Higher Engineering Mathematics
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Lecture 30
Undirected and Directed Graphs

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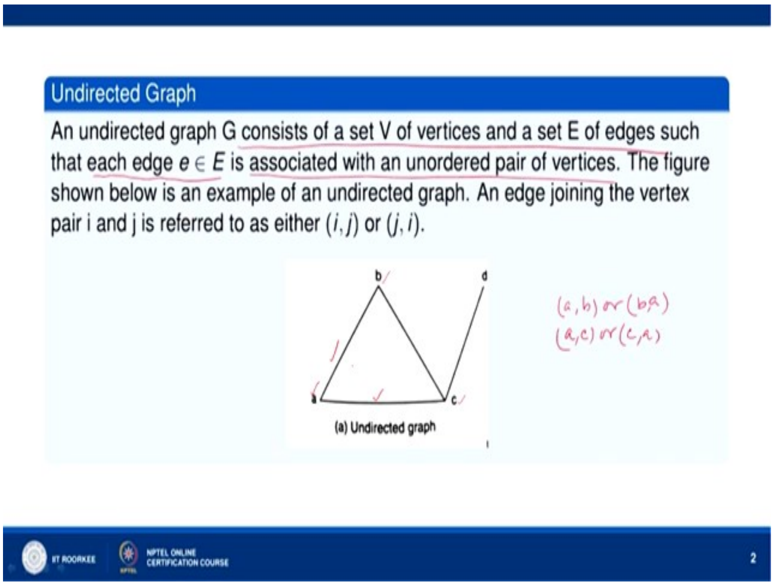

Hello friends, welcome to my lecture on Undirected and Directed Graphs. Let, us first define an undirected graph. An undirected graph G consists of a set V of vertices and a set E of edges such that each edge e ∈ E is associated with an unordered pair of vertices. Say for example here, you can see the vertex ‘a’, the vertex ‘a’ is associated with the vertex ‘b’ and the vertex ‘c’ (so) and it is an unordered pair of vertices, so we can write (a, b) the edge, this edge can be written as (a, b) or (b, a) because there is no direction associated with it and this edge which is, by which a is associated to c can be written as (a, c) or (c, a).

So, the figure shown below is an example of an undirected graph, an edge joining the vertex pair i and j is referred to as either (i, j) or (j, i).

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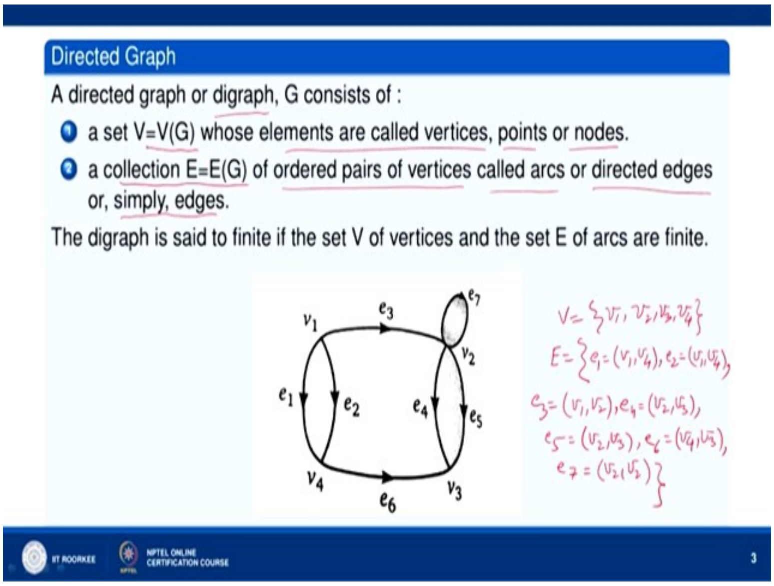

Now, on the other hand a directed graph or we also call it a diagraph, (consists) G consists of, a set V = V (G) whose elements are called vertices, points or nodes. A collection E = E (G) of ordered pairs of vertices called arcs or directed edges. So, here you can see the edges are directed, there is a direction associated with it, so it is called a directed graph or simply edges. So, in the case of a directed graph we have a set V whose elements are called vertices, and a collection E of ordered pairs of vertices called arcs or directed edges.

Say for example in this figure, you can see there are four vertices, v_1, v_2, v_3, v_4. So V = { v_1, v_2, v_3, v_4} and E is equal to, okay so, now, you can see there is a direction associated with each edge v_1, v_2 is joined by the edge e_3 and e_3 is in the direction from v_1¿v_2 . So, we have e_3, let us say begin with e_1 . e_1 joins v_1 to v_4, so e_1 = (v_1 , v_4), e_2= (v_1 , v_4), e_3 =(v_1 , v_2), (v_1 , v_2),and e_4 is (v_2 , v_3), e_5 is (v_2 , v_3), e_6 is (v_4 , v_3), e_7 is (v_2 , v_2), it is a loop at v_2 . So there are seven edges $e_1, e_2, e_3, e_4, e_5, e_6, e_7$, which are ordered pairs of vertices.

