Higher Engineering Mathematics
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Lecture-34
Isomorphic and Homeomorphic Graphs

Hello friends, welcome to my lecture on Isomorphic and Homeomorphic Graphs. Let us first discuss Konigsberg work problem, bridge problem.

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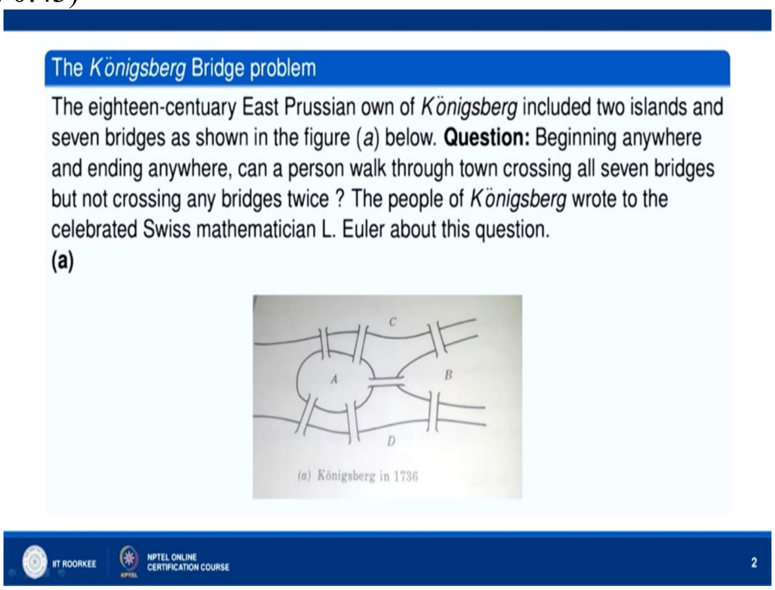

In the 18th century, East Prussian town of Konigsberg included 2 islands. This is island A, this is island B. okay. And 7 bridges shown here in this figure, 1, 2, 3, 4, 5, 6, 7. So there are 7 bridges. Okay. And we have 2 towns, A and B and C and D are the riverbanks. This the river. In the river, there are 2 islands, A and B. There are 7 bridges and the riverbanks RC and D. So the problem is beginning anywhere and ending anywhere, can a person walk through the town crossing all 7 bridges but not crossing any bridge twice? Okay. This problem was asked to a Swiss mathematician, L Euler by the people of Konigsberg.

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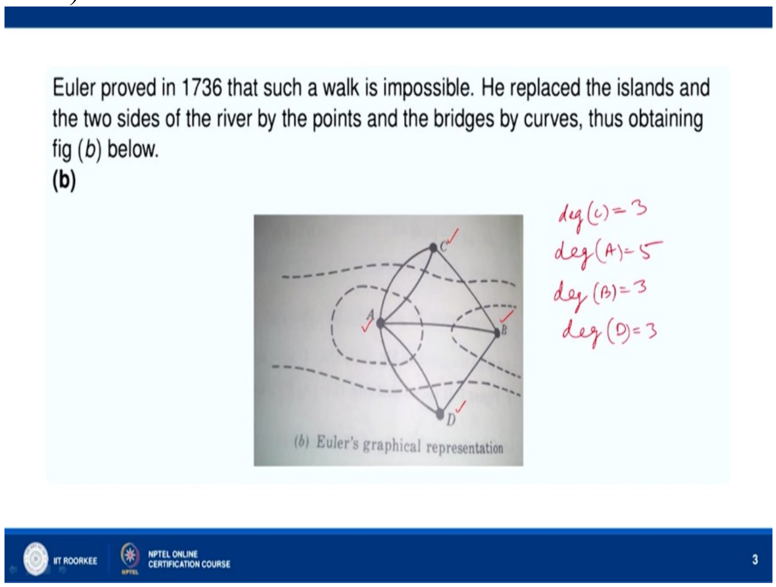

And then Euler in 736 proved that such a walk is impossible. He replaced the island by means of points C and D and the 2 sides islands by means of points A and B. Okay? He represented islands by means of points A and B and the 2 sides of the river by means of points, C and D. And the bridges, he represented by curves. Then he obtained this graph. Okay? This graph he obtained. So in this graph, we can see that all the vertices of this graph, A, B, C, D, you have odd degrees. Okay? degree (C) = 3, degree of (A) =5, degree of (B) = 3, degree of (D) =3.

