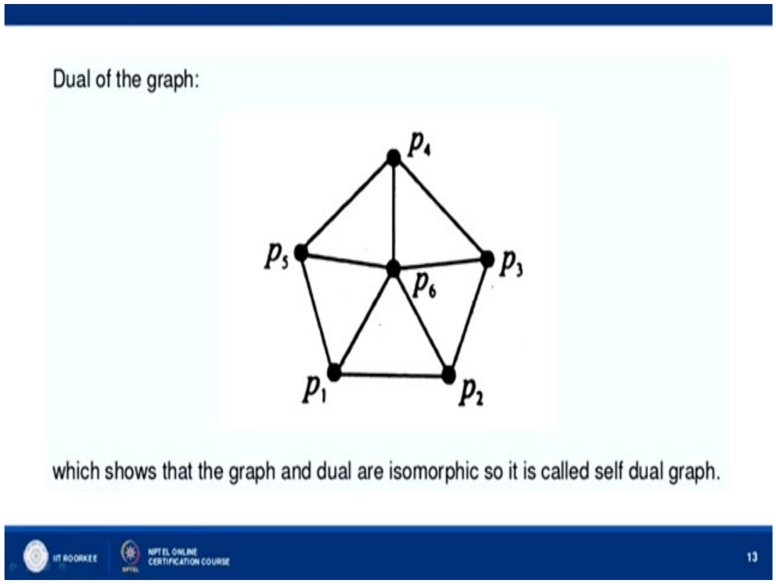


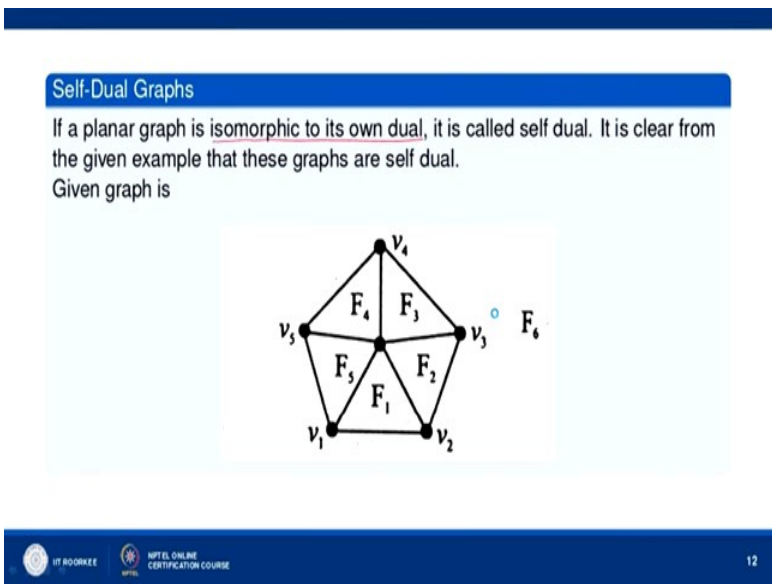




Okay, now self-dual graphs, if a planar graph is isomorphic to its own dual okay it is called self-dual, now it is clear from this example, let us look at this example, this one okay, this graph we had okay we obtain the dual graph like this okay and you can see that this graph, dual graph and this graph they are the given graph, they are isomorphic. So this graph, given graph is called as self-dual okay, if a planar graph is isomorphic to its own dual we will call it as self-dual okay, so the example 1 was a self-dual graph.

(Refer Slide Time: 26:07)





Now, this was the dual so because they are isomorphic so it is a self-dual graph, since this graph is dual of this graph and they are isomorphic okay, the given graph is a self-dual graph. So that is the end of my lecture, thank you very much for your attention.