Higher Engineering Mathematics
Lecture – 03
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Tautologies and Contradictions

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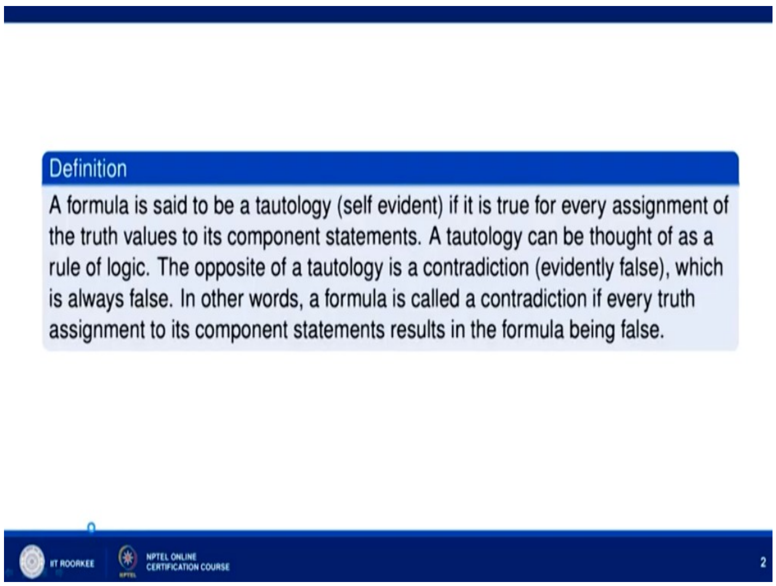

Hello friends. Welcome to my lecture on Tautologies and Contradictions. First let us define what do we mean by tautology. A formula is set to be a tautology, tautology is also known as self-evident, if it is true for every assignment of the truth values to it is compound statement. A tautology can be thought of as a rule of logic. The opposite of a tautology is a contradiction, and contradiction means evidently false. So the opposite of the tautology is a contradiction which is always false. In other words a formula is called a contradiction if every truth assignment to it's component result in the formula being false consider the propositions, 'the professor is either a woman or a man'.

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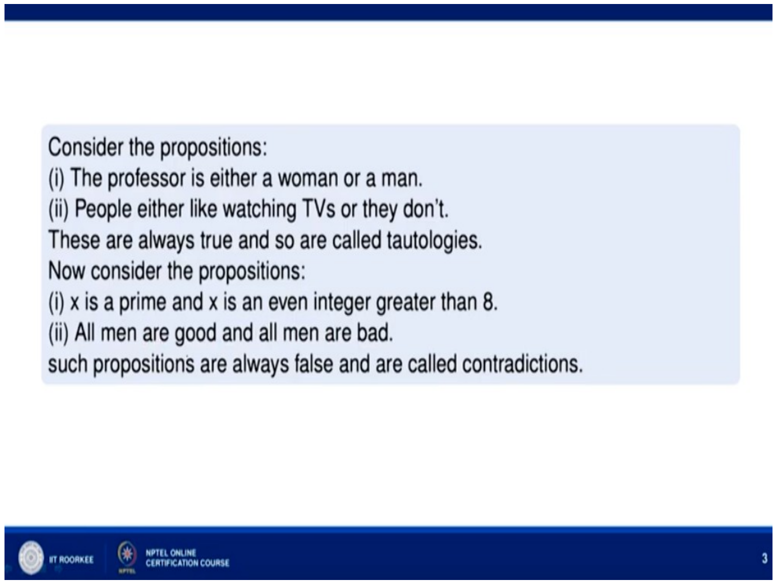

The next proposition 'people either like watching TVs or they do not', now you can see these two propositions are both are always true, and so they are called tautologies. Now let us consider the propositions x is a prime and x is an even integer greater than 8, the next proposition 'all man are good and all men are bad' such propositions are always false and therefore are called contradictions.