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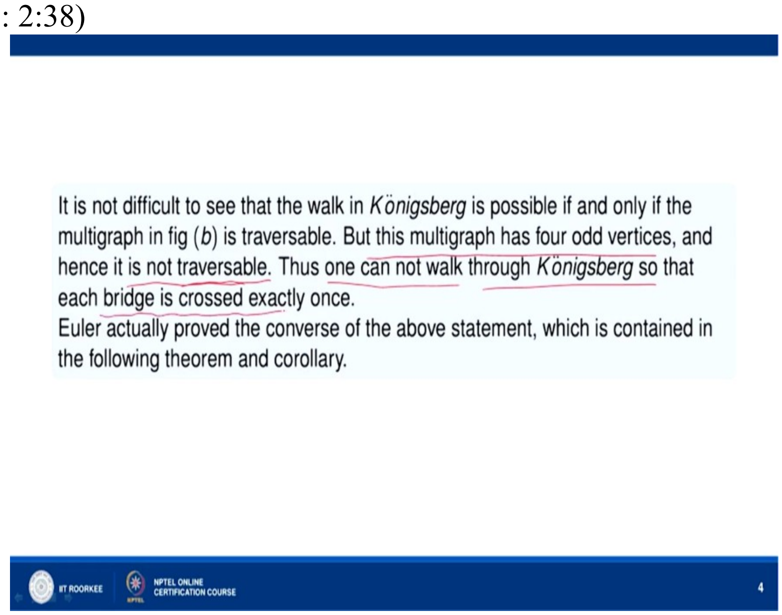

So all the vertices have odd degrees and therefore, this graph is not traversable. Thus, one cannot walk through Konigsberg so that each bridge is crossed exactly once. Okay.

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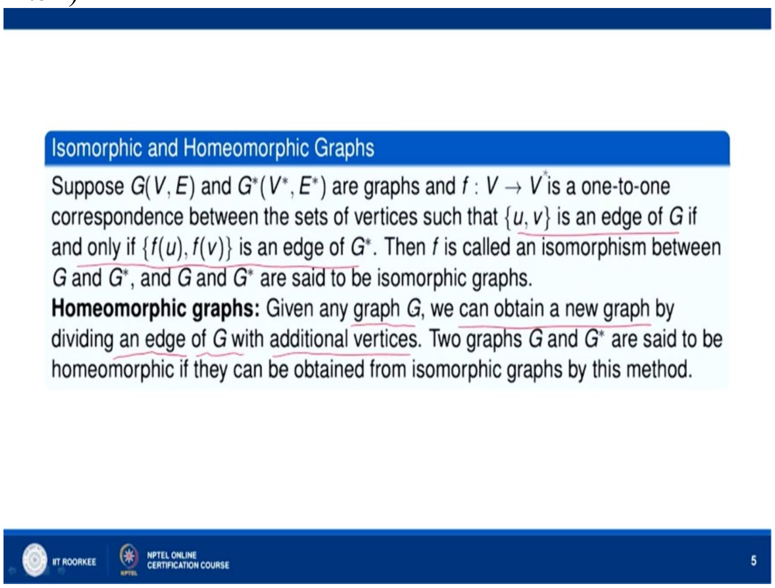

Now isomorphic and homeomorphic graphs. Suppose G(V,E) and $G^{¿}(V^{¿}, E^{¿})$ are graphs and f is a one-one mapping from V to $V^{¿}$ between the sets of vertices V and $V^{¿}$ such that U, V is an edge of G if and only if f(u), f(v) is an edge of $G^{¿}$, okay. Then f is called an isomorphism between G and $G^{¿}$, and G and $G^{¿}$ are said to be isomorphic graphs. Let us also see the definition of homeomorphic graph. Given any graph G, we can obtain a new graph by dividing an edge of G with additional vertices. Okay?

Two graphs G and $G^{¿}$ are called homeomorphic okay if they can be obtained from isomorphic graphs by this method. So if we are given a graph G, we can obtain a new graph by dividing an edge of G by by adding vertices okay with additional vertices. So two graphs G and $G^{¿}$ are called homeomorphic if they can be obtained from isomorphic graphs by this method. By this method means, by adding vertices.

