Higher Engineering Mathematics
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Lecture-35
Kuratowski's Theorem

Hello friends, welcome to my lecture on Kuratowski's Theorem. We have seen that a graph or a multi-graph is planar if it can be drawn in a plane so that its edges do not cross. Kuratowski gave a Theorem which characterises non-planar graphs.

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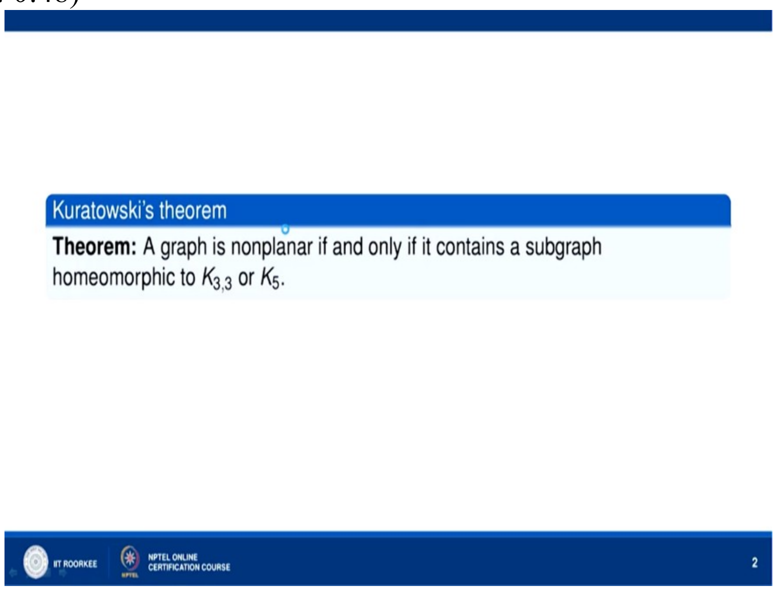

A graph is not planar if and only if it contains a subgraph homeomorphic to K_{33} or K_5. We have also seen that K_{33} and K_5 are nonplanar graphs. Now, so in a way this theorem otherwise can also be interpreted as a graph is planar if and only if it has no subgraph which is homeomorphic to K_{33} or K_5. So, he, Kuratowski gave a characterisation of planar graphs.

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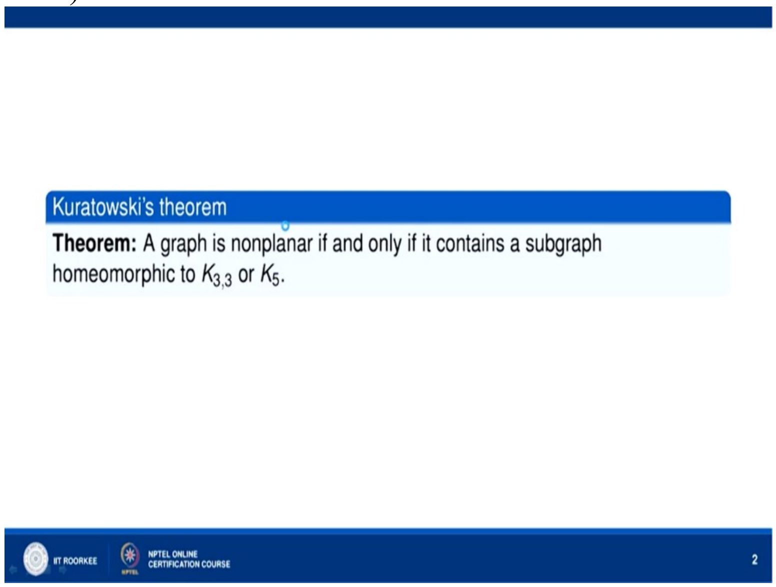

