Higher Engineering Mathematics
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Lecture 31 – Eulerian and Hamiltonian Graphs

Hello friends! Welcome to my lecture on Eulerian and Hamiltonian Graphs. Let us first define an Eulerian graph. A graph G is called an Eulerian graph if there exists a closed traversable trail, called an Eulerian trail. Now let us recall the definition of an Eulerian, let us recall the definition of a traversable graph. A multigraph is called traversable if it can be drawn without any breaks in the curve and without repeating any edge. That is, if there is a path which includes all vertices and uses each edge exactly once.

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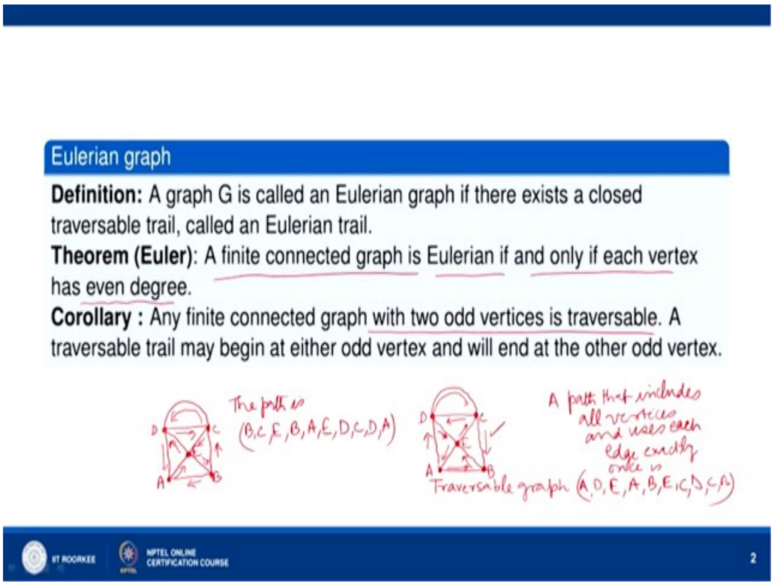

Say for example, we can consider this multigraph. Let us consider this multigraph. We are going to show that it is a traversable graph. So we must show that there is a path which includes all vertices of this graph and uses each edge exactly once. The vertices are, five vertices are there: A, B, C, D, E. So let us show that it is a traversable graph. So we will show that there is a path which includes all vertices and uses each path exactly once.

So let us follow this path—AD, then DE, then we follow EA, and then we go from A to B, and then we go BE, then EC, then CD and then DC and then CB. So a path we can take as, we can similarly start with the point B instead of the point A and end at A. So a path that includes all vertices and uses each edge exactly once is: A, D, E, then EA, then AB, then BE, then EC, then CD, then DC and then CB.

So A to D, then D to E, then E to A, then A to B, then B to E, then E to C, then C to D, and then D to C and then C to B. So this is one path. And you can see there is a path which includes all the vertices, five vertices; A, B, C, D, E and also uses each edge exactly once. The other path could be, the other path can be taken as: We can move along BC, BC, then CE, then EB, then BA, then AE, then ED, then DC, then CD, and then DA. So this another path which begins at B and ends at A.

And you can see both A and B are odd vertices, vertices of order three each. So any finite connected graph with two odd vertices is traversable. A traversable trail may begin at either odd vertex and ends at the other odd vertex. So here we started at the odd vertex A and ended at the other odd vertex B. Here we can start at the odd vertex B and end at the odd vertex, can end at the other odd vertex A.

So the path here is: B, C, E, then B, CE, EB, BA, then AE, then ED, then DC, then CD and then DA. So since we are able to find a path which includes odd vertices and uses each exactly once, so this graph is a traversable graph.

Now there is a theorem by Euler—A finite connected graph is Eulerian if and only if each vertex has even degree. So there is necessary and sufficient condition for an Eulerian graph which is given by Euler, it says that a finite connected graph is Eulerian if and only if each vertex has even degree. So we can see whether a given graph is Eulerian or not by using this condition.

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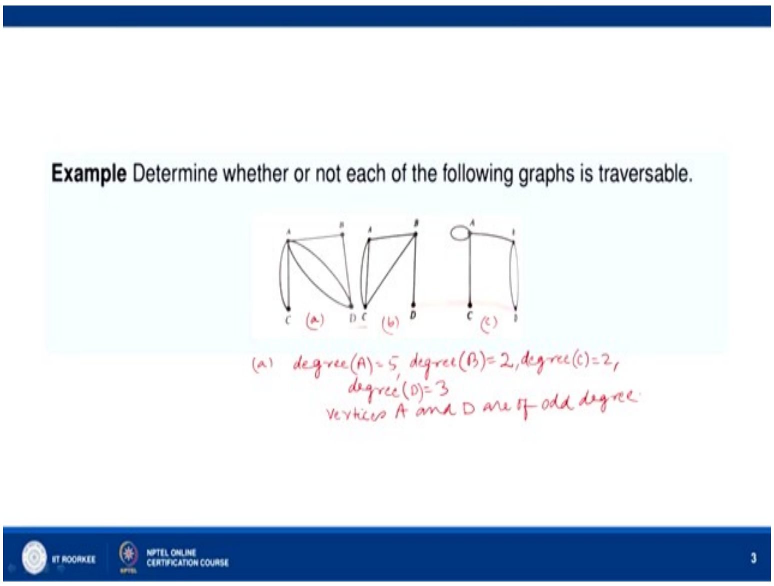