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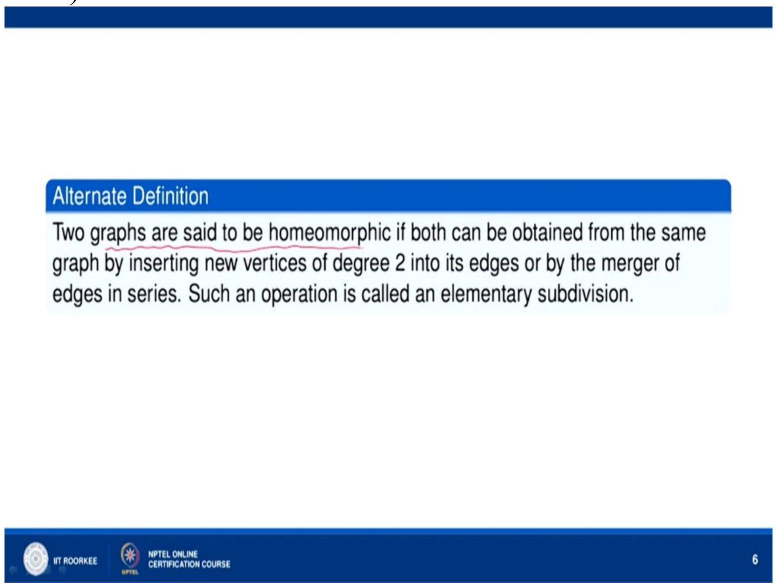

Now alternate definition of homeomorphic graph. Two graphs are called homeomorphic if both can you obtain from the same graph by inserting new vertices of degree 2 into its edges. Okay? So such an operation is called an elementary subdivision.

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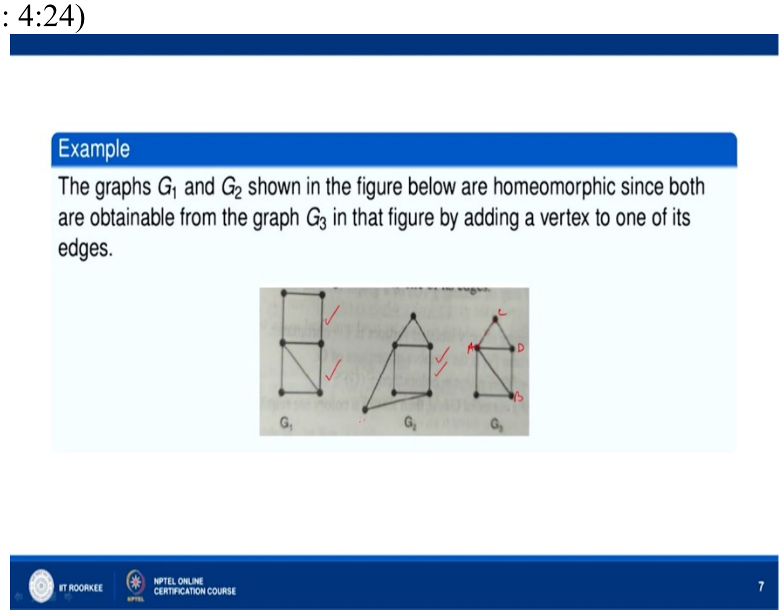

Now, let us look at the graphs, G_1 and G_2 here. This is graph G_1, this is graph G_2. Okay. We will see that both of them are homeomorphic because both can be obtained from the graph G_3 okay, by adding a vertex to one of its edges. If you add one vertex here okay, let us say either on this edge or on this edge of graph G_3 okay, so then you can get this graph, okay. So by adding one vertex either to this edge or to this edge okay we can arrive at this graph G_1 from G_3 okay. And we can arrive at the graph G_2 by adding one vertex over this edge. Okay.

If you add one vertex over this edge, then you can get this, these edges okay these edges and the other edges are as such. So this graph G_2 can be obtained from G_3 by adding a vertex on this edge okay. Let us say this is AB. Adding a vertex on the edge AB and the graph G_1 can be obtained by adding a vertex on the edge AC or on the edge CD okay. So the graph G_1 and G_2 are homeomorphic therefore.