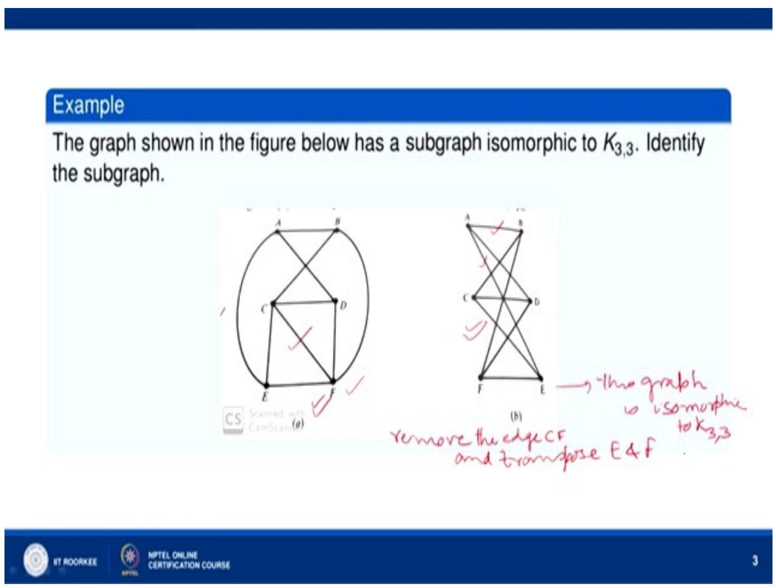

Now, let us see some examples on this Kuratowski's Theorem. Let us look at this example, here we have a graph, this one, this graph we have. We have 6 vertices here, A, B, C, D, E, F okay and they are joined by these edges. Now this graph has a subgraph which is isomorphic to K_{33}. Let us see that that means we want to show that this graph is nonplanar graph, okay. If we can show that this graph has a subgroup which is homeomorphic to K_{33} or K_5 then it will be a nonplanar graph by the Theorem of Kuratowski. Okay. So, let us see how we can do that.

We remove the edge CF Okay, let us remove the edge CF. okay. Let us remove the edge CF from this graph. By removing vertices or edges from a graph we can arrive at a sub graph, we have earlier seen that. So when we remove the edge CF, okay, and arrange E and F as F and E, okay redraw the graph then what do we notice? The graph becomes in this form okay. So let us remove the edge CF This edge, this edge we remove and transpose E and F. okay. Transpose E and F. Okay. So, E vertex now comes here and F vertex now is here. Okay.

And then we see that A which is joined to E okay, it is joined to E like this, okay so this is AE and then A is joined to B, this is joined to B here. Then A is joined to D. This is A joined to D and C is joined to B, C is joined to B here. C is joined to E, okay. This is CE. So we have CE here okay and then C is joined to D. And then EF okay F is joined to D. So this is F is joined to D here. And then F is joined to E, so this is F is joined to E and then F is joined to B, okay this one F is joined to B okay. So F is joined to B.

So D, 6 vertices, A, B, C, D have been partitioned into 2 sets ACF and BDE okay. And each vertex of one set, ACF is joined to every other vertex of the other set BDE okay. So this graph, this graph is K_{33} okay. This graph is isomorphic to, this graph is isomorphic to K_{33}. Okay. So the given graph, okay this given graph has a subgroup, okay the given graph has a subgraph which is isomorphic to K_{33}. Okay. And therefore the given graph has a subgraph which is homeomorphic to K_{33} because by the Theorem, by Kuratowski Theorem, it follows that since this graph is isomorphic to K_{33} okay, we have K_{33} is a nonplanar graph okay K_{33} is a nonplanar graph.

And so it contains a subgraph which is homeomorphic to K_{33} or K_5. Okay so it is this graph is nonplanar K_{33} is nonplanar. So we have a subgraph which is homeomorphic to K_{33} and therefore, this graph okay this graph had a subgraph which is isomorphic to K_{33} and K_{33} has a subgraph which is homeomorphic to K_{33}. So this graph has a subgraph which is homeomorphic to K_{33}.

And therefore this graph is nonplanar, the given graph. We can remove the edge CF and transpose the vertices E and F and arrive at this graph okay which is isomorphic to K_{33} okay. And then use Kuratowski's Theorem to obtain a subgraph of K_{33} which is homeomorphic to K_{33} and therefore the given graph has a subgraph which is homeomorphic to K_{33}.

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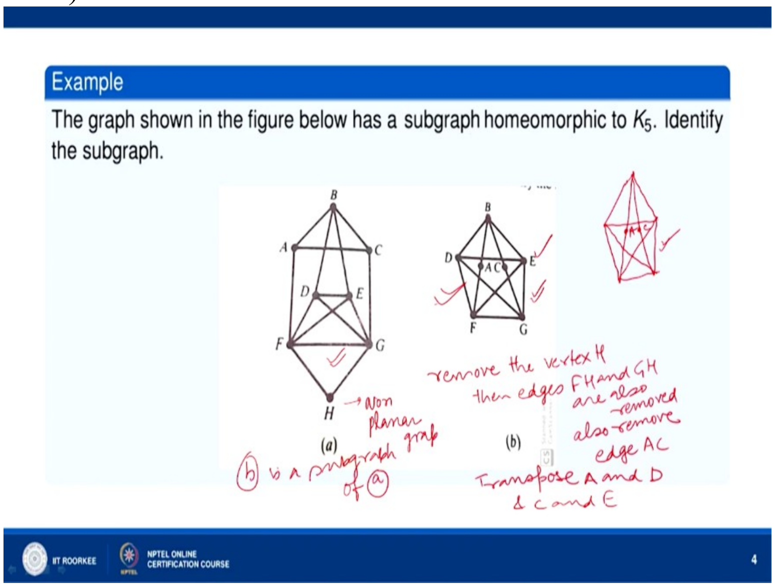