Now let us see, we can see whether each of the following graphs is traversable or not. So let us see, let us note the degrees of each of these graphs. So this is let us say part a, this is b, this is c. So in part a, degree of A, degree(A) = 5, degree(B) =2, degree (C) =2. And

degree (D)=3. So there are two vertices with odd degree. Vertices A and D are of odd degree.

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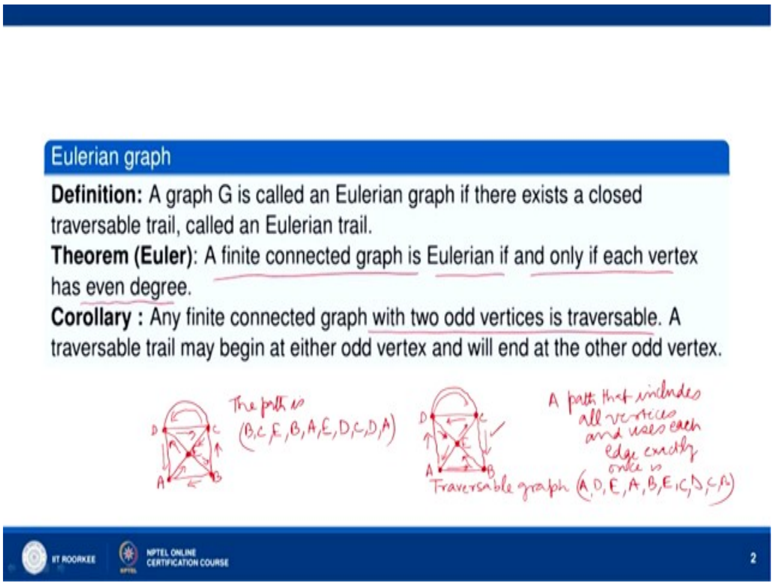

Now let us see, a finite connected graph is Eulerian if and only if each vertex has even degree. So each vertex must be of even degree. A finite connected graph with two odd vertices is traversable.

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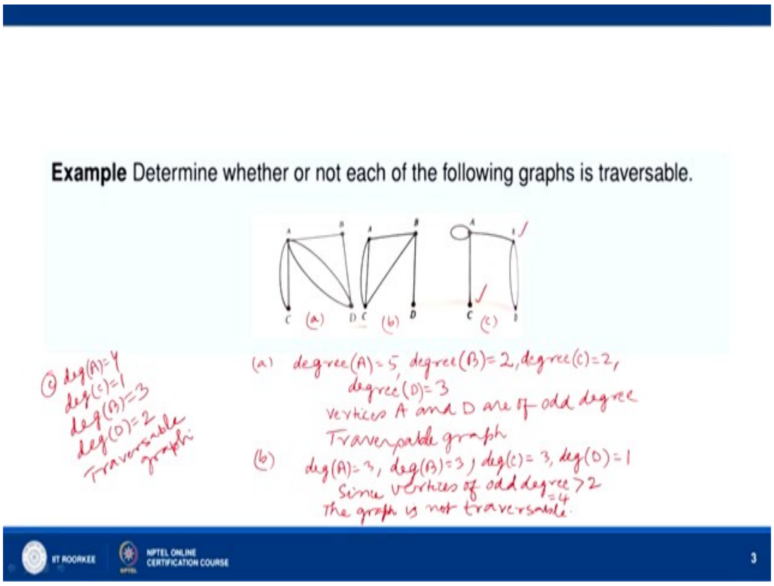

So here we have two odd vertices and therefore this is a traversable graph. The graph in part a is a traversable graph. Now in the case of part b, the degree(A) =3. Degree(B) =3 And degree(C) =3, degree(D)=1. So all vertices are of odd degree. Now there are more than two vertices. So this graph has more than 2 odd vertices, therefore it is not a traversable graph.