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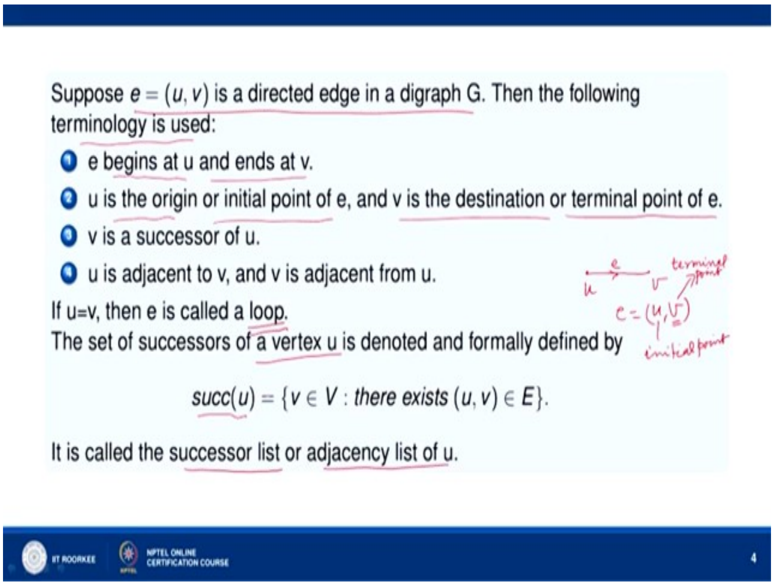

Now, let us say e = (u, v) is a directed edge in a diagraph, then we use the following terminology, e begins at u and ends at v, u is the origin or initial point of e, and v is the destination or terminal point of e. Say for example, this is u and this is v, so this is your edge e and the direction is u, v so e = (u, v). So u is the origin or this is initial point of e and this is terminal point of e, so this is terminal point of e, terminal or destination point of e, v is a successor of u, this v is a successor of u, or we also say u is adjacent to v, or v is adjacent from u.

If u = v, then e is called a loop. Now, the set of successors of a vertex u is denoted by this succ (u) = {v∈V: there exists (u,v) ∈ E}. So this set consists of all those v belonging to V such that there is an edge from u to v and v is a successor of then u, v is a successor of u, so this is successor list of u or adjacency list of u, this set is called as successor list or adjacency list of u.

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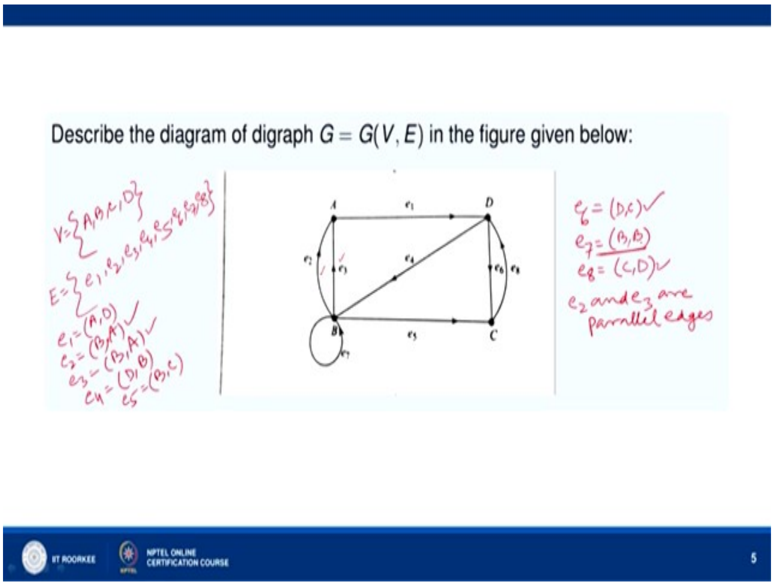

Now, describe the diagram of diagraph G, G = G (V, E) in the figure below. So, let us see what are the vertices here. First we describe the vertices, so V is the set of vertices here, vertices are A, B, C, D and then set of edges, edges are e_1 ,e_2, e_3 ,e_4 ,e_5 , e_6,e_7 ,e_8.So e_1 ,e_2, e_3 ,e_4 ,e_5 , e_6,e_7 ,e_8. Now, what is e_1 ? e_1 is (A, D), e_2 is (B, A), e_3 = (B, A), e_4 is (D, B), $e_5$is (B, C,) e_6 is (D, C), e_7 is (B, B) it is a loop and e_8 = (C, D). We had earlier defined parallel edges, parallel edges were those edges which were, which had the same join the same points.