Now let us look at this one, 2nd one. The graph shown in this figure has a subgraph okay homeomorphic to K_5. Identify the subgraph. So what we do here? Let us remove the vertex H, the edges FH and HG. Okay. So let us remove the vertex H okay. When we remove the vertex H, the edges FH and GH will also be removed okay. Then edges FH and GH are also removed. Okay. And then remove this edge CA okay. Remove edge CA, also remove edge AC, okay this AC. Then transpose A and D, okay transpose A and D and C and E okay.

When we transpose A and D, are transposed A and D okay, you see D goes to the position of A and A goes to the position of D. You can see here. Okay. So transpose A and D and C and E. okay. We arrive at this subgraph okay. This is a sub graph okay because it has been obtained by deleting, removing the vertex H. Thereby removing the edges FH and GH and also the edge, AC. Okay. So this is a subgraph of this given graph, of (b) is a subgraph of (a). okay. Moreover, it is homeomorphic to K_5 because in K_5, we have this graph.

K_5 is this graph. Okay. Just K_5. So we are adding two vertices, A here and C here. Okay. When we add the vertices A and C on the edge BF and BG okay the graph that we get, that graph is homeomorphic to this graph. So this, so in K_5 okay we add the vertices A and C to arrive at this graph okay and we know that K_5 is a nonplanar graph, okay. So this graph is a subgroup of this graph. This graph is a subgraph of this graph. Okay. And moreover, this is homeomorphic to K_5.

And therefore, the given graph A has a subgraph which is homeomorphic to K_5 and so the given graph is nonplanar. The graph in (a) is, this graph is nonplanar graph.

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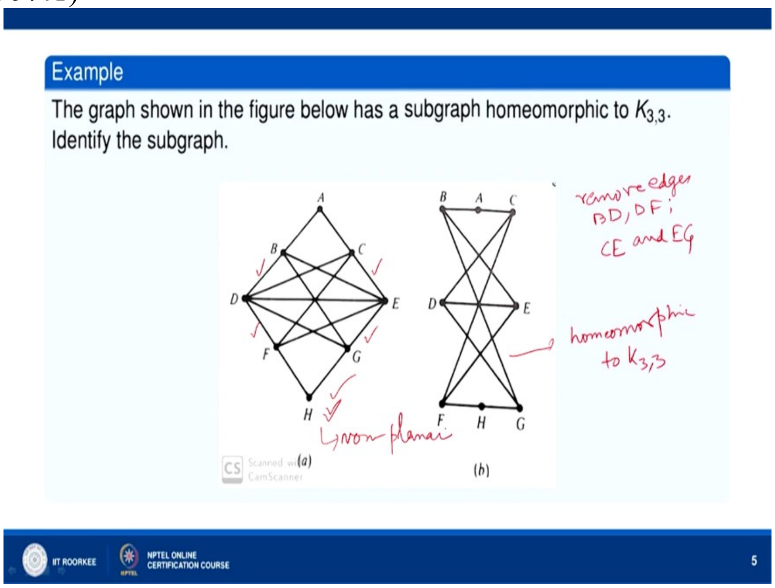

Okay. The graph shown in the figure below has a graph homeomorphic to K_{33}. Let us identify that. Okay. So what we do in this? We delete, we remove the edges BD. We remove edges BD, DF, okay and CE and EG okay. And then redraw this graph okay. okay. After removing the edges BD, this one, BD and DF, CE and EG okay. Let us redraw this graph. When we redraw this graph, it becomes like this. You see BC can be drawn as BC and then the vertex A here okay. Then this FHG can be drawn like this, this FH and then HG. Over this edge FG with put this vertex H, so FHG, we can draw like this and then once we remove BD, DF and CE and EG okay, the graph takes this form.

