Higher Engineering Mathematics
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Lecture 32 – Planar Graphs

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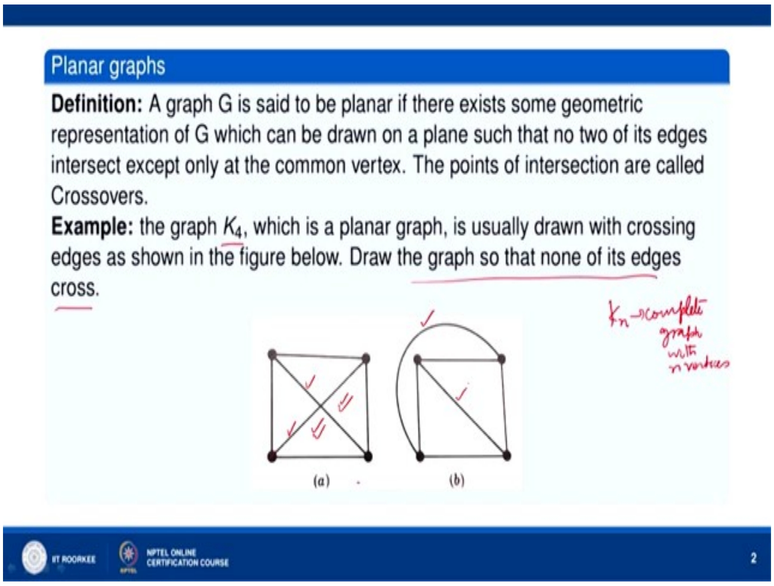

Hello friends! Welcome to my lecture on Planar Graphs. Let us define a planer graph. A graph G is called a planer if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect except only at the common vertex. The points of intersection are called Crossovers. The graph K_4, okay, let us look at the graph K_4. K_4, as we remember, is K_n is called a complete graph with n vertices. And a graph is called complete if every vertex is related to, joined to every other vertex.

So here let us look at the graph K_4. Here you can see this is the graph K_4. Here every vertex is joined to every other vertex. Now it is a planar graph because you can see no two of its edges, now here this is not a planar graph because two edges here, this edge and this edge, they are intersecting each other. But this can be made a planar graph by taking this edge as like this. We can, if we draw this edge like this, then it is, it becomes a planar graph.

So this is a complete graph, this is a graph K_4. It can be made planar graph like this. So the graph K_4 which is a planar graph is usually drawn with crossing edges as shown in the figure below. Draw the graph so that none of its edges cross. So you can see this here in the complete graph K_4, these two edges cross each other, so here we can make it a planar graph when we join this vertex and this vertex in this manner. Then this edge and this edge do not cross each other. So graph K_4 is a planar graph in this case.

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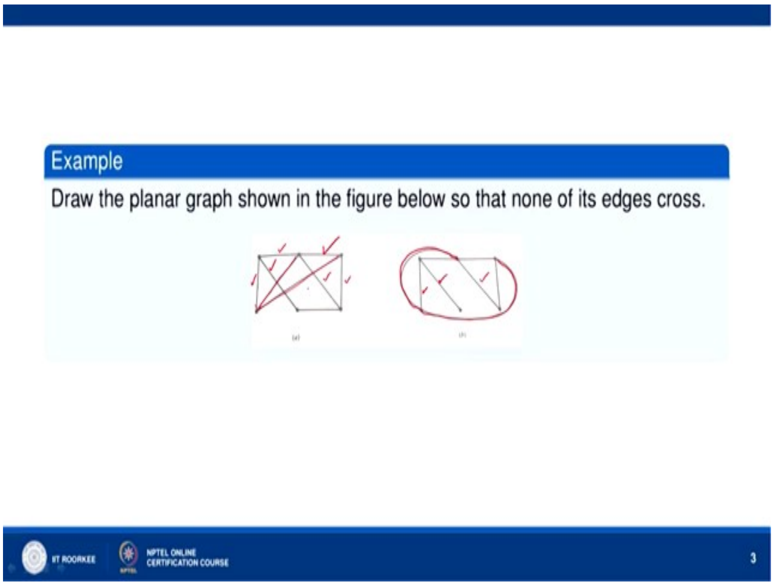

Now let us see this one, this graph. This graph is a planar graph because none of its edges cross. As such we are seeing that its edges are crossing but we can draw it like this. So this, this and this, they are, these ones, this one, this one and this one, they are these ones and this one. And this edge we can join like this. This edge is like this, this is joined like this. And this one, this edge is joined like this. So since this graph can be made in such a way that none of its edges cross, so it is a planar graph.