So since vertices of odd degree exceed 2, vertices of odd degree are 4. Vertices of odd degree are 4 which is greater than 2, the graph is not traversable. Because if there are more than two vertices which are of odd degree, the graph cannot be traversable. So the graph is not traversable. Now let us see the part 3. In part 3, the degree (A)=4. So degree(A) =4, degree (C) = 1.

degree(B)=3 and degree(D) = 2. So there are two vertices B and C which are of odd degree. So there are two vertices which are of odd degree. This graph is traversable. So the graphs in part a and c are traversable while the graph in part b is not traversable.

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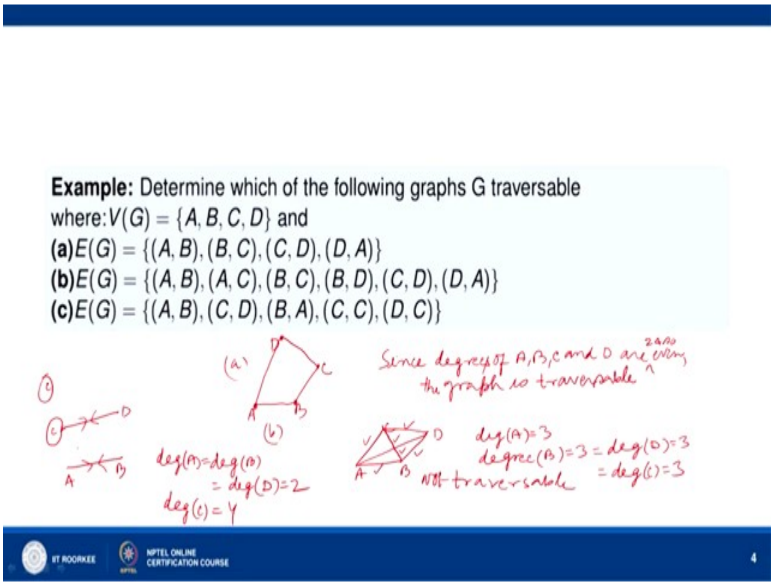

Now let us determine which of the following graphs G are traversable. So V(G) are A, B, C, D. E(G) is set of edges of the graph G are AB, BC, CD, DA. So AB, BC, CD and then DA. So now the degree of the vertex (A) = 2, degree(B) = 2, degree(C)=2, degree(D) =2. Since degrees of A, B, C, and D are even, are 2 the graph is traversable.

The graph is traversable because we have said that, okay, we can also say a finite connected graph with two odd vertices is, either with zero vertices it is traversable or with two odd vertices it is traversable. So here it is a graph which is of zero odd vertices. So since the degree of A, B, C, D are all even, the graph is traversable. Now let us see the part (b). In part (b) we have AB, AC, BC, BD, CD, DA. You see, AB, then AC, then BC, this is BC, then we have BD, then we have CD, we have CD here and then DA.

So now let us see what are the degrees of the vertices here. degree(A)=3. degree(B)=3. degree (D)=3. And degree(C)=3. So all the vertices are of odd degree, that is 3 and therefore it is not traversable. Okay, now let us see the part (c). We have AB, CD, BA; so AB, BA. Okay, CC, so we have a loop here, CC. At C there is a loop and then we have CD, then D. Now you can see each vertex here, A is of degree 2, B is of degree of 2, C is of degree 2, D is of degree 2.

Now degree(A) =degree(B)= degree(D)= 2. While degree(C)=4 because 2 for loop and 2 for CD and DC. So degree(C) = 4. Now degrees of all the four vertices are even but the graph is not traversable because it is not connected. It is not connected graph, so it cannot be traversable.

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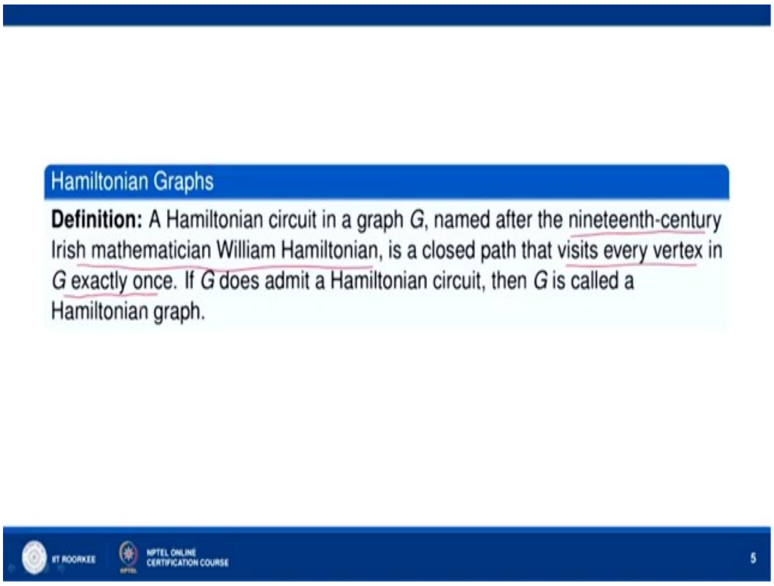

Now let us discuss Hamiltonian Graphs. A Hamiltonian circuit in a graph G, named after the nineteenth-century Irish mathematician William Hamiltonian, is a closed path that visits every vertex, that is that includes every vertex in G exactly once. If G does admit a Hamiltonian circuit, then G is called a Hamiltonian Graph. So here what is the main thing that it is the closed graph and it visits every vertex in G exactly once. In the case of Eulerian graph, you use any edge only once. Here we have, we use any vertex only once. You can skip edge here.