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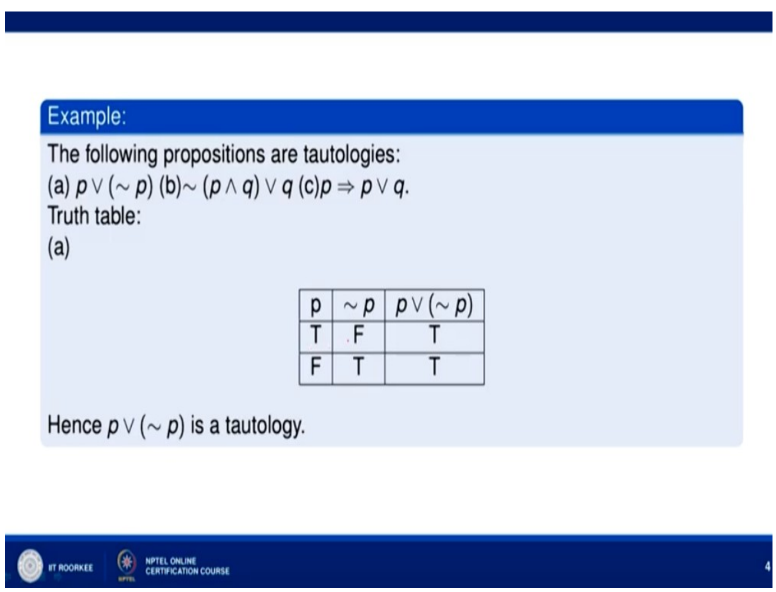

The following propositions, let us look at the propositions $p \vee (\sim p)$, $\sim(p \wedge q) \vee q$, $p \Rightarrow p \vee q$, these propositions are all tautologies, we can check that by writing their truth table so let us write the first truth table for the first proposition p∨(~p), p has truth value T and F, ~p will have the corresponding F and T and there for p∨(~p) will have the truth value T in the case when p has truth value T and ~p has truth value F and then in the case when p has truth value F, ~p has truth value T, then again p∨(~p) will have truth value T. So p∨(~p) has truth values T for all the possible ties here and therefore, we say that, p∨ (~p) is a tautology.

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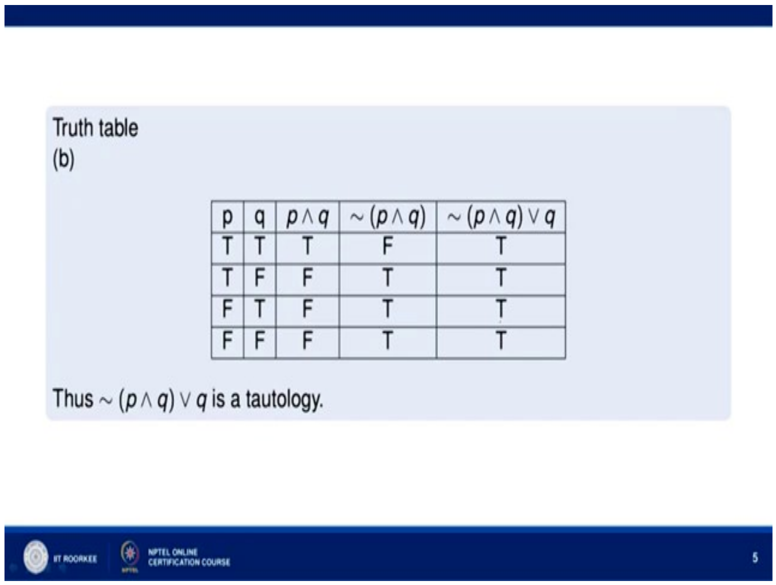

Truth table

(b)

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim(p \wedge q) \vee q$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Thus $\sim(p \wedge q) \vee q$ is a tautology.

Let, us take the second proposition ~(p^q)∨q and let us form the truth, so first we found the truth table here for the p and q, p and q has truth value TT TF FT FF and when p and q are both T, p^q have truth value T, for the case TF when p has truth value T, q has truth value F, p^q will have truth value F and similarly for the other cases p has truth value F, q has truth value T, p^q will have truth value F, and when p has truth value F and q has truth value F, p^q has truth value F , then negation of p^q will have truth values F T T T. Now ~(p^q)∨q will then have the truth values.

So, let us consider ~(p^q)∨q so q has truth value F so when ~(p^q) truth value F and q has truth value T then ~(p^q)∨q will have truth value T and similarly for the case when ~(p^q) has truth value T and Q has truth value F, the ~(p^q)∨q will have truth value T and when both have truth values T, the ~(p^q)∨q will have truth T and when ~(p^q) has truth value T and q has truth value F the ~(p^q)∨q will have truth value T, so you can see the column of the truth values for ~(p^q)∨q has all Ts and therefore it is a tautology.