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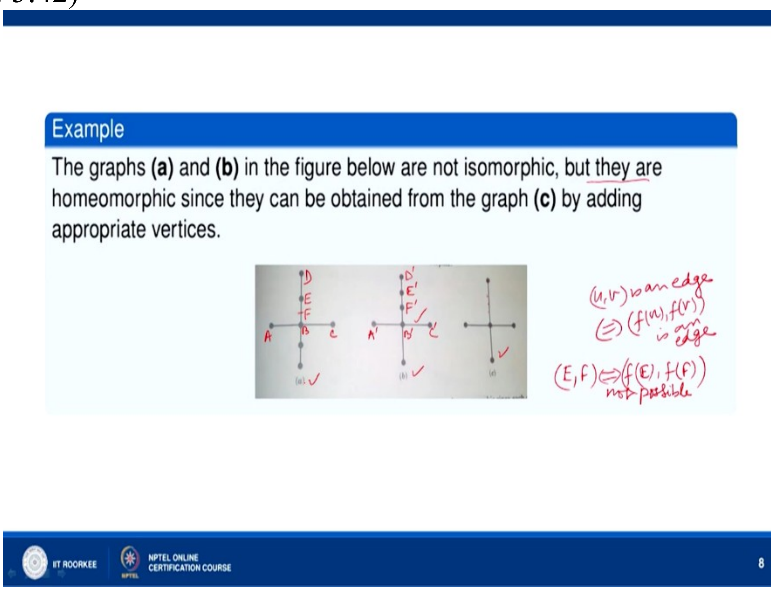

Now, let us look at this figure. Graphs A and B in the figure are not isomorphic. Now you can see, the graphs A and B are not isomorphic. Why they are not isomorphic? Okay. Because see we see that two graphs are isomorphic if (u,v) is an edge in a graph G, if and only if (f(u), f(v)) is an edge of in the graph G dash. So here you see, this is an edge. Okay. Say this is AB, so corresponding to AB we have edge A′B′ okay. Then corresponding to BC, we have B′C′ okay. Now let us see here, we have D okay. So DE, corresponding to edge DE, we have D′E′.

Now corresponding to the edge this one say here we have F′, corresponding to the edge E′F′ okay we do not have here EF okay. Say if you take E′F′ edge, then we do not have EF here okay. The vertices E and F so that uEF corresponds to FE′, FB′, FF′ okay. (u,v) is an edge if and only if (f(u), f(v)) is an edge. Okay. So we do not have a function F one-one onto function F such that EF corresponds to FE, FF okay. FEF corresponds to FEFF okay. So this is not possible. We do not have any function F such that whenever (u,v) is an edge okay of the graph A, (f(u), f(v)) is an edge of the graph B.

So A and B are not isomorphic. No C, they are homeomorphic because they can be obtained from the graph C by adding appropriate vertices. Now, you can see if you look at the graph C, by adding vertices one vertex here, one vertex here, you can get the graph B okay. So by adding two vertices on this edge, we can get the graph B. So B can be obtained from the graph C by adding some vertices and similarly here if you add one vertex here, on this vertex, on this edge and one vertex here on this edge okay, then you get this graph. Okay. So A and B are obtainable from the graph C by adding additional vertices and therefore we can say that the graphs A and B are homeomorphic. Okay.

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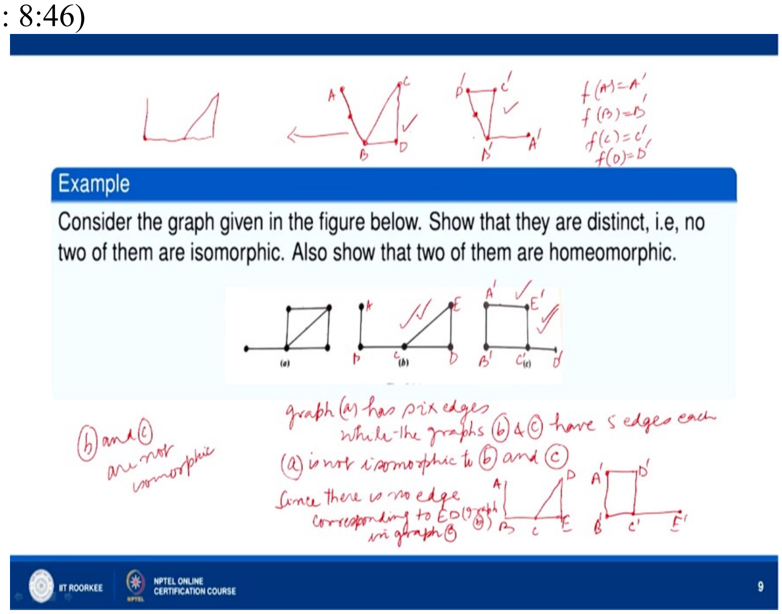

Now, let us consider the graph given in the figure below. So that they are distinct, that is no 2 of them are isomorphic. Now if you can see in the graph A, there are 1, 2, 3, 4, 5, 6 edges okay. Graph A has 6 edges while the graph B and C have 5 edges each. Graph B and C have you can see here, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5. So in the graph A there are 6 edges while in the graph B and

C, there are 5 edges. So A is not isomorphic to B and A is not isomorphic to C. So A is not isomorphic to B and C. okay. Now let us see that B and C are also not isomorphic, okay.