Now, this graph is homeomorphic to K_{33} okay, so because why it is homeomorphic to K_{33}? Because in the K_{33} graph this A vertex and H vertex are not there okay. The remaining graph is isomorphic to K_{33}. So this graph has been obtained from the graph which is isomorphic to K_{33} by adding vertices A and H and therefore this is homeomorphic to K_{33}. This graph is homeomorphic to K_{33}. So the given graph has a subgraph which is homeomorphic to K_{33} and therefore this graph is nonplanar. Okay.

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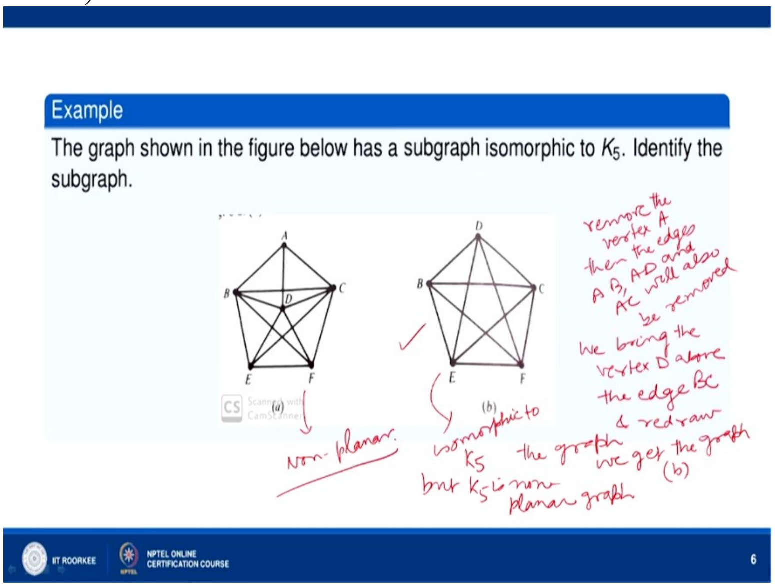

Now, let us look at this graph. The graph shown in the figure below has a subgraph isomorphic to K_5. Okay. So what we do here, let us remove the vertex A and then the edges AB, AC, and AD will be removed okay. So let us remove the vertex A okay. Then the edges AB, AD and AC will also be removed. Okay. And then let us redraw the graph. We bring this vertex D above this edge BC okay. So we bring the vertex D above the edge BC okay this edge above this edge BC and redraw the graph.

We see that we arrive at, we redraw the graph, we get the graph (b). And we know that because here every vertex is connected to every other vertex okay then therefore this graph is isomorphic to K_5. This is isomorphic to K_5. But K_5 is a nonplanar graph. So the given graph is also nonplanar. This is also nonplanar.

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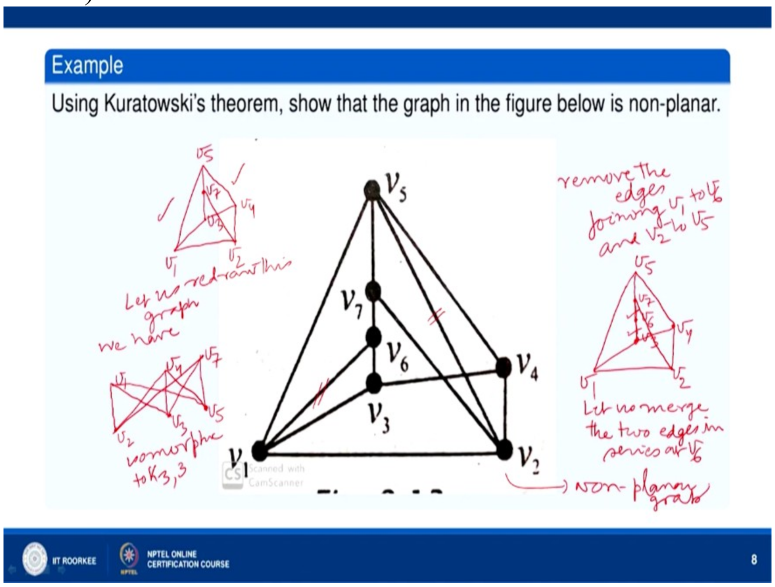

Now, let us use Kuratowski's Theorem to show that the graph below given below is nonplanar. Okay, what we do is let us remove the edges joining v_1 to v_6. Okay. Let us remove the edges joining v_1 to v_6 and v_2 to v_5. Okay. So we remove this edge okay and also the edge joining v_2 to v_5 okay and we arrive at this subgraph. Okay. We have removed v_1 the edge joining v_1 to v_6 okay this edge and we have also removed the edge joining v_2 to v_5 okay. So we have v_1 to v_3, v_3 to v_4, we have v_1 to v_6. v_1 to v_6 we have removed. Okay. And we have also removed v_2 to v_5.