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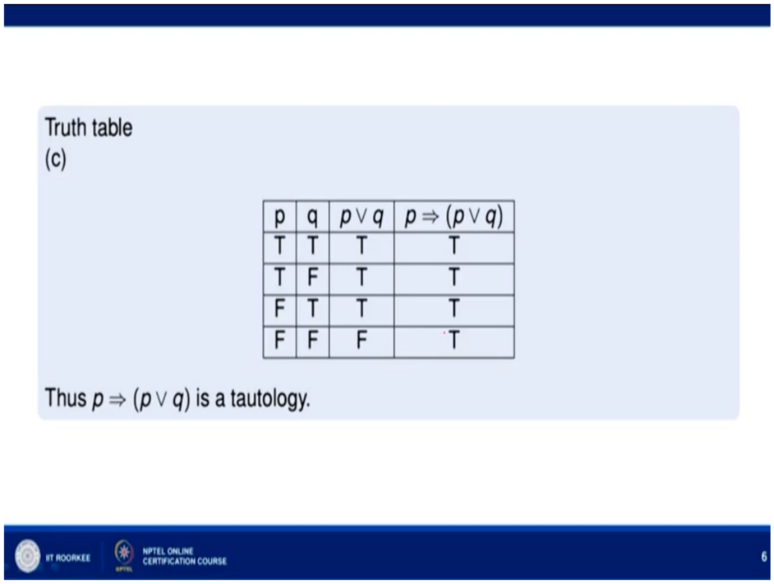

p	q	$p \vee q$	$p \Rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Now, let us consider the third case in the third case we have to show that $p \Rightarrow (p \vee q)$ is a tautology, so this are the truth value of p and q TT TF FT FF then $p \vee q$ has truth values TT TF and then $p \Rightarrow (p \vee q)$ so $p \Rightarrow (p \vee q)$ means when p is T and $p \vee q$ is T, $p \Rightarrow (p \vee q)$ has truth value T, and again here p has truth value T and here this has truth value T, so it is T and one P has truth value F, $p \vee q$ has truth value T then $p \Rightarrow (p \vee q)$ truth value T so again the column for the proposition $p \Rightarrow (p \vee q)$ or all the value truth values are T therefore it is a tautology.

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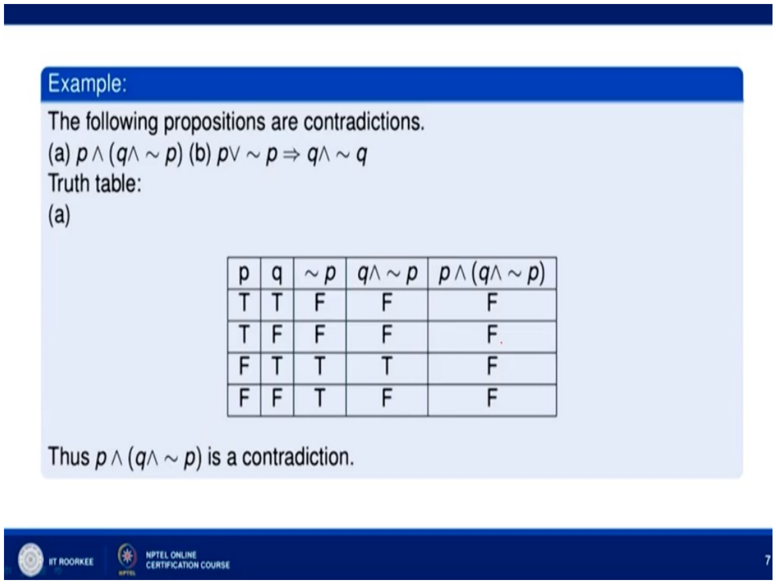

Now, let us look at the propositions $p \wedge (q \wedge \sim p)$, $p \vee \sim p \Rightarrow q \wedge \sim q$, we will see that both of them are contradictions that means the column corresponding to these propositions will have all truth values F, so pq has truth values TT TF FT FF these are the combinations, now then ~p has truth values FF TT, now $q \wedge \sim p$, $q \wedge \sim p$will have truth values for TF it will be F, for FF it is F, for TT it is T, for FT it is F and then $p \wedge (q \wedge \sim p)$, so it is T here, it is F here so we have F here, it is T here, it is F.

So this is F and then F and T gives F and F and F gives F, so the column corresponding to the proposition has F $p \wedge (q \wedge \sim p)$ throughout for all possible combinations and therefore it is a contradiction.