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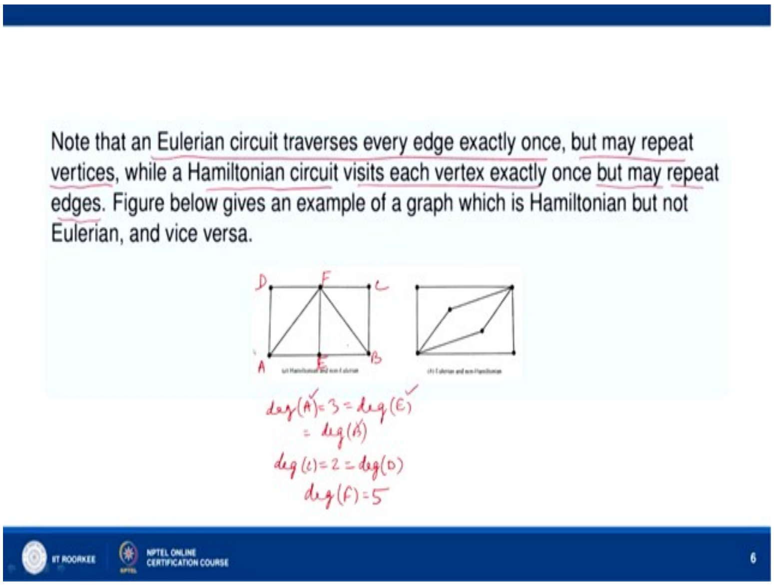

So let us say for example, an Eulerian circuit traverses every edge exactly once but may repeat vertices. In the case of Eulerian graph, we traverse every edge exactly once but we may repeat vertices. In the case of Hamiltonian circuit, we visit each vertex exactly once but may skip edges. We may skip edges. So let us see this figure here, the first figure. So this figure is a graph which is Hamiltonian but not Eulerian.

Now let us see how it is not an Eulerian graph. So let us say this is A, B, C, D, E and F. Then, if you want to show that it is an Eulerian graph, we must show that one thing that is, that we have for an Eulerian graph: A graph is Eulerian if and only if each vertex has even degree. So let us see whether each vertex has even degree here.

You can see, yeah, this vertex, each vertex does not have an even degree here. You can see, degree(A)=3 here, degree(E) =3 here, degree(B)=3 here and degree(C)=2, degree(D) =2. And degree (F) =5, okay. So by the Euler theorem which shows that graph is Eulerian, connected graph is Eulerian if and only if it has even degree, each vertex has even degree. So here the vertices A, E and B do not have even degrees. Therefore, this graph is not Eulerian. Now let us see whether it is a Hamiltonian graph.

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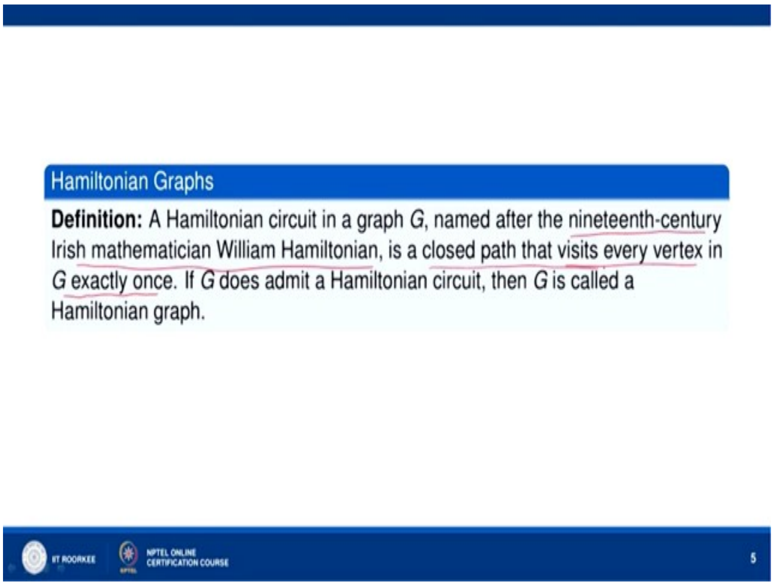

So for Hamiltonian graph what we have said ?. It is a closed path that visits every vertex in G exactly once and it can skip edges.