Now, what we do, we merge the two edges, these two edges. There are 2 edges at v_6 okay. Let us merge the 2 edges. Series edges in series at v_6 okay. We have this, these 2 edges, this edge and this edge, they are in series at v_6 . So let us merge these 2 edges which are in series at v_6. We arrive at this graph. So v_5, we have v_1, v_2 here,v_4 ,v_3 , v_1 to v_3 and then and then v_3 to v_4 okay, v_7. Okay. And we have this. So we have this graph okay. When we merge these 2 edges, this one and this one which are in series at v_6 okay we arrive at this graph okay. v_1,v_2 , v_4,v_5 , v_7, v_1, v_3 and yes, this one. Okay. And so, what we notice is that now this can be redrawn, let us redraw this.

We have the following. We have v_1, v_4, v_7,v_2 ,v_3 , v_5. Okay. So we have v_1 v_2, and then we have $v_1 v_3$, and then we have $v_1 v_5$. Okay. Like this. Then $v_4 v_2$, $v_4 v_3$,$v_4 v_5$. And this $v_2 v_7$. Okay $v_7 v_3$ and $v_7 v_5$. You can say here, we have v_7. v_7 is joined to v_5 okay. $v_7$is joined to v_3. v_3 we have okay. And then $v_7$is joined to v_2 that is there. So v_2 should have been there. v_7 is there, yes $v_7$is joined to v_2. So this one here. Okay. That we left out. So v_7 is joined to v_2. v_7 is joined to v_5. v_7 is joined to v_3.

Then v_4, v_4 is joined to v_3 okay. $v_4$is joined to v_5, v_4 is joined to v_2 and then v_1. v_1 is joined to v_5 okay, v_1 is joined to v_3, v_1 is joined tov_2 . So this graph can be redrawn and we see that this graph is isomorphic to K_{33} . This graph is isomorphic to K_{33} and K_{33} we have seen is A nonplanar graph and therefore, the given graph is also nonplanar because it has a subgraph okay with this subgraph which is isomorphic to K_{33} and K_{33} is nonplanar. So this graph is nonplanar.

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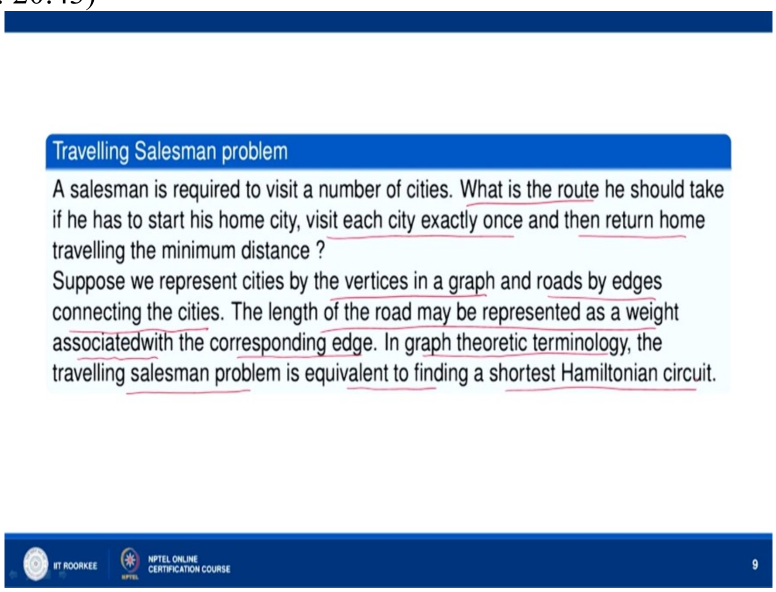

Now, let us consider travelling salesman problem. A salesman is required to visit a number of cities. What is the route he should take if he has to start from his home city, visit each city exactly once and then return home travelling the minimum distance? Okay. So suppose we represent cities by the vertices in a graph and roads by edges connecting the cities. The length of the road may be represented as a weight associated with the corresponding edge. Okay. Now in graph theoretic terminology the travelling salesman problem is equivalent to finding a short test Hamiltonian circuit. A Hamiltonian circuit is, a Hamiltonian cycle is a which visits each vertex exactly once.