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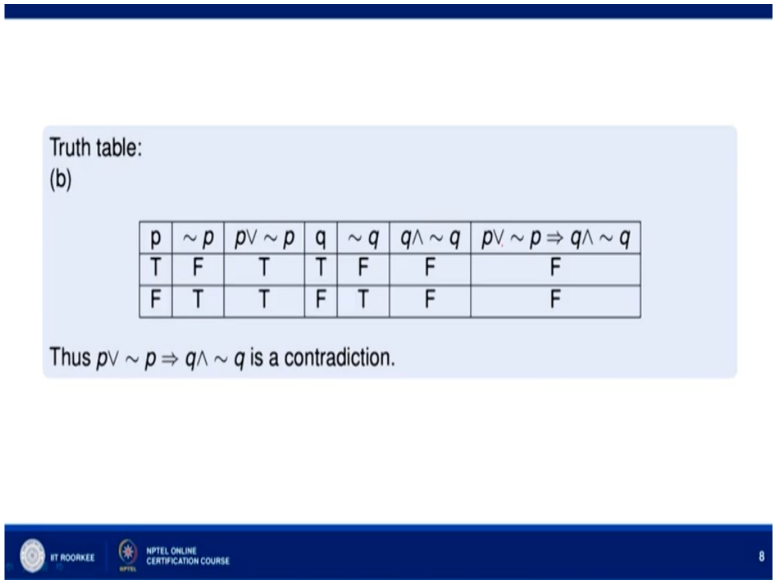

Truth table:

(b)

p	$\sim p$	$p\vee \sim p$	q	$\sim q$	$q\wedge \sim q$	$p\vee \sim p \Rightarrow q\wedge \sim q$
T	F	T	T	F	F	F
F	T	T	F	T	F	F

Thus $p\vee \sim p \Rightarrow q\wedge \sim q$ is a contradiction.

Now, let us go to the second proposition $p\vee{\sim}p \Rightarrow q\wedge{\sim}q$, so the truth table let us form here again, so p has truth values T and F, ~p has truth values F T, $p\vee{\sim}p$, $p\vee{\sim}p$ will then have truth values T here, F here, so it will give you T, F and T here, so it will give you T and q has truth value T and F, ~q has truth value F and T, so and $q\wedge{\sim}q$ will have truth values and F here and this is also F therefore now let us see $p\vee{\sim}p$ has truth value T, $q\wedge{\sim}q$ has truth value F.

So, when it is T here, it is F here we have F in the case of this implies, so implication and then $p\vee{\sim}p$ has truth value T, $q\wedge{\sim}q$ has truth value F, so again we have F, so the column corresponding to the proposition $p\vee{\sim}p \Rightarrow q\wedge{\sim}q$ has F for all possibilities here so it is a contradiction.

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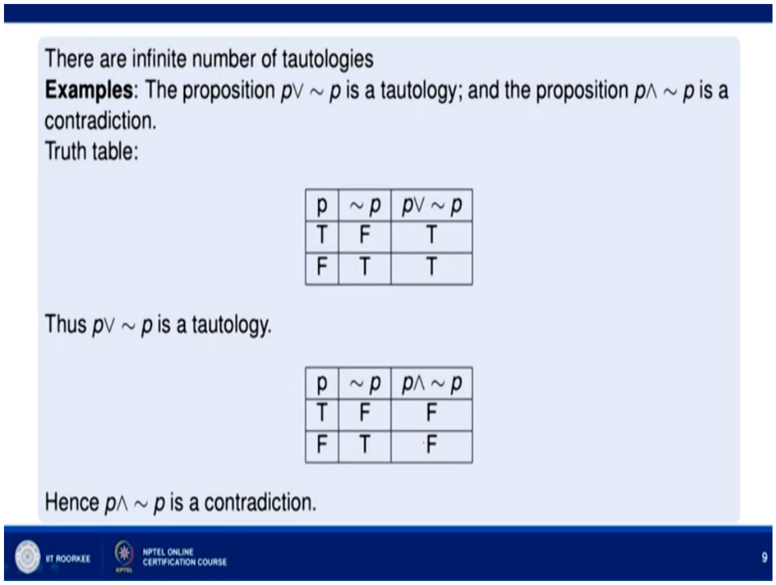

Now, there are infinite number of tautology we shall see, first let us see the proposition p∨~p is a tautology and the proposition p∧~p is a contradiction, so you can see here p∨~p, so p is having truth value of TF, ~p has truth value F and T, so p∨~p will then have truth value T and here also T and therefore, it is a p∨~p is a tautology, in the case of the other situation p∧~p, so p has truth value TF, ~p has truth value FT, then p∧~p will have truth value F, for the case TF for the case FT also it is F and therefore p∧~p is a contradiction.