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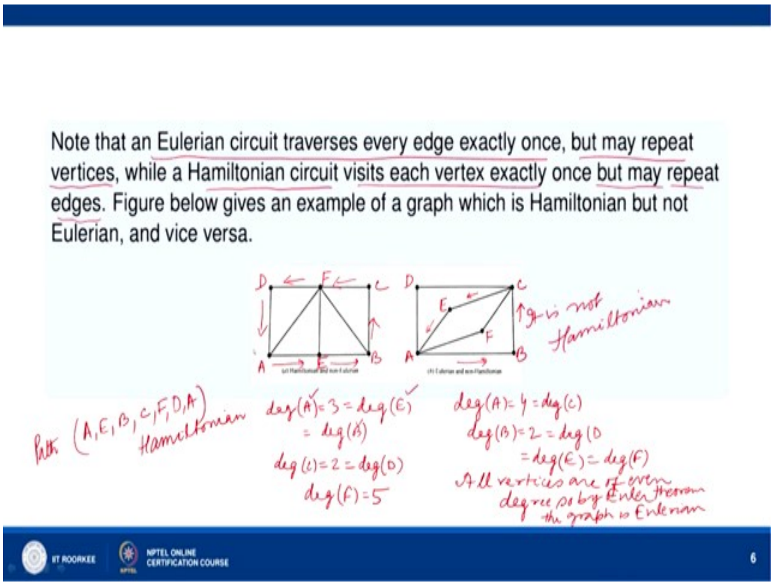

So we can follow the path: A, E, B, C, F, D, A, this path. We go from A to E, E to B, B to C, C to F, F to D and then D to A. So it is a closed path and uses every vertex only once. It skips edges AF, EF and BF. So we have skipped the edges AF, BF and EF here. But we have found a closed path which includes every vertex exactly once, so it is a Hamiltonian graph. Now let us look at this. It is an Eulerian graph but it is not Hamiltonian.

Let us see why it is Eulerian. Let us say this is A, B, C, D, E, F. Then degree(A) =4.

degree (B) = 2; degree(C)=4. degree(D)=2. degree(E)=2 and degree (F)=2. So all vertices have even degrees. All vertices are of even degree and so by Euler theorem, it is Eulerian, the graph is Eulerian.

Now let us see why it is not Hamiltonian. You see, we cannot find any closed path which includes all the vertices, A, B, C, D, E, F even if we skip edges. We can follow any closed path, we will not be able to get all vertices, a path which includes all vertices exactly once. So let us say if we go from A to B, then we go from B to C, then we come from C to E and then we come from E to A. So we are using the vertex A twice. If we go from A to B, then B to C, then C to F, again we will have to come to A. So we are not using exactly once every vertex.

If we can go from A to B, then B to C, then C to D, then D to A, again we are not getting a path which is using every vertex only once. So this is not a Hamiltonian graph. It is not Hamiltonian. It is not Hamiltonian graph.

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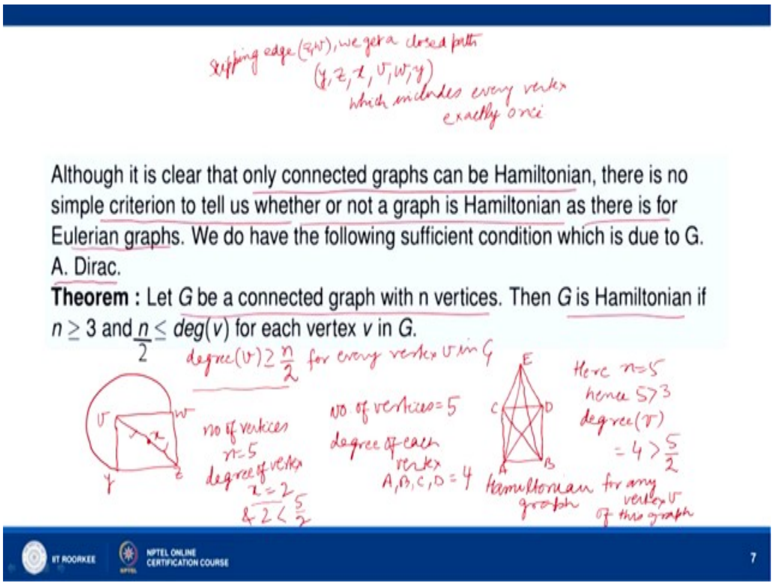

Now let us see, only connected graphs can be Hamiltonian. This we have seen, only connected graphs can be Hamiltonian. Now there is no simple criterion to tell us whether or not a graph is Hamiltonian as there is for Eulerian graph. For Eulerian graphs we have seen the theorem of Euler which says that a connected graph Eulerian if and only if every vertex of the graph is of even degree. So such a criterion does not exist for a graph to see whether it is Hamiltonian or not.