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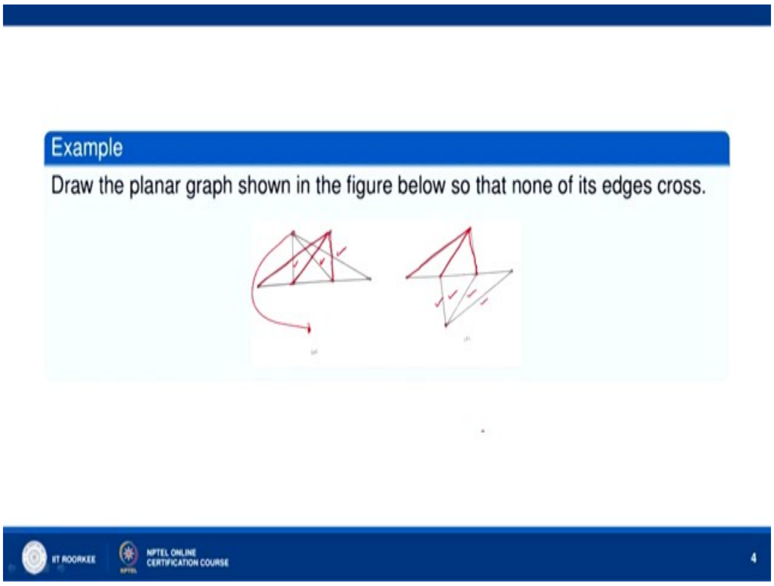

Draw the planar graph shown in the figure below so that none of its edges cross. Now here we can see this, as such we are seeing that the edges cross here each other but it can be made in such a way that its edges do not cross each other. So this, this and this, they are here. And this vertex, let us bring down here so that when it comes down this can be joined to this and this can be joined to this and this can be joined to this. If you draw this vertex down here, then these edges which are joining to it, this one and okay this one and this one, they will come here like this.

And by drawing graph in a different way like this, we see that none of its edges cross. So what do we see here that it looks like that the edges cross each other but they are not crossing each other.

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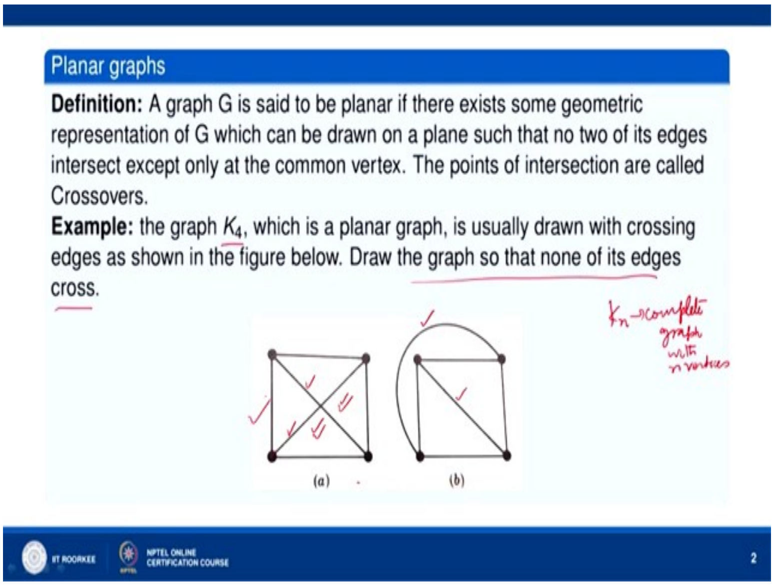

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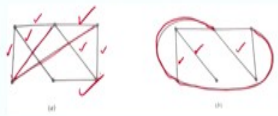
Example

Draw the planar graph shown in the figure below so that none of its edges cross.