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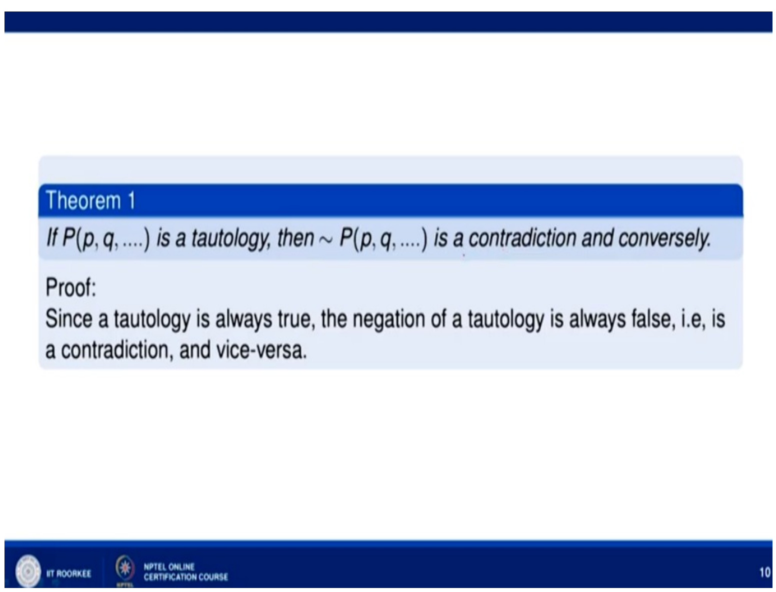

Now we have a theorem which is very useful theorem, if P(p,q,…) is a tautology then ~P(p,q,…) is a contradiction and conversely, so this can be easily proved since a tautology, we have seen that all the truth values in the case of the tautology or T, so since a tautology is always true the negation of a tautology will give you false, so the negation will give all values false that means negation of the tautology is a contradiction.

So and conversely if we consider the ~P(p,q,…), which is the contradiction then P will be a tautology because ~P, if it is a contradiction that means all values in the column corresponding to the contradiction all values will be false and therefore P(p,q,…), P(p,q,…) will be a tautology, so this theorem now can be used to this can be used later.

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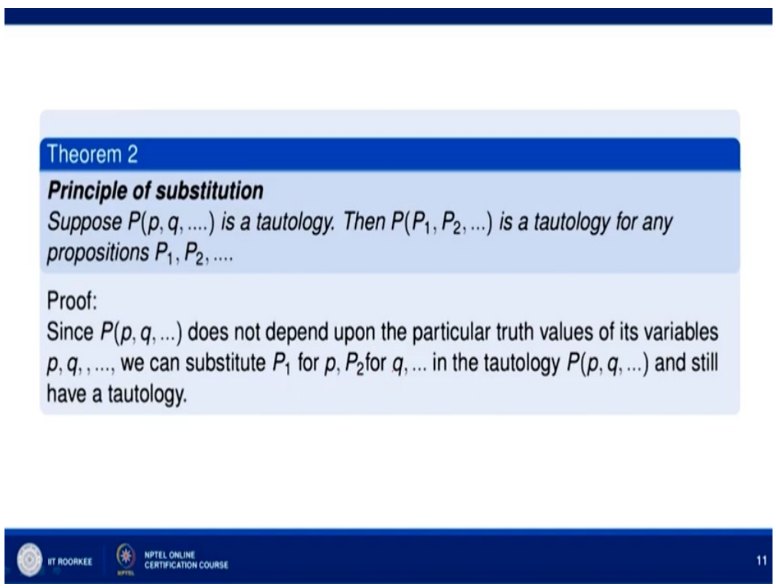

So, let us do another theorem principle of substitution. Suppose P(p,q,…) is a tautology then P($P_1, P_2,\ldots$) is a tautology for any proposition P_1 P_2 …, since P(p,q,…) does not depend on the we have seen that P(p,q,…) does not depend on the particular truth values of its variables p,q, …, it is always T in the column for a tautology, so it does not depend on the particular truth value of its variable p,q… therefore, we can substitute P_1 for p, P_2 for q,… in the tautology P(p,q,…) still have a tautology.

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