So what we can do let us say this is these are the vertices A, B, C, D. Okay and these are say vertices A′, B′, C′ , and D′ okay. So then we can see AB corresponds to A′ B′ here, BC corresponds to B′ C′ here, CE corresponds to C′ E′ here okay, CD corresponds to C′D′, but there is no edge corresponding to the edge ED here. Corresponding to edge ED, there is no edge here. Okay. So we can say that if you consider CD goes to C′D′, CE goes to C′E′, then ED there is no edge corresponding to ED in the graph C.

Similarly, if you take say for example let us change, we can take another situation. We can consider another situation whereby we can take like this say A, B, C, then we can consider this one, this is another situation which is possible, CD, I can take here, D, I can take here, E, I can take here, then I will have so A′, B′, C′, then we can write say D′. Okay. So CD goes to C′D′ and CE goes to C′E′. Okay. Then ED okay or DE has no edge corresponding to, I mean there is no edge corresponding to DE in the graph C.

So therefore, B and C are not isomorphic. So since there is no edge corresponding to ED in graph C okay there is no edge corresponding to ED, ED of graph B, ED of graph B okay there is no edge corresponding to ED in graph C, B and C are not isomorphic. Okay. But let us show that 2 of them are homeomorphic, okay. So let us see we can consider okay, we can see that B and C can be obtained okay let us see we can let us consider this graph.

This is one graph okay. So, now another one is, first of all we see that these 2 graphs are isomorphic, let us consider this. Okay. 1^{st} we consider, so that this graph and this graph are isomorphic. Say this is A, B, C, D, okay then what we will have? So C goes to, this is C′ okay. She has degree 2 here, so here C′ has a degree 2 okay. And here B has degree 2, so this is B′ we can take and then D. okay. So D has degree 2, so I can write it as D′ and then A has degree 1, so I can write this as A′ okay.

So, let us consider these 2 graphs. Then we can see that there is one correspondence okay. We can define the mapping f okay where F is FA which goes to A′ okay, FB which goes to B′, FC which goes to C′, FD which goes to D′. Okay. A′ here, A has degree 1 here, here A′ has degree 1.

Here B has degree 3. Okay. oh, okay okay, so not like this. This should be B′ here, the degree should be same. So this is B′ and there we shall have C′ okay. C has degree 2 there. So here C′ has degree 2, here B has degree 3, so B′ has degree 3. Okay. D has degree 2 here, so D′ has degree 2 here.

So we can define mapping f which takes A to A′, B to B′, C to C′, D to D′ and F is a one one onto mapping okay. So these 2 graphs are isomorphic and we can see that this graph and this graph can be obtained from these 2 graphs by taking additional vertices. So what we do? You take an additional vertex here. Okay. If you take an additional vertex on AB then you will get this graph okay. This graph will give us this one. Okay. This additional vertex will correspond to this one.

So when we take an additional vertex on the edge AB in this graph, we get this graph, okay. And here, if we take an additional vertex on this B′D′ okay then we will get this graph okay. So the graphs B and C can be obtained from 2 isomorphic graphs okay by taking appropriate vertices and therefore that, therefore we can conclude that the graphs B and C here are homeomorphic.

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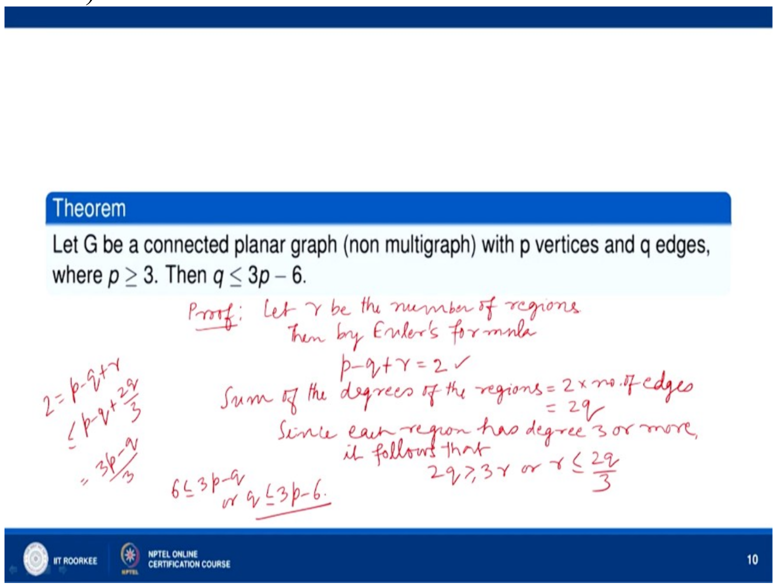