There are theorems which give us sufficient conditions for Hamiltonian graph. For example, let us look at this theorem by A. Dirac. By A. Dirac, it tells that, let G be a connected graph with n vertices. Then G is Hamiltonian if the number of vertices ≥ 3 and $degree(v) \geq n/2$. $degree(v) \geq n/2$ for every vertex v in G. So this is the theorem which gives sufficient conditions to determine whether a given graph is Hamiltonian or not.

Till today we do not know necessary conditions for a graph to be Hamiltonian. Now let us look at some examples here. So let us consider this graph. We have A, B, C, D, E. Now we can see here in this graph, the number of vertices is equal to 5. Number of vertices in this graph is equal to 5. And then degree of each vertex you see, degree(A) =4; degree (B) =4; degree (D) =4; degree (E) =4; degree (C) = 4; degree of each vertex A, B, C, D is equal to 4.

Now you can see here the number of vertices is five. So number of vertices which is n here in this theorem that must be more than 3. So number of vertices here is 5, so$n \geq 3$, so here n is equal to 5. Hence 5>3. And degree of every vertex, degree (v)= 4 which is greater than 5/2, that is n/2, and for any vertex v of this graph v is either A, or B, or C, or D, or E. So for each vertex we notice that its degree exceeds n/2. It is greater than n /2, so this graph is Hamiltonian graph by Dirac's theorem.

Now let us look at another example. Let us look at this graph. Now here we see there are five vertices; 1, 2, 3, 4, 5. So number of vertices, n = 5. And degree of, let us see degree of vertex. degree (x)=2, degree of vertex x is 2. And 2 is not greater than 5/2, and 2 is less than 5/2. So this condition, n ≤ degree (v), n = 5 here and degree(v) =2, degree of x. v, I am taking as x, so degree (x) =2, therefore degree (v)≥ n/2, this condition is not valid here. So we cannot conclude that this graph is Hamiltonian from this Dirac's theorem because the conditions are not satisfied.

But we shall notice, we can notice that this graph is Hamiltonian. Because what we have to see is that it is a path, it is a path which visits each vertex exactly once, so let us see whether we can do this. So we can follow the path, we can go from y to z, then z to x, then x to v, then v to w and then w to y. We can skip the edge, zw . So skipping edge zw we get a path, a closed path: y, z, x, v, w, y; yz, xv, wy; which visits, which includes every vertex exactly once. And so, it is a Hamiltonian graph.

So we are not getting this conclusion that it is a Hamiltonian graph by using Dirac's theorem because in the Dirac’s theorem only sufficient conditions are given to see whether a given graph is Hamiltonian. We will have to see directly from the definition that it is a Hamiltonian graph. Okay. Now let us see some more examples.

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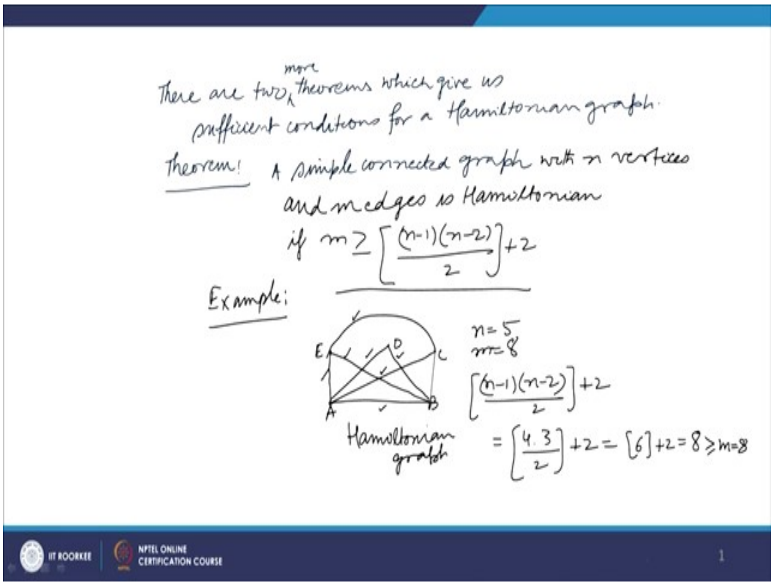

So there are two more theorems. There are two more theorems which give us sufficient conditions for a Hamiltonian graph. So let us first see the theorem. It says that a simple connected graph with n vertices and m edges is Hamiltonian if $m \geq \left[\frac{(n-1)(n-2)}{2}\right]+2$. This is the integer part of $\left[\frac{(n-1)(n-2)}{2}\right]$. So $m \geq \left[\frac{(n-1)(n-2)}{2}\right]+2$. If this condition holds true, where n is the number of vertices in the simple connected graph and m edges are there, then the graph will be Hamiltonian.