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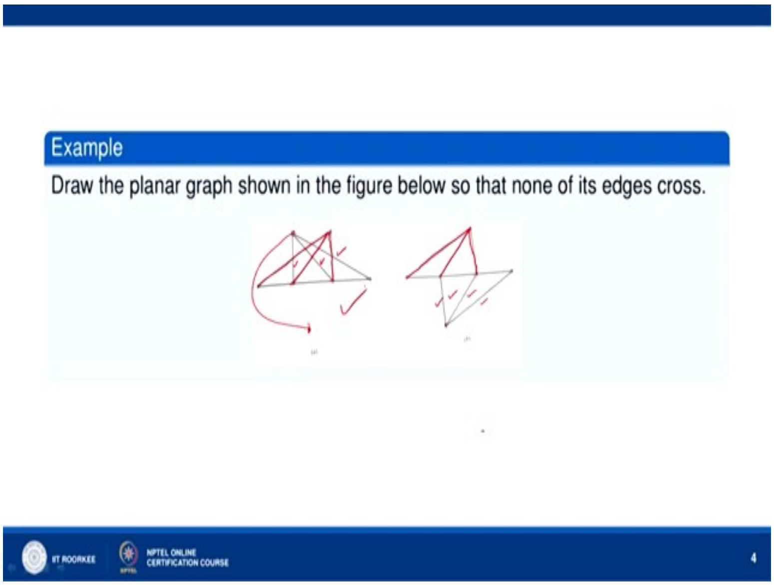

And so in the case of this graph which is K_4, in the case of this graph and in the case of the third one, this one, all graphs are planar graphs. We can draw them in such a way that none of their edges cross.

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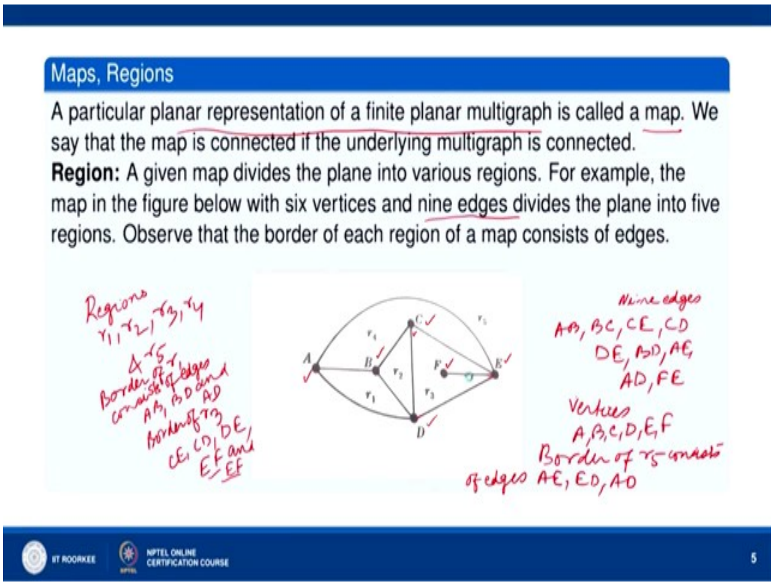

Now let us discuss maps and regions. A particular planar representation of a finite planar multigraph is called a map. We say that the map is connected if the underlying multigraph is connected. Now region, a given map divides the plane into various regions. For example, the map in the figure below with six vertices, you can see there are six vertices, A, B, C, D, E and F. There are six vertices and there are nine edges. Edges are: See, AB, BC, CE, we have CD, we have DE, we have BD, we have AE, we have DE; DE we have taken, AD we have, AD, and then FE.

So 1, 2, 3, 4, 5, 6, 7, 8, 9; so nine edge are there. And there are six vertices. Vertices are: A, B, C, D, E, F. Now this graph divides the plane into five regions. Regions are: r_1, r_2, r_3, $r_4$and r_5. The border of each region of a map consists of edges, we can see that border of r_1. Border of r_1 is border of $r_1$consists of edges AB, BD and AD. If you look at r_2, then the border of r_2 region consists of edges BC, CD, BD.

Likewise, if you see the r_3 region, border of: r_3 , border of $r_3$consists of edges CE, then CD, then DE, then EF, and FE. Because it can be considered as a closed path if you start with same vertex C, then you follow CD, DE, EF, then FE, then EC. We can start with C and reach C. Only what we have to do is we have to move in the direction EF and then again in the direction, then back from F to E. So by following the path CD, DE, EF, FE, EC, we can go to C. And therefore it is a closed path.

So r_3 is bounded by, I mean the border of r_3 consists of edges CD, DE, EF, FE and CE. And similarly if you take say your r_4 ,r_4 is, has border, I mean the edges on its border as AB, AC, AE, CE, and BC. The border of r_4, border of r_4 consists of the edges AB, AE, AB, BC, and CE. And if you look at the border of r_5, border of r_5 consists of edges AE, then ED, and then AD. It consists of edges AE, ED, AD. Now here the edges EF and FE will be considered as same because we are not associating direction with it.