Here A and B these two join the same points, B and A, but here in the case of directed graph those edges are called parallel which join the same points and their directions are also same. Say for example here, e_2 ,and e_3 they join B and A and moreover their directions are also same. So you can see e_2 , is (B, A) right? And e_3 is also (B, A), so e_2 ,and e_3 since they join B and A and their directions are same so they are parallel edges, so e_2 ,and e_3 are parallel edges.

Now, in the case of e_6 ,and e_8 , e_6 is (D, C), e_8 is (C, D) although they join same points C and D, but their directions are opposite, e_6 , is (D, C) from D to C, e_8 is from C to D, so e_6 ,and e_8 are not parallel edges. As we said earlier this e_7 , e_7 is B, B so it defines a loop at B, it gives a loop at B.

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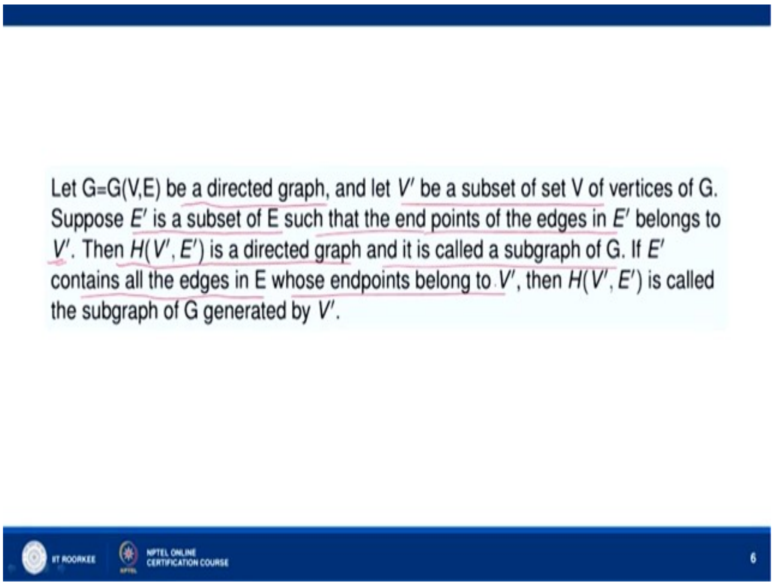

Now, let G = G (V, E) be a directed graph, and V' be a subset of V, V is the set of vertices of G. Then E' is a subset of E such that the, suppose E' is a subset of E such that end points of the edges in E' belongs to V', the end points of the edges in E' belongs to (E) belongs to V'. Then H(V', E',) is a directed graph and it is called a subgraph of G. If E' contains all the edges in E whose endpoints belong to V', then we say that H(V', E',) is a subgraph of G generated by V'.

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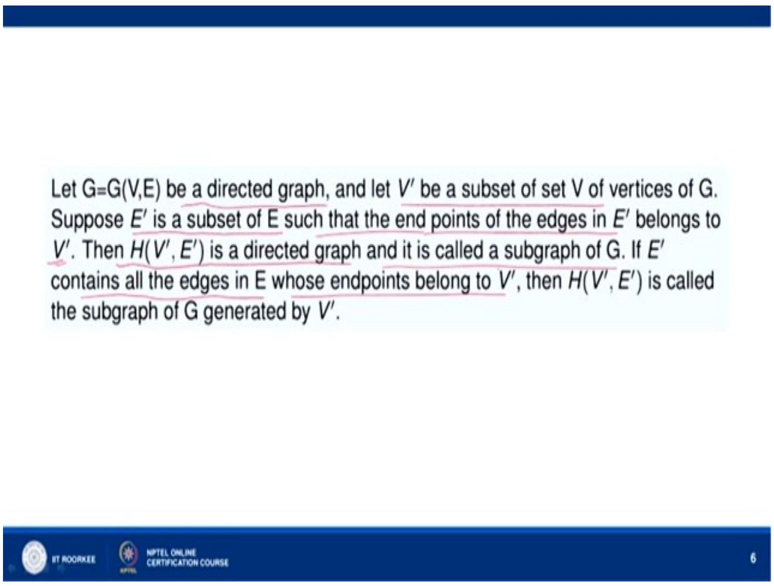