So let us see for example, let us consider this graph. So we have A, B, C, D, E. So there are five vertices in this graph, n is equal to 5. And how many edges are there ? 1, 2, 3, 4, 5, 6, 7. We have AB, BD, AD, AC, EB, we have 8 there. So m is equal to 8, there are 8 edges here in this figure. So now let us see whether integer part of $\left[\frac{(n-1)(n-2)}{2}\right]+2$is less than or equal to m or not. So this is, n is equal to 5, so $\left[\frac{4.3}{2}\right]+2$.

So integer part of 6, so this is 8. So this is ≥m which is 8. So this graph satisfies the sufficient condition and therefore it is a Hamiltonian graph. It is a Hamiltonian graph.

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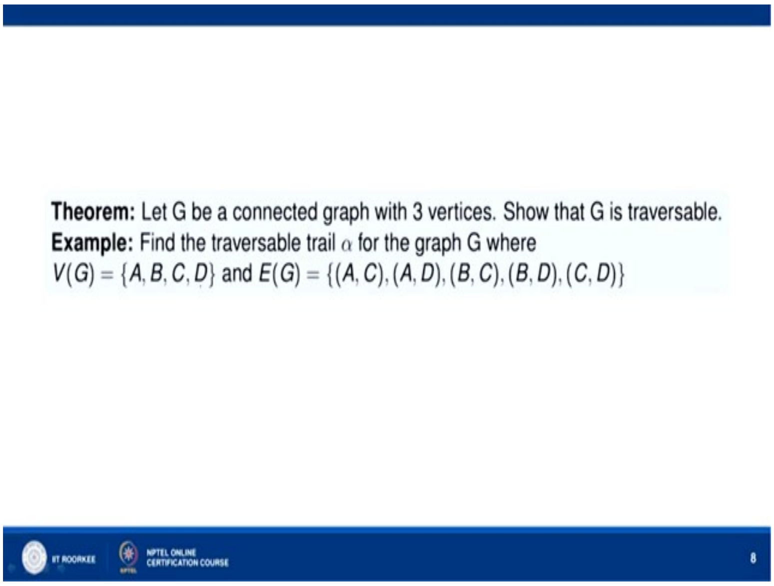

Let us go to this problem: Let G be a connected graph with 3 vertices. Show that G is traversable. Find the traversable trail alpha for the graph G where V(G) is equal to A, B, C, D. Now we have a connected graph with three vertices.

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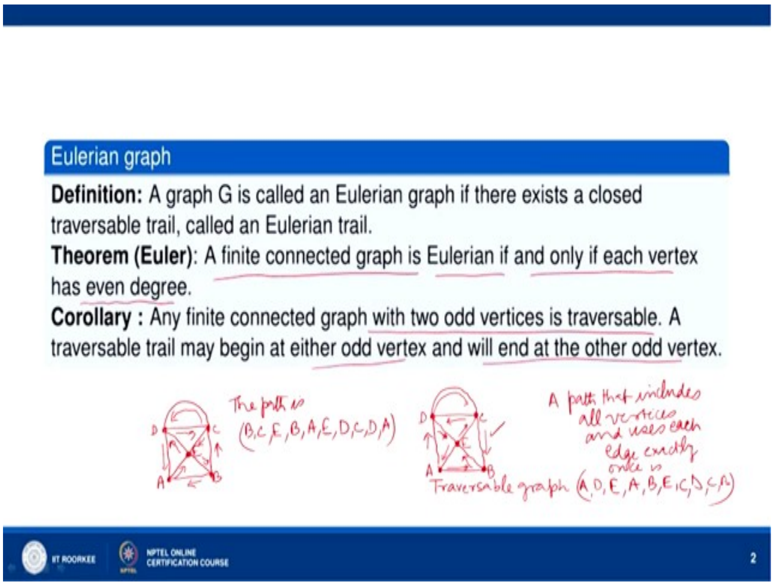

Then we know that the graph is traversable provided any finite connected graph with two odd vertices, is traversable. A traversable trail may begin at either odd vertex and will end at the other odd vertex. So if there are three vertices, we can enter one vertex which is an odd vertex. If there are two odd vertices, we can enter one odd vertex and then end at the other odd vertex. If there are zero odd vertices, then all the vertices of the graph will be of even degree, so the graph will be traversable.