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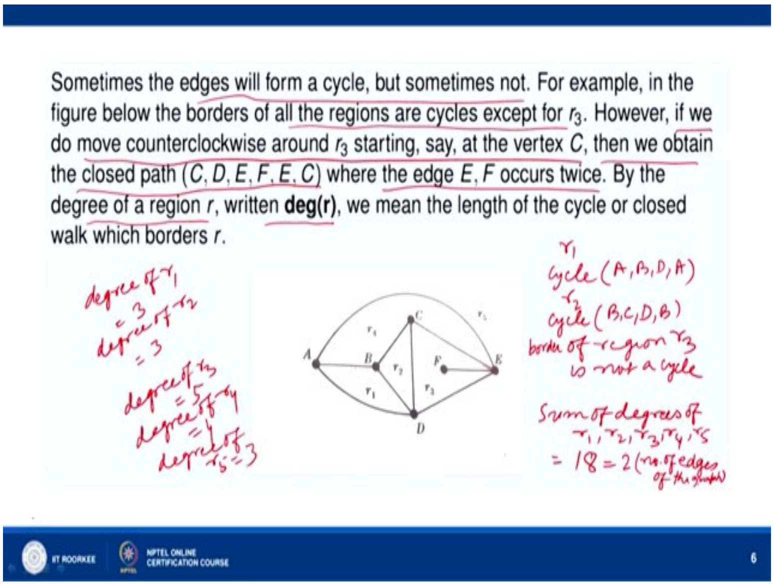

Now sometimes the edges will form a cycle but sometimes not. The edges will form a cycle but sometimes not. For example, in the figure below the borders of all the regions are cycle except for r_3. So let us see how they are cycles. For r_1 , let us see the cycle. So r_1, the cycle is A, B, D, A. For r_2; B, C, D, again cycle, B, C, D, B; because cycle is a closed path. And then we have r_3. In the case of r_3, we have, although it is a closed path but it is not a cycle.

Because we follow CD, then DE, then EF, then FE, so that we cannot do in the case of a cycle, so r_3, the border of $r_3$is not a cycle. While if you take r_4, the region is again a cycle because we can follow AB, BC, CE, and then EA. So all others are cycles. The borders of all other regions are cycles except r_3.

Now however if we do move counter clockwise around r_3 starting, say, at the vertex C, then we obtain the closed path: C, D, E, F, E, C where the edge EF occurs twice. By the degree of a region r, degree of r, we mean the length of the cycle or closed walk which borders r. So let us see we can find the degree of the vertex here, degree of the region. Degree of r_1 is how much? Degree of r_1, you see we have, by the degree of region r, written as degree of r, degree r, we mean the length of the cycle or closed walk which borders r.

So length of the cycle here is what? AB, BD, DA; that is 3. Length of degree ofr_1 is equal to 3. And degree of r_2, region $r_2$is also 3; B, C, D, B. Degree of r_4, now we mean the length of the cycle or closed walk. So here in the case of r_3 so CD, then DE, then EF, then FE, then EC. So we have 1, 2, 3, 4, 5; so degree of r_3, okay we are writing degree ofr_3 , degree of r_3 is equal to, yeah so, CD, DE, EF, FE, that means we are going from E to F and then F to E, so 1, 2, 3, 4, 5.

And then we have degree of r_4 , so AB, BC, CE, and then EA, so 1, 2, 3, 4. And degree of r_5, you see r_5, AD, DE, EA, so 3. Now some of the degrees of the regions is how much? Some of degrees of r_1, r_2, r_3, r_4 ,r_5 is equal to 3 + 3 = 6; 6+5, 11; 11+4, 15+3=18. And number of edges here are, number of edges here is 9. So it is twice the number of edges, two times number of edges of the graph. So in the case of graph, the sum of the degrees of the regions of the graph is equal to twice the number of edges of the graph.