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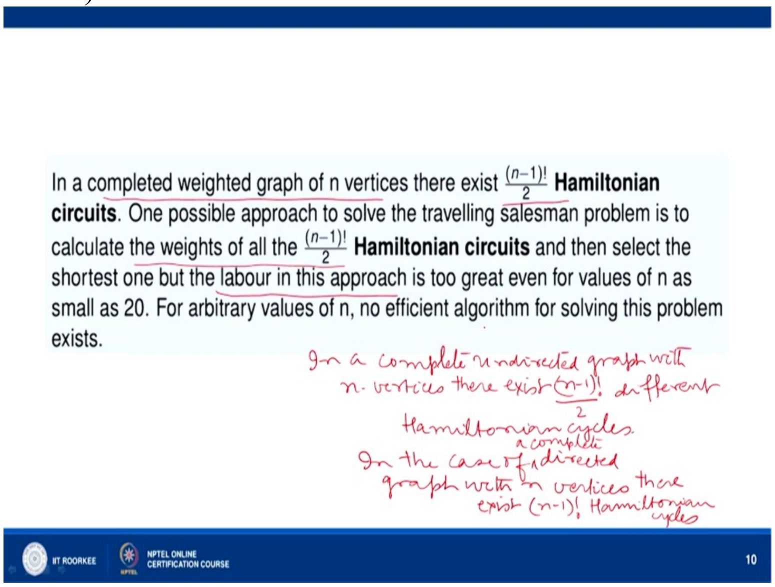

Now, if you take a complete weighted graph of n vertices, then there exists $\frac{(n-1)!}{2}$ Hamiltonian circuits. So if you take n vertices, okay we consider a complete undirected graph, complete undirected graph where n are the number of vertices in a complete undirected graph with n vertices, there are, there exists $\frac{(n-1)!}{2}$different Hamiltonian cycles. If we consider a complete directed graph, then there exists (n-1)! different Hamiltonian cycles because their direction is associated. In the case of undirected, complete graph, there is no direction. So they become half and therefore we get $\frac{(n-1)!}{2}$ Hamiltonian cycles in the case of complete undirected graph okay.

In the case of a complete directed graph with n vertices, there exists (n-1)! Hamiltonian cycles. Okay. So in a complete weighted graph of n vertices, of course here we are talking about undirected graph, okay. There exists $\frac{(n-1)!}{2}$ Hamiltonian circles. Now one possible approach to solve the travelling salesman problem is to consider calculate the weights of all the $\frac{(n-1)!}{2}$ Hamiltonian circles and then select the shortest one that is but in that case what will be this problem will be very difficult to solve because the when n is large, the number of Hamiltonian circuits will be too large.

So it is difficult to determine the shortest path to solve the travelling salesman problem and therefore, because of this (n-1)! factorial, even for values of n which are not very large, it is difficult to solve the consider all the Hamiltonian circuits and thereby determine the the circuit Hamiltonian circuit along which travelling salesman problem has a solution. So it can be done only manually by for a very small value of n okay. For arbitrary value of n, no efficient algorithm to solve this problem is known till today.

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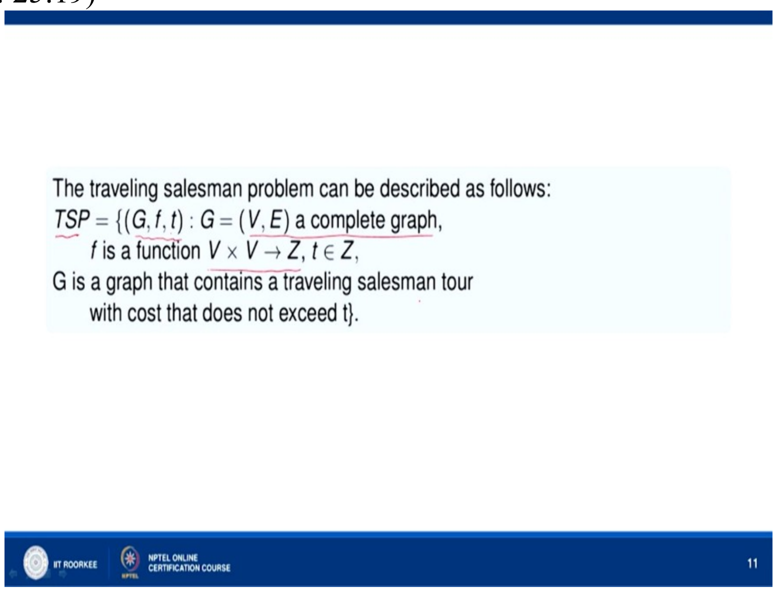