Now, let us consider this result. Let G be a connected planar graph not multi-graph, okay. Let G be a connected planar graph not multi-graph with p vertices and q edges okay where p≥3, then q ≤ 3p-6. So, let us prove this theorem. Let r be the number of regions in this connected planar graph, then by Euler's theorem, Euler's formula $p-q+r=2$ okay. Now since sum of the degrees okay sum of the degrees of the regions is equal to twice the number of vertices, we get sum of the degrees of the regions equal to 2q okay, p is the number of vertices, q is the number of edges, twice q number of edges.

Sum of the degrees of the regions is equal to twice number of edges. So we get sum of the degrees of the regions equal to 2q. Okay. Now, we since each region has degree 3 or more, we know that each region has degree 3 or more, okay, it follows that 2q ≥3r okay. There are r regions, the degree of a region can be 3 or more okay. So and total degree is equal to 2q, so 2q ≥ 3r okay. r. r is less than or equal to $r \leq \frac{2q}{3}$. Okay. Now let us consider this equation, $p-q+r=2$.

So, we have $2=p-q+r$ and $r \leq \frac{2q}{3}$, so $p-q+$¿ $\frac{2q}{3}$, and this gives you $\frac{3p-q}{3}$. Okay so that means 6 ≤3p-q or we can say q ≤3p- 6. Okay. So if G is a connected linear graph, not multi-graph with p vertices and q edges where p≥3, then number of edges (q) ≤3p- 6. So this result we shall use later on.

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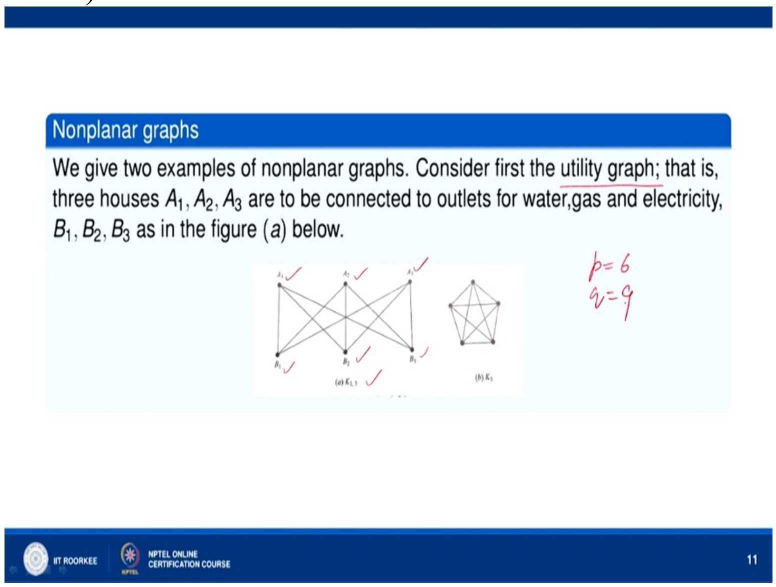

Let us consider now nonplanar graphs. Let us take 2 examples. Look at the example, in this example A, in the example A we have a utility graph okay. There are 3 houses, A_1, A_2,A_3okay which are to be connected to outlets for water, gas and electricity okay B_1, B_2, B_3 as shown in this figure okay. Then in the figure (a) we have 6 vertices and we have 9 edges okay. You can see 1, 2, 3, 4, 5, 6. So 6 vertices are there. p = 6 and number of edges, each vertex is connected to every other vertex. So 3 for A_1, 3 for A_2, 3 for A_3. So q = 9. Okay.