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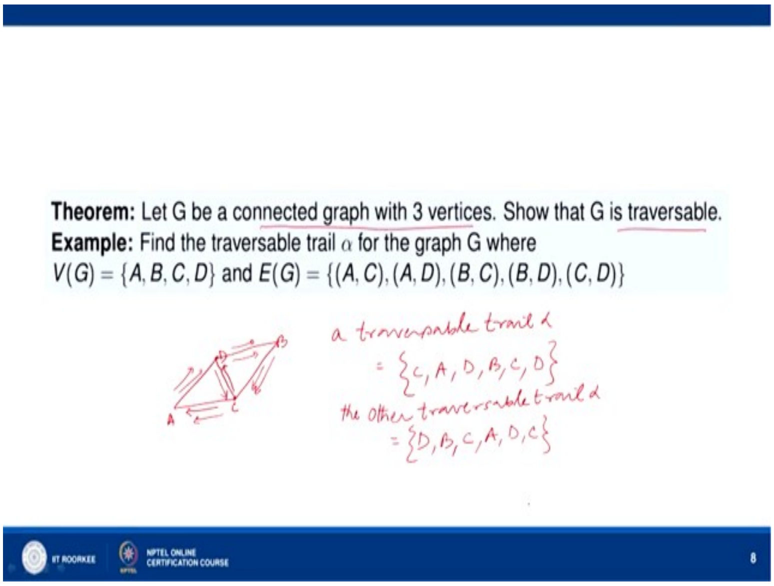

So a graph with three vertices, connected graph with three vertices will always be a traversable graph. Because if we have zero vertices which are of odd degree, then all vertices will be even degree, so it will be traversable. If there is two odd vertices, they are of odd degree, then we can enter one odd vertex and leave the graph from the other odd vertex. So the graph with three vertices is always, this thing, traversable.

Now find the traversable trail alpha for the graph G where V(G) is A, B, C, D; E(G) is AC, AD, BC, BD, CD. So find the traversable trail alpha for the graph G where V(G) is equal to A, B, C, D. So traversable trail means we have to find a path which includes every edge exactly once. So we can follow this path: C, A, D, B, C and then this. Yes, so we can start with C because it is a forward degree. C is a forward degree and D is a forward degree. So we can end at D and we can start, begin at C. Or we can start with D and end up at C.

So when we have two vertices of odd degree, we can enter one vertex and then leave at the other vertex. So we can follow the path, traversable trail. Traversable trail could be, alpha here is C, A, D, B, C, D. Okay, this is one traversable trail. The other traversable trail could be, alpha could be taken as, we can start with D, so we have, we will enter at D and leave at C. So we will have D, B, C, A, D, C. We enter at D and then move along DB, then BC, then CA, and then AD and then C. So we enter at D and leave at C. So this is another traversable trail.

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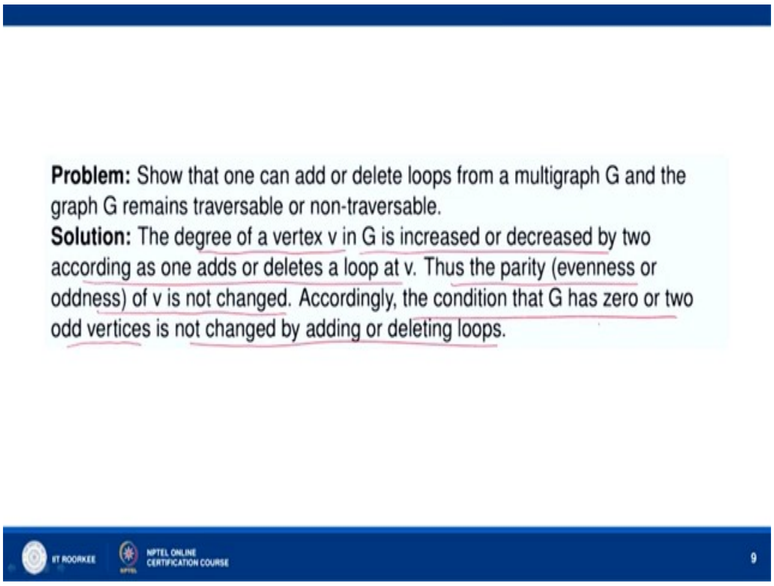

Show that one can add or delete loops from a multigraph G and graph G remains traversable or non-traversable. The degree of a vertex v in G is increased or decreased by two according as one adds or deletes a loop at v. If you add or delete a loop at vertex v, then if you add, you add it degree 2 to the graph to the vertex. And if you remove or delete one loop at vertex, then you decrease the degree there by 2.

So thus, the parity, evenness or oddness of the vertex v is not changed. If it was of even degree, it will remain of even degree. If it was of odd degree, it will remain of odd degree. So the condition that G has zero or two odd vertices, because G remains traversable or not, G is traversable means it has zero or two vertices, or if it is not traversable, then it is having other than that. So it is not changed by adding or deleting loops.

If it has zero odd vertex or two odd vertices, then the nature of the vertex whether it was even or odd is not changed by adding or deleting a loop. And therefore the graph if it was traversable, it will remain traversable. If it was not traversable, it will remain non-traversable graph. So that is all in this lecture. Thank you very much for your attention.