Now the travelling salesman problem can be described as follows. This TSP means travelling salesman problem. It can be written as (G,f,t), where G = (V, E) a complete graph. Okay V is the set of vertices, E is the set of all edges in the complete graph, f is a function from $V \times V$ to Z. okay. $V \times V$ means all the pair of vertices of V. Okay. So f is a function from $V \times V$ to Z okay. And $t \in Z$ okay. So G is a graph. This $t \in Z$, G is a graph that contains a travelling salesman tour with cost that does not exceed t. okay. So G is a graph which contains a travelling salesman tour with cost not exceeding t.

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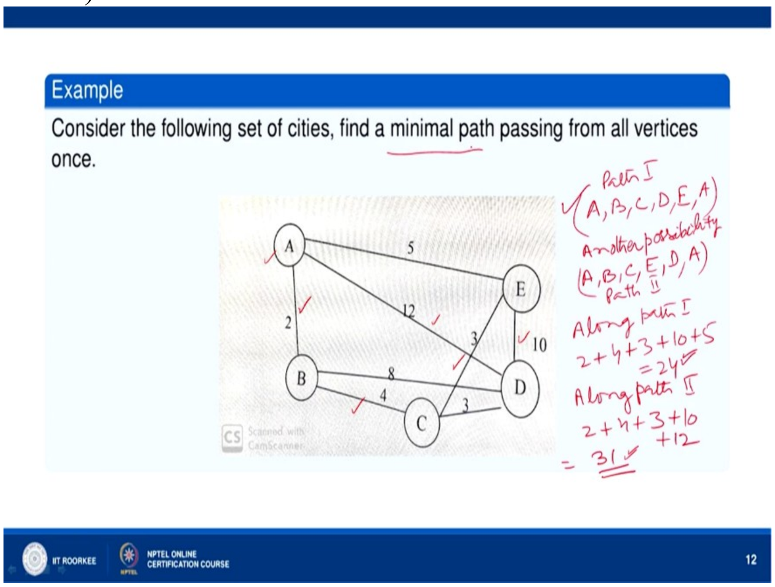

Now let us consider this case, this example. We have , consider the following set of cities, find a minimal path passing from all vertices once. So there are cities presented by A, B, C, D, E, Okay. And the weights are given to each path, okay. To path AB, weight 2 is given, to path BC, weight 4 is given. To path Could, the weight 3 is given. So let us find the path along which the we get the minimum value of this minimum along which the travelling salesman problem have the solution. So let us see, we have 2 paths okay.

Because we have to consider Hamiltonian cycle so we have to consider exact, each vertex exactly once okay while going from along the this graph. Say suppose we start with A okay then we must consider the Hamiltonian cycle ending at A. So let us see which path we can take. We have the path A, B, C, D, A, B, C, D, E, and back to A. okay. This is one path. We can also consider A, B, C, D A, B, C, E, D okay we can add another possibility could be, another Hamiltonian cycle is A, B, C, then we go to E, then we go to D and then we go to A okay.

So if we follow this path, ABC, so what is the total of weights to this path 1 and this is path 2. Both these paths when we follow, we visit each vertex exactly once. So along Path 1, some of the weights is 2 plus 4 okay plus 3 plus 10 plus 5. And we get 2 +4 =6, 6 + 3 =9, 9 + 10 =19 + 5 =24 okay along path 1. And along Path 2 okay, we have the total as 2 plus 4 plus and then from C to E we go. So 3 and then we go to D okay. From E to D. So we get 10. Okay. And then from D, we go to A. So 1, 2, 3, no 1, 2, 3, 4 and then we go to this 5. Okay. So 2 plus 4 plus 3, then

10. I have missed the 3 here. 3 okay. AB, BC, CE okay. Then ED and then DA. So when we follow this path, how much we get? 2 + 4 =6, 6 + 3= 9, 9 +10= 19 okay 9 + 10= 19 +12 this is 31 okay. So here we have 2 Hamiltonian cycles okay. One gives 24 value, another one gives 31 value. So 24 is the smallest okay. Therefore we should follow the path 1. Okay. That is A, B, C, D, E and A okay.

So that is the solution of the, that is the minimal path here that gives the solution of the travelling salesman problem in this case.