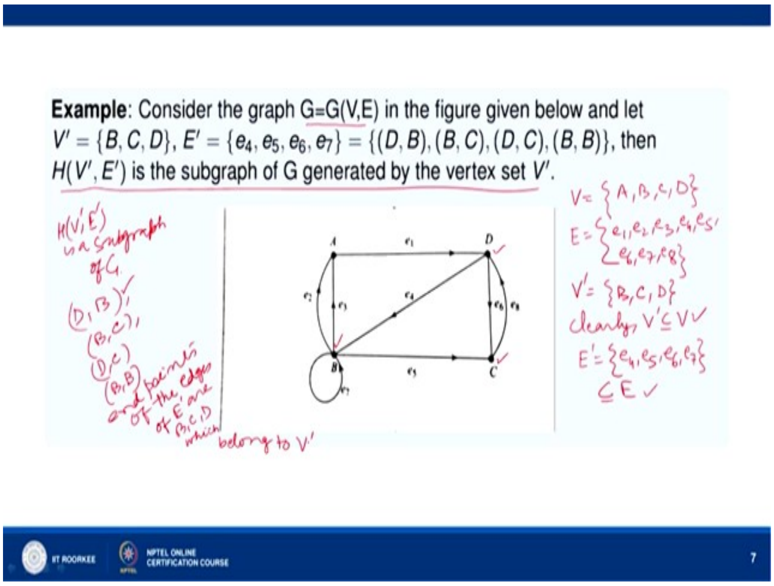

Now, let us say this example. Let us look at this, consider the graph G = G (V, E) in the figure given below, let V' = { B, C, D}. So B, C, D. See, what is V here? V ={ A, B, C, D}, and what is E here? E = { $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ } just now we had studied this example. Now V', what is V'? V' is {B, C, D}. We are given V' = {B, C, D}. So clearly $V' \subseteq$ V.

Now, E' , E' is consists of e_4, e_5, e_6, e_7 and it is a subset of E, E consists of $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ and e_8. So,e_4 is (D, B); (B, C) is e_5, (D, C) is e_6, and (B, B) is e_7 . Now, we have to show that H(V', E') is the subgraph of G generated by the vertex set V'. Now, let us look at, go to the definition of subgraph of G generated by the vertex V'.

Here, we say that H(V', E') is a directed graph and it is called a subgraph of G, that we have seen $V' \subseteq$ V, $E' \subseteq$ E, so H(V', E') is a directed graph and it is a subgraph of G.

So, H(V', E') is a subgraph of G because $E' \subseteq$ E , $V' \subseteq$ V. Now, we have to show that it is subgraph of G generated by the vertex V'.

So, let us see how we define that, H(V', E') is called the subgraph of G generated by V', if E' E dash contains all the edges in E whose endpoints belong to V'.

So, let us see what is V' here? V' is {B, C, D}, okay B, C, D are the endpoints B, C, D.

So, we say that E' contains all the edges in E whose endpoints belong to V'. So E' contains all the edges in E,E' contains, now E' contains what edges?E' contains D, B, B, C, and then D, C, and then B, B. So endpoints of these vertices are you see endpoints are here for this edge endpoints are D and B, for this endpoints are B, C, for this endpoints are D and C, and for this endpoints are B, so endpoints are B, C, D the endpoints here, endpoints of the edges of E', endpoints of the edges of E' are B, C, D and they clearly belong to which are, which belong to V'. So, H(V', E') is the subgraph of G generated by the vertex set V'.

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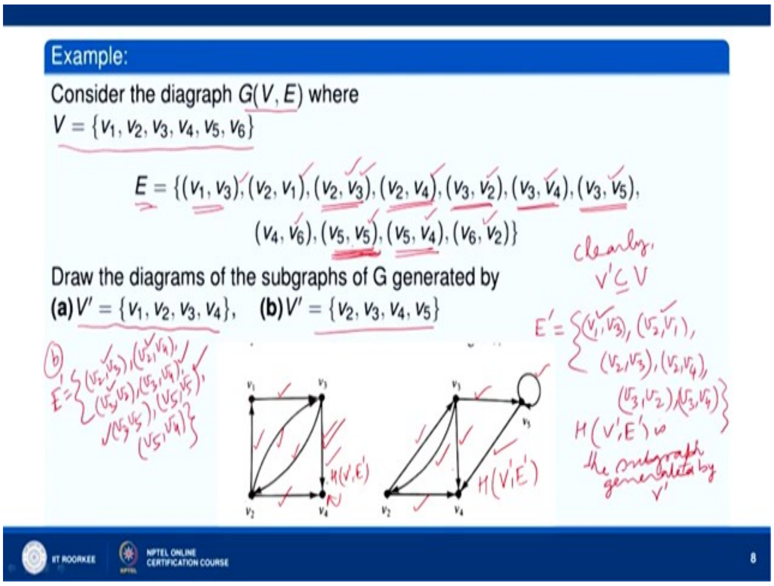