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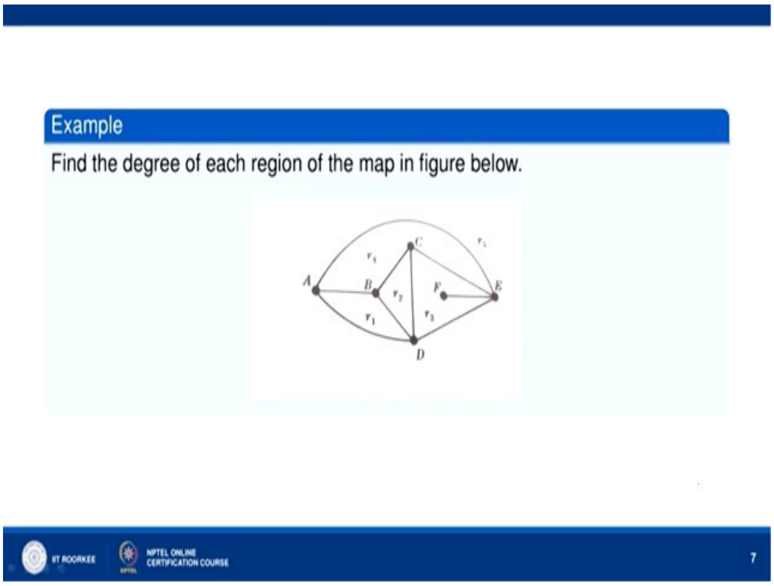

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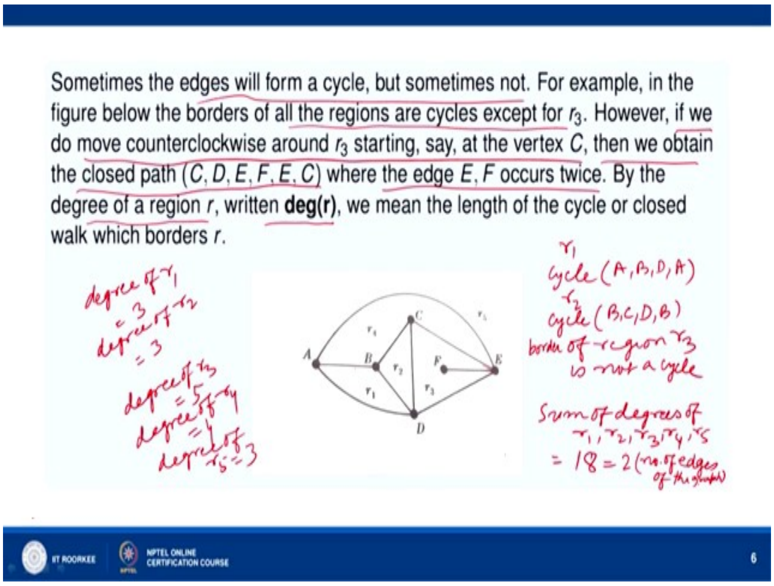

So we have this. Find the degree of each region of the map. In the figure below we have found the degrees of the various regions here. The same graph we have here.

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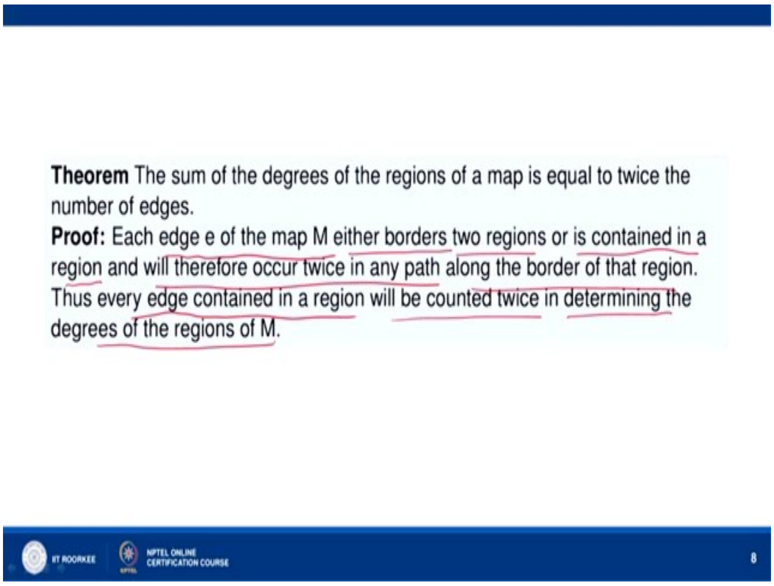