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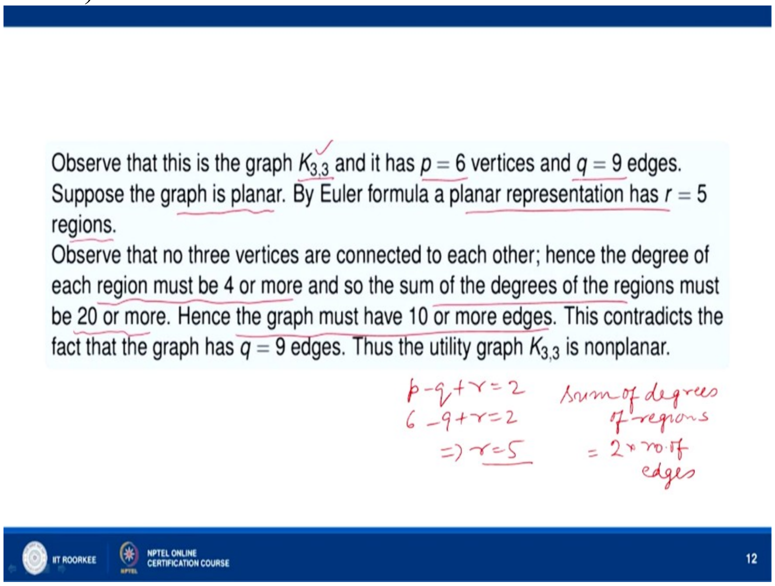

Now, so this is now each vertex is connected A_1, A_2, A_3 is connected to every other vertex. So this is the graph K_{33} okay which has p = 6 vertices and q =9 edges. Now, we have to prove that this graph is not planar graph okay. Let us assume that the graph is a planar graph. Then by Euler's formula, Euler's formula says, p -q +r = 2. Okay. p = 6, q = 9, okay so we will have r =5. So if this graph is planar, by Euler's formula, planar presentation will have 5 regions okay, r= 5. Now we can notice here that no 3 vertices, okay you take any 3 vertices, no 3 vertices are connected to each other.

Say, you can take A_1, A_2, B_1 okay, then A_1 is not connected to A_2. Okay. A_1 is connected to B_1, A_2 is connected to B_2 but A_1 is not connected to A_2. So you can take any of the 3, say A_2, B_2, B_3 you can take. A_2 is connected to B_2, $A_2$is connected to B_3 but $B_2$is not connected to B_3. So no 3 vertices are connected to each other and hence the degree of each region must be 4 or more, okay. If 3 vertices are connected to each other, the degree will be 3. Okay. The degree of the region will be 3 but if the, if there are no 3 vertices are connected to each other, then the degree of the region must be 4 or more.

And so, sum of the degrees of the regions must be 20 or more because there are 5 regions okay. We have seen that r = 5. Each region has degree 4 or more. So sum of the degrees of the regions must be 20 or more and therefore, the graph must have 10 or more regions okay. Sum of the degrees, we have seen that sum of the degrees of the regions is equal to twice the number of edges. So some of the degrees of the regions if it is 20 or more, then the number of region must be, number of edges must be 10 or more okay. And this is what we have here, number of edges are only 9. So there is a contradiction and therefore, the utility graph K_{33} is nonplanar.

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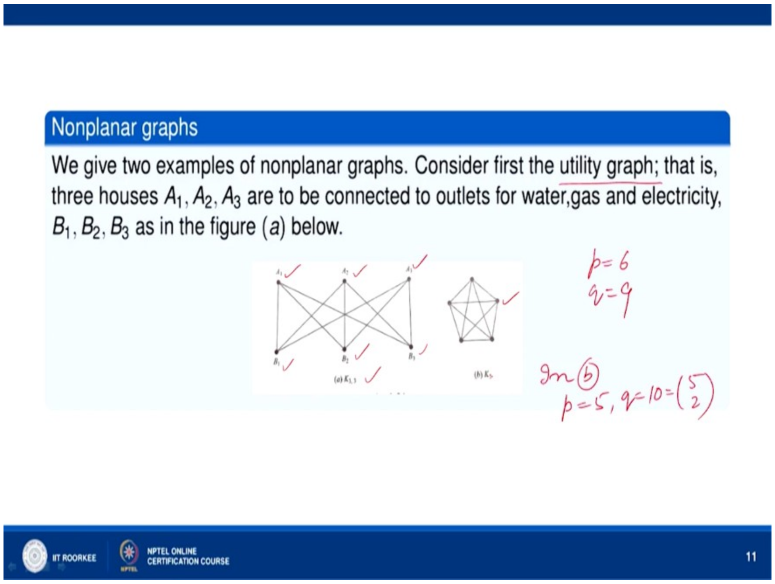

Now, let us consider the star graph here okay. This is star graph, this one. Okay there are 1, 2, 3, 4, 5. 5 vertices in in graph B, p = 5 and q is equal to we have 1, 2, 3, 4 okay and then we have here every vertex is connected to every other vertex. Okay. So this here q is equal to number of vertices that is equal to 10. $5C_2$ okay. This is $5C_2$. Okay. So number of vertices is equal to 10 and since each vertex is connected to every other vertex, it is a K_5 graph.