Now, let us consider this graph G (V, E), where V is the set of vertices v_1, v_2, v_3, v_4, v_5, v_6, and E is the set of edges (v_1, v_3), $(v_2, v_1$¿, ¿ v_3¿, ¿ v_4¿, $(v$¿¿$3, v_2)$¿ $(v$¿¿$3, v_4)$,¿ $(v$¿¿$3, v_5)$,¿ (v_4, v_6), (v_5, v_5), (v_5, v_4), (v_6, v_2), so that means at this (v_4, v_4, v_4,, v_4 v_6, no this v_5, v_5) this (v_5, v_5),gives us a loop at the vertex v_5. Now, let us draw the diagrams of the subgraphs of G generated by v_1, v_2, v_3, v_4. So v_1, v_2, v_3, v_4 is clearly a subset of V, clearly $V' \subseteq$ V , this V dash which is subset of V, we have to draw the diagram of the subgraphs of G generated by v_1, v_2, v_3, v_4.

So, we have to find all those edges of E whose endpoints are v_1, v_2, v_3, v_4 so (v_1, v_3), we will take (v_1, v_3), then we will take $(v_2, v_1$¿, then ¿ v_3¿, then we have ¿ v_4¿, then we will have $(v$¿¿$3, v_2)$¿ then we have $(v$¿¿$3, v_4)$¿ and then we have, yeah. So, let us take E' to be equal to this. So E' consists of those edges whose endpoints belong to V' and therefore, the graph of, graph G(V', E') will be the subgraph of G (V, E) generated by V', generated by V'.

And its figure could be like this, you see this figure, look at this figure it satisfies all these I mean the edges here are nothing but the edges of E'. So, you can see a ¿¿), this is ¿), okay (v_2, v_1¿,so this is $(v_2, v_1$¿, then¿ v_3¿, so we have ¿ v_3¿, here this is ¿ v_3¿, then we have ¿

v_4¿ this is ¿ v_4¿ then we have $(v$¿¿$3, v_2)$¿ so this is $(v$¿¿$3, v_2)$¿ this one, and then we have $(v$¿¿$3, v_4)$¿ so this is $(v$¿¿$3, v_4)$¿, this one.

So this graph, okay this graph is the graph which is generated by the vertices V' v_1, v_2, v_3, v_4 consisting of v_1, v_2, v_3, v_4. So H(V', E') this is H(V', E') this one and it is generated by V' is the subgraph. Now, in the other part V' is $, v_2, v_3, v_4, v_5$. So, let us consider those ordered pairs here in E, whose endpoints arev_2, v_3, v_4, v_5 so v_2, v_3 so we can take this one (v_2, v_3¿ , then we can take$(v$¿¿$3, v_2)$¿, we can take$(v$¿¿$3, v_2)$¿, we can also take ¿ v_4¿ and then $(v$¿¿$3, v_4)$¿ we can consider this one $(v$¿¿$3, v_4)$¿ and then $(v$¿¿$3, v_5)$¿ this is $(v$¿¿$3, v_5)$¿ and then$(v$¿¿$5, v_5)$¿ we can consider, okay $(v$¿¿$5, v_5)$¿ and then $(v$¿¿$5, v_4)$¿, we can take.

So, 1, 2, 3, 4, 5, 6, 7, okay there are (6) 7 edges and a graph showing these edges and vertices v_2, v_3, v_4, v_5 is this one, so v_2, v_3, v_4, v_5.

Now, let us see the edges of this graph or these edges (v_2, v_3¿ you see (v_2, v_3¿ , okay (v_2, v_3¿ here. So, here E' for part (b), okay for part (b), E' is (v_2, v_3¿ and then ¿ v_4¿ , then we have $(v$¿¿$3, v_2)$¿, then we have$(v$¿¿$3, v_4)$¿ , then we have $(v$¿¿$3, v_5)$¿ and then we have $(v$¿¿$5, v_5)$¿ and $(v$¿¿$5, v_4)$¿, okay so (v_2, v_3¿ , okay (v_2, v_3¿ is here, then ¿ v_4¿ is there, then we have $(v$¿¿$3, v_2)$¿ this one, so $(v$¿¿$3, v_2)$¿, is there $(v$¿¿$3, v_4)$¿, is there,$(v$¿¿$3, v_5)$¿ this is $(v$¿¿$3, v_5)$¿ , and then we have $(v$¿¿$5, v_5)$¿ this is $(v$¿¿$5, v_5)$¿, okay and then $(v$¿¿$5, v_4)$¿, so this one. So this is the graph of H(V', E') it is the diagram of the, diagram of H(V', E') this subgraph of G generated by V'.