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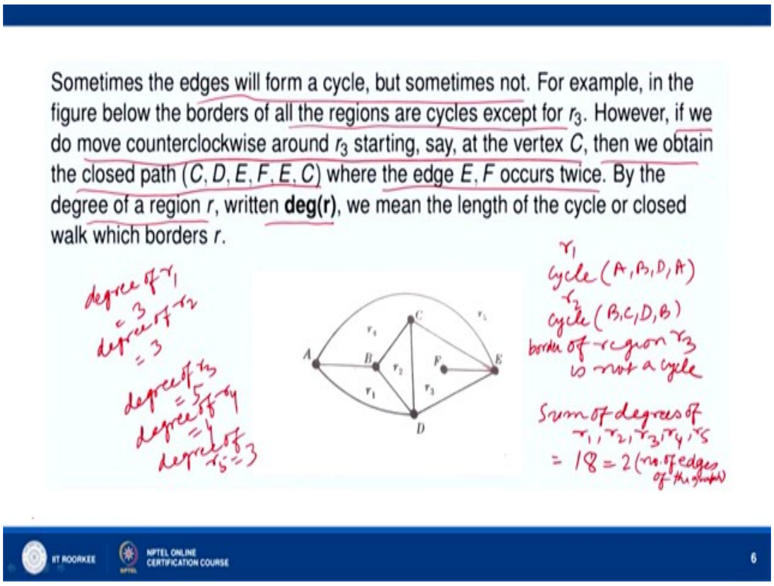

And then the some of the degrees of the regions of a map is equal to twice the number of edges, that we have verified here for this case. Now let us see the proof: Each edge e of the map either borders two regions or is contained in a region and will therefore occur twice in any path along the border of that region.

For example, here you can see this one, this BC you can see, BC is a part of the boundary of this r_4 and also it is a part of the boundary of r_2. So it will be counted twice towards the sum of the degrees of the regions. And then you see, if an edge is contained in a region like FE, FE is contained in this region, then again it is counted twice. You can see here. We moved from E to F, then back from F to E, so it is counted twice. So in order to count the, find the degree of r_3, we had to count EF twice.

So each edge e of the map M either borders two regions or is contained in a region and will therefore occur twice in any path along the border of that region. Thus, every edge contained in a region will be counted twice in determining the degrees of the regions of M. And so some of the degrees of the regions of a map is equal to twice the number of edges.

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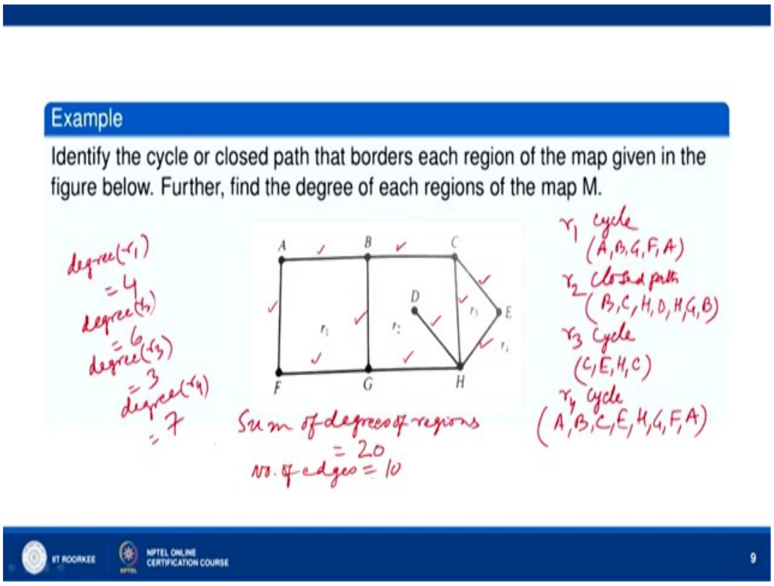

Now identify the cycle or closed path that borders each region of the map given in the figure below. Further, find the degree of each regions of the map. So let us first find for $r_1$what is the cycle. r_1, okay, cycle is A, B, G, F, A. And for r_2, for the region $r_2$we have closed path because DH is contained in the region r_2. So we see, so it is a closed path. It is not a cycle. We have B, C, H, D, H, G, B. That is the boundary of region r_2.

And then r_3, so we have cycle: C, E, H, C. r_4, we have taken r_1,r_2 , r_3, now this isr_4 , so we have A, B…It is again a cycle, and we have A, B, C, E, H, then G, then F and then A. So this is the…Now for degree, as regards to degree, degree of r_1 equal to 4; 1, 2, 3, 4. Degree of r_2 , 1, 2, 3, 4, 5, 6; so we have 6 because DH has to be moved twice. So it is 6. Degree of r_3 ; 1, 2, 3. And degree of r_4, so AB, BC, CE, so 1, 2, 3, 4, 5, 6, 7.

So sum of degrees will be equal to, sum of degrees of regions of this map is equal to 4 plus 6, 10; 10 plus 3, 13, plus 7, 20. And edges are how many? Number of edges? So 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. Number of edges are 10. So sum of degrees of the regions of this map is equal to twice the number of edges.