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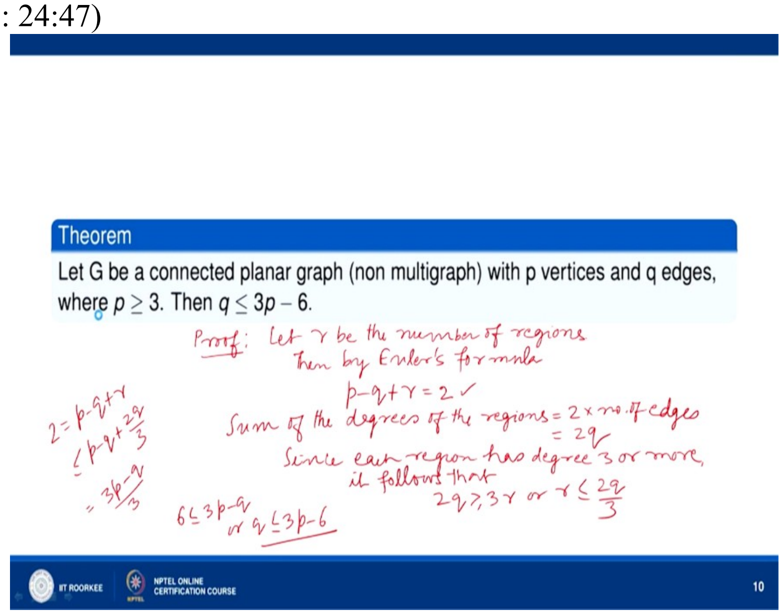

Now that is, it is the complete graph K_5 okay. p = 5, q = 10. Now, let us show that this graph is not a planar graph. Suppose the graph is planar. Then what we will have? Let us apply the theorem which we have just now proved, this one. If the graph is planar with p vertices and q edges, where p ≥3, then q ≤ 3p- 6. So let us see what happens here.

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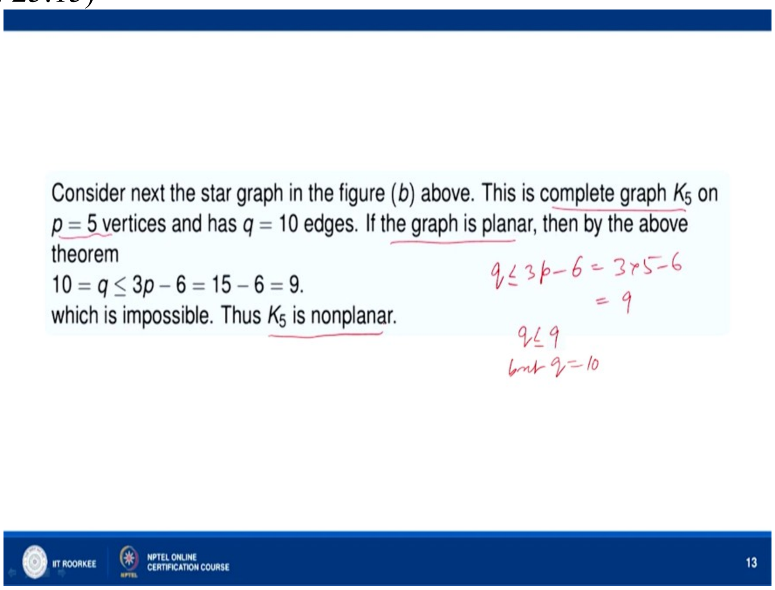

Here we have p =5. So p ≥ 3, condition is satisfied. Okay. So therefore q ≤ 3p- 6 by that theorem, okay per planar graph. So here 3p−¿6 is how much? 3 × 5 - 6. That means 15- 6, that is 9, so q ≤9. But q =10. Okay. So there is a contradiction and therefore the graph is, K_5 is not a planar graph. That is the end of my lecture, thank you very much for your attention.