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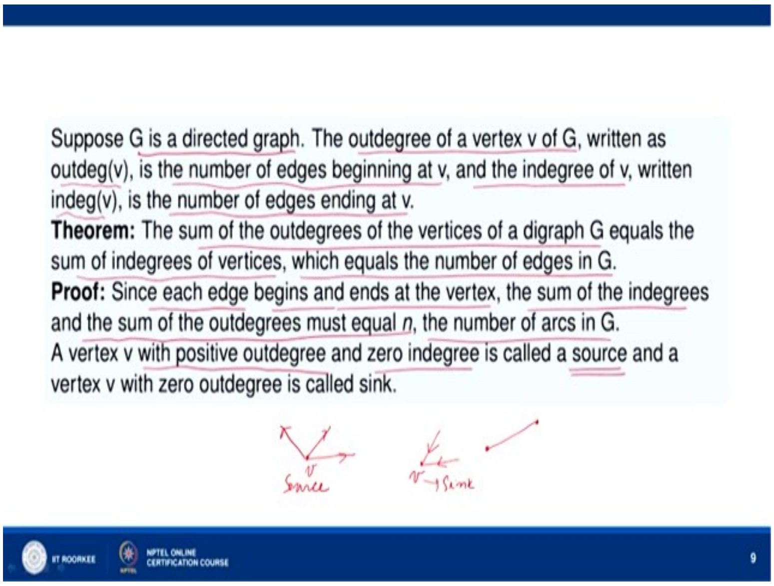

Now, suppose G is a directed graph, suppose G is a directed graph, the out degree of a vertex v, the out degree of a vertex v of G, written as out degree v is the number of edges beginning at v, the number of edges which begin at v they give us the out degree of v, and the indegree of v is written as indeg(v), so this is the number of edges that are ending at v. So out degree is the number of edges which begin at v and end degree is the number of edges ending at v.

The sum of the outdegrees of the vertices of a diagraph G equals the sum of indegrees of vertices. So sum of outdegrees of the vertices of a diagraph is equal to the sum of indegrees of the vertices and it is also equal to the number of edges in G. Now, let us see how we proof this. Since each edge begins and ends at the vertex, see, you take any vertex, a vertex begins and ends at vertex, any edge begins at a vertex and ends at a vertex, so the sum of the indegrees and the sum of the outdegrees must be equal to n, if there are n vertices, the number of arcs in G.

So, now a vertex v with positive outdegree, if we have a vertex with positive outdegree, and zero indegree, that zero indegree means there are no edges ending at v, but there are edges beginning at v. So, if v has a positive outdegree and zero indegree it is called a source. So, if you take a vertex such that the edge begin at this vertex, the edge begin at this vertex, there is no edge which is ending at this vertex then it is called a source, it will be called a source and if there is a edge there is a vertex with zero outdegree and positive indegree, zero outdegree means you have edges ending at v, no edge is beginning at v, so this is called sink.

So, if positive outdegree is there and zero indegree is there then the vertex will be source and if positive indegree is there and zero outdegree is there then it will be called a sink.

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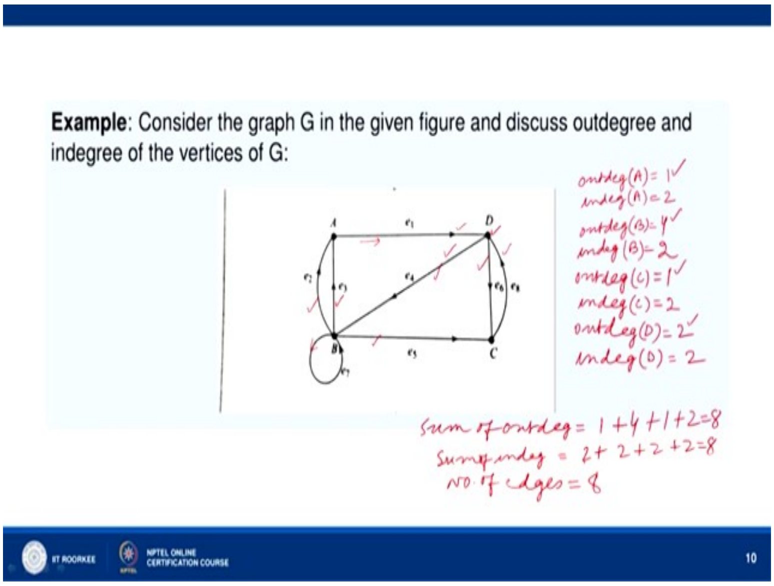

Now, consider the graph G in the given figure and discuss outdegree and indegree of the vertices of G. Let us see, let us first find the outdegree of A, so how many vertices are beginning at A? You see, this vertex the vertex this e_1 is beginning at A, there is no vertex other than e_1 which is beginning at A, so outdegree of A is 1. Indegree of A will be how much? The edges that are ending at A, so e_2 is ending at A, e_3 is ending at A, so (outdegree) indegree is 2.