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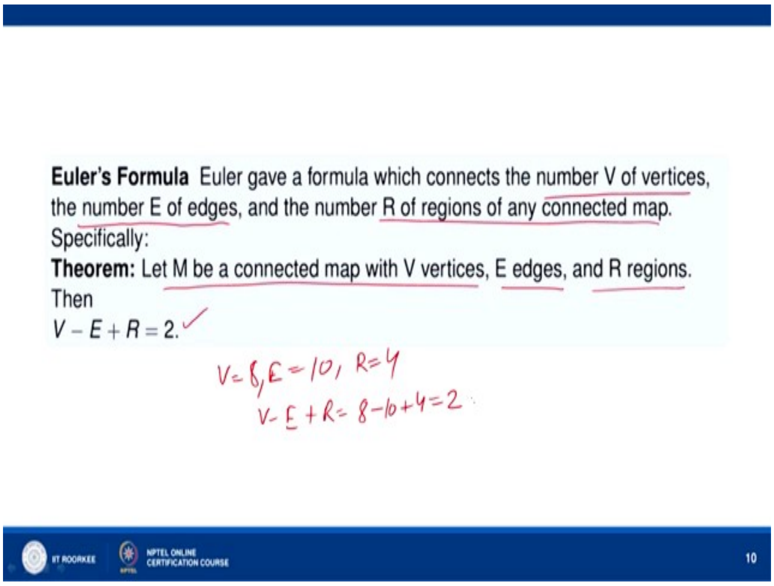

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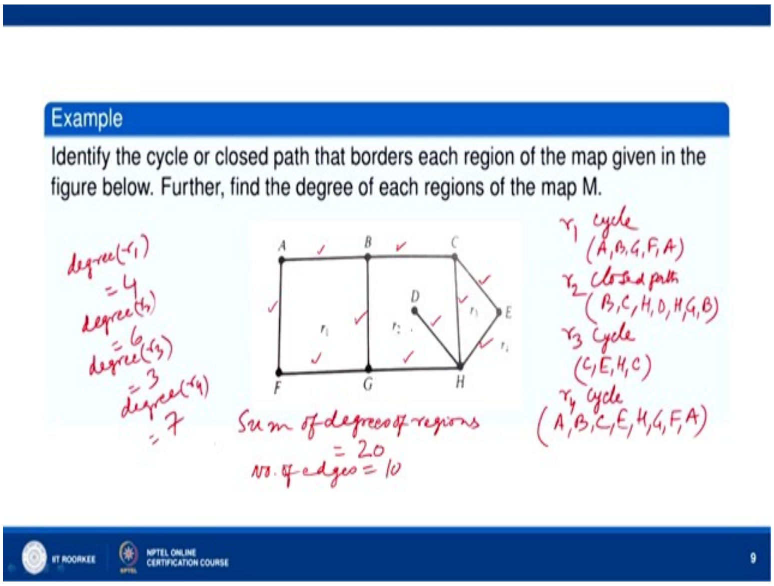

Now Euler's Formula, Euler gave a formula which connects the number V of vertices, the number E of edges, and the number R of regions of any connected map. Specifically, he proved that let M be a connected map with V vertices, E edges, R regions; then V minus E plus R equal to 2. We can verify this formula for the map we have just now discussed.

So V is the number of vertices. So there are how many vertices? 4, okay, 1, 2, 3, 4, 5, 6, 7, 8. Eight vertices are there, so V equal to 8. And then E, E is the number of edges. Okay, number of edges are 10, so E equal to 10. And R is number of regions. There are 1, 2, 3, 4; 4 regions are there, so R equal to 4. So V minus E plus R equal to 8 minus 10 plus 4, equal to 2. So that this example verifies this theorem.