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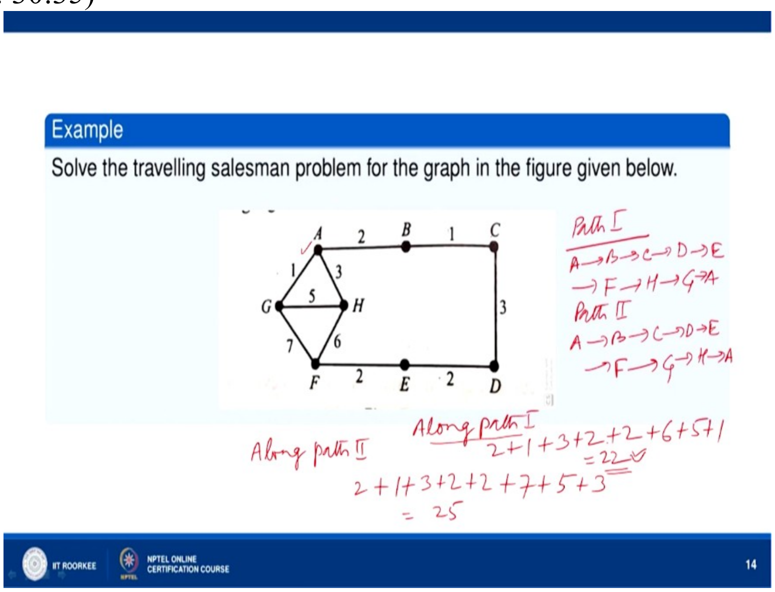

Now let us consider another one. Let us consider this problem here okay. So again we have let us let us start with A okay. So then which path we have to take? Hamiltonian cycle. So path 1. Let us see path 1. So path one could be A to B, B to C, C to D, D to E, E to F, F to H, H to G, okay and G to A. So we can follow this path. Another path could be again A to B, B to C, C to D, D to E, E to F, F to G, G to H, okay and H to A okay. Now let us see along Path 1, what total we are getting. Along Path 1. Sum of weights is 2+1 + 3 + 2 +2 so after D we come to E, from E we go to F, F to H. So we have 6 here. F to H, then H to G so we have 5 and then G to A, so we get 1.

So this 2 + 1 =3, 3 +3 =6 +2 =8 + 2= 10 + 6 =16 + 5 =21 +1= 22 okay. The sum is A to B, so we have 2. Then B to C, we have 1. Then we have C to D, so 3 and then D to E, that is 2. Then E to F, again 2 and then F to G, we get 7. And then we have G to H, so we get 5 and then H to A, so we get 3. Okay. So this is 2 +1 =3+ 3= 6+ 2 =8 + 2= 10+7 =17 +5=22 and 3 =25 okay. So out of these 2 Hamiltonian cycles, the one which gives the least value is the path 1 okay. So we should follow path 1. So that is the solution of this travelling salesman problem.

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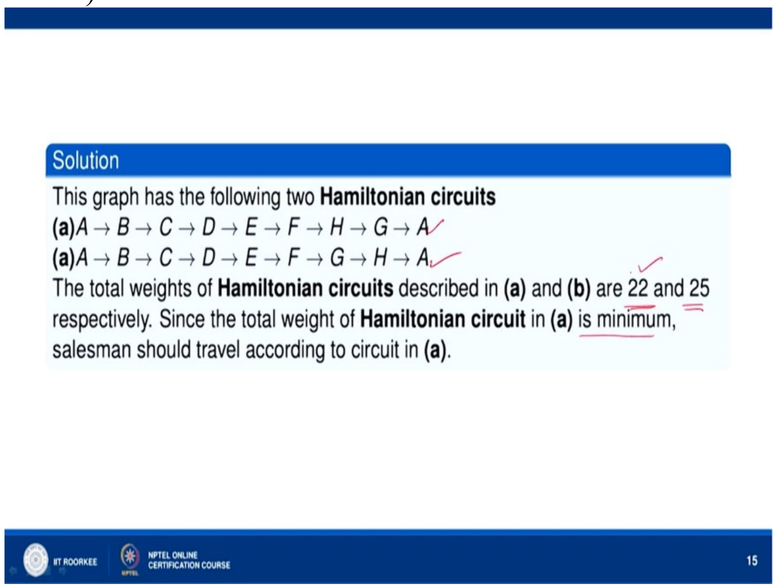

So the total weights of Hamiltonian circuits described in these 2 paths are 22 and 25 okay. So in the total weight Hamiltonian circuit in A is minimum, this one is minimum. The sales man should travel along 2 circuit in A. So that is the end of my lecture. Thank you very much for your attention.