Now, similarly outdegree of B, at B you see we have e_5 , e_5 beginning at B, e_3 beginning at B, e_2 beginning at B, and moreover e_7 beginning at B because e_7 is going like this soe_7 actually is beginning at B as well as ending at B. So, when we are counting the edges that are beginning at B, then we will count e_5, e_3, e_2 and also e_7. So, we have 4. And indegree of B will be, indegree of B will be equal to, indegree will be equal to, now 1 because this edge is ending at B, so $e_7$is ending at B, so we have 1 for this and then e_4 is also ending at B, so 2.

Then outdegree of C, outdegree of C. So, at C, you can see e_8, e_8 is beginning at C and only e_8 is beginning at C, so outdegree of C is 1, indegree of C, how many are ending at C? e_5 is ending at C, e_6 is ending at C, so 2.

And, then outdegree of D let us see, how much? Yeah, so outdegree of D means e_6, e_6 is beginning at D and also e_4 is beginning at D, so we have 2. Indegree of D, indegree of D will be e_1 is ending at D, e_8 is ending at D, so indegree is 2. So we have considered A, B, C, D all of them.

Now, let us find the sum of outdegrees, sum of outdegrees is, sum of outdegrees is 1, 4, 1, 2, so 1 + 4 + 1 + 2 = 8. And sum of indegrees of the vertices of this diagraph, this is 2 + 2 + 2 + 2 = 8, and number of edges of this diagraph, number of edges, see we have $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ so there are 8. So sum of outdegrees of the vertices of a diagraph is equal to sum of indegrees of the vertices of the diagraph and is equal to so number of edges of the diagraph.

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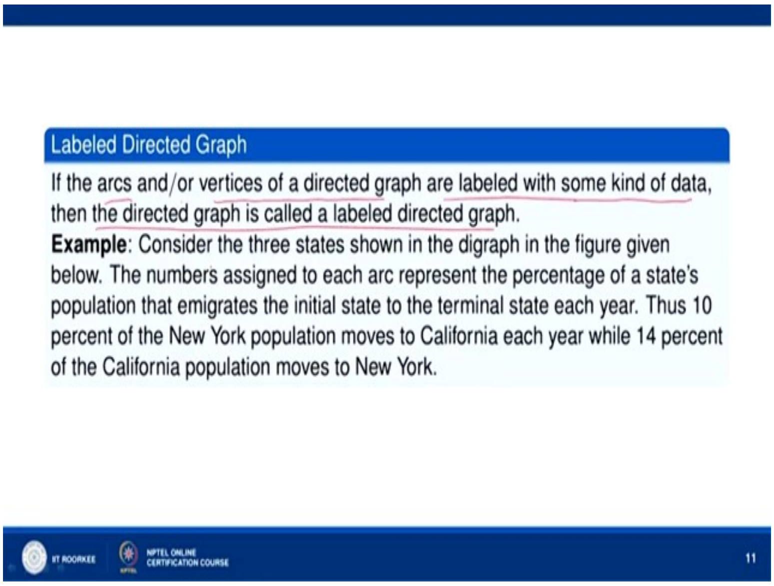

Now, labelled directed graph. If the arcs and or vertices of a directed graph are labelled with some kind of data, then the directed graph is called a labelled directed graph. So if the edges, edges or vertices or both of them are labelled in the case of a directed graph are labelled with some kind of data, then we will call the directed graph as a labelled directed graph. Now, let us consider the three states shown in the diagraph in the figure given below.

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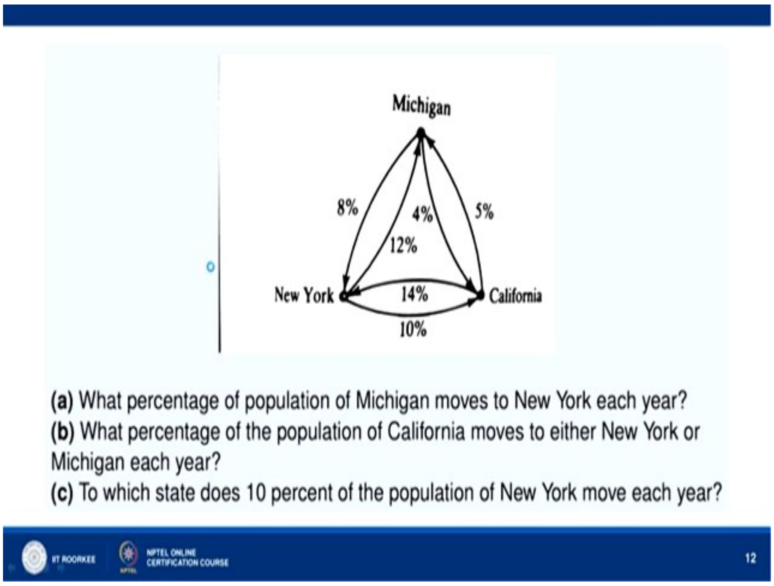

So, let us see this one, you see this diagraph, here the edges are labelled, edges and the vertices are labelled, this vertex is New York, this vertex shows Michigan, this vertex shows California, and these edges (show) are labelled with 8 %, 12 %, 4 %, 5 %, 14 %, 10 %, so it is a labelled directed graph.