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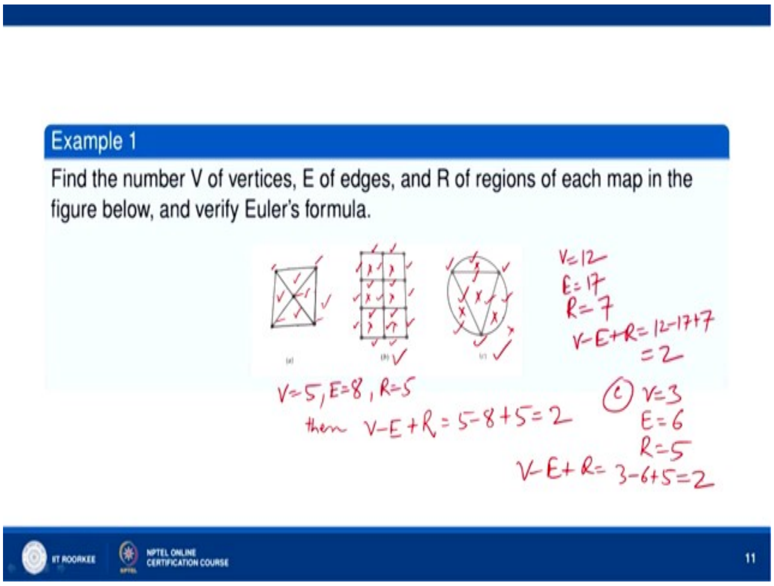

Find the number V of vertices, E of edges, R of regions of each map in the figure below and verify Euler's formula. Now let us see here. So there are five vertices; 1, 2, 3, 4, 5; V equal to 5. And then number of edges, number of edges are 1, 2, 3, 4, 5, 6, 7, 8; so we have edges equal to 8. Regions are 1, 2, 3, 4 and one region outside, 5; so R equal to 5. So then V minus E plus R, edges are….So 5 minus 8 plus 5 equal to 2. So that verify the formula.

And here how many vertices are there? 1, 2, 3; 3, 6; 3, 9; 12. So V equal to 12 for the part (b). And then edges are how many? 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17. So, 17, so edges are 17. And then regions are how many? 1, 2, 3, 4, 5, 6, and one region outside, so 7, so what do we have here? V minus E plus R equal to 12 minus 17 plus 7, so that is equal to 2.

And then for this, so vertices R for part c, vertices are 1, 2, 3, there are three vertices. And E is equal to 1, 2, 3, and this one 4, this one 5, this one 6; so we have six edges. 1, 2, 3, 4, 5, 6. And then regions are 1, 2, 3, 4, and one outside, 5. So we have V- E +R =3- 6+ 5, and we get 2. So these maps verify the Euler's Formula.

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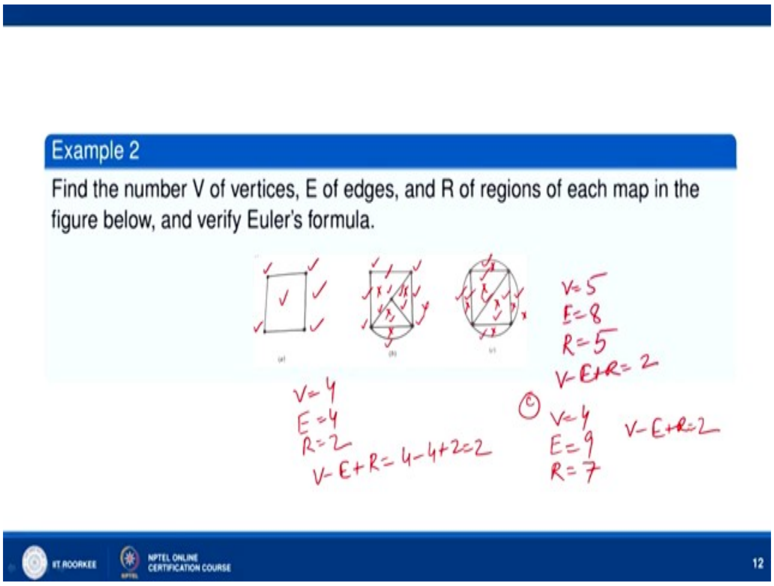

Find the number V of vertices, E of edges, R of regions of each map in the figure below and verify Euler's formula. For example, this one here we have four vertices, so V equal to 4, E equal to 4 because there are 4 edges. Regions are two, one here, one outside, so R equal to 2. We have four vertices, four edges and regions are two. So V - E + R, yeah, 4- 4 + 2 = 2.

Now here 1, 2, 3, 4, 5; V equal to 5. And E, edges are 1, 2, 3, 4, 5, 6, 7, 8; 8 edges are there. And regions are 1, 2, 3, 4, and 5; so we have V -E + R = 2. And now let us take the last case. We have 1, 2, 3, 4; V equal to 4 in part (c). And we have E equal to 1, 2, 3, 4, 5, and then 6, 7, 8, 9; nine edges are there. And regions are 1, 2, 3, 4, 5, 6, and one outside 7. So we get V-E + R = 2. So this also verifies the Euler's Formula. So that is all in this lecture. Thank you very much for your attention.