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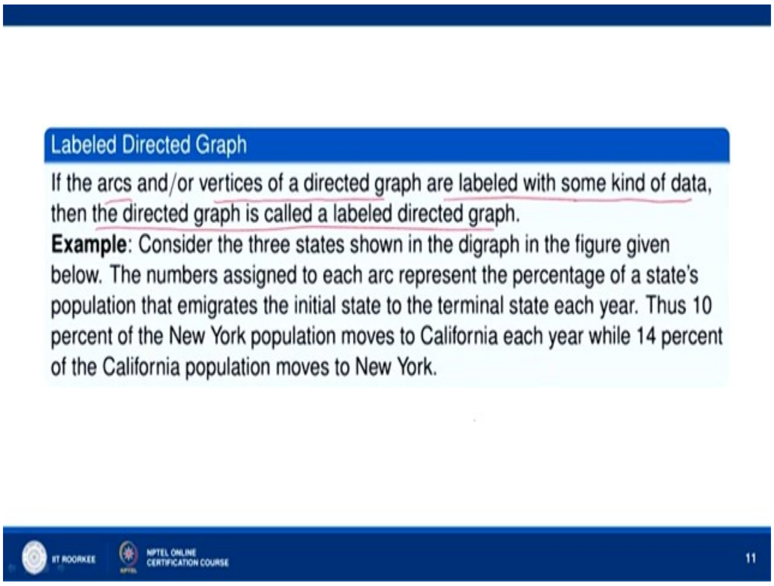

Now, let us see. So, what is the problem? Consider the three states shown in the diagraph in the figure given below. The numbers assigned to each arc represent the percent of percentage of a state's population that emigrates the initial state to the terminal state each year. Thus, 10 % of the New York population, you see 10 % of the New York population moves to California each year.

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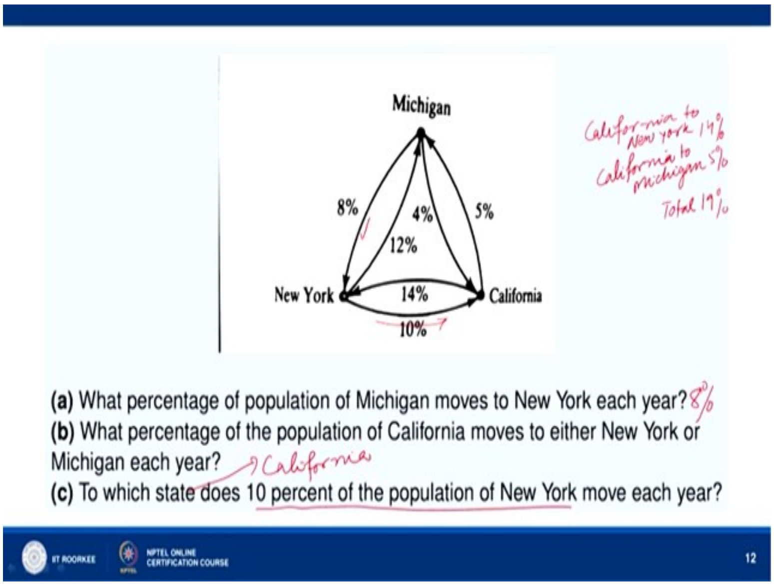

You can see here, you see this one, this edge, this edge shows that 10 % of the New York population emigrates to California each year, and 14 % of the California population emigrates to New York each year, so it is a labelled directed graph. Now, let us answer these questions, what percentage of population of Michigan moves to New York each year?

So, let us say Michigan to New York, Michigan to New York, this one, so 8 %, 8 % of the population of Michigan moves to New York each year. Then what percentage of population of California moves to either New York or Michigan? So California population moves to New York, California population moves to New York is 14 %, California to New York 14 % and California to Michigan we have, California to Michigan 5 %.

So percentage of population of California which move to New York or Michigan is 19 %. So, total is 19 %, 19 % of California population moves either to Michigan or to New York each year.

Now, to which state does 10 % of the population of New York move each year? Let us see from New York 10 % population moves to, yeah New York to California. So 10 % population of New York moves to California each year, so this, the state here is California. So, that is the end of my lecture. Thank you very much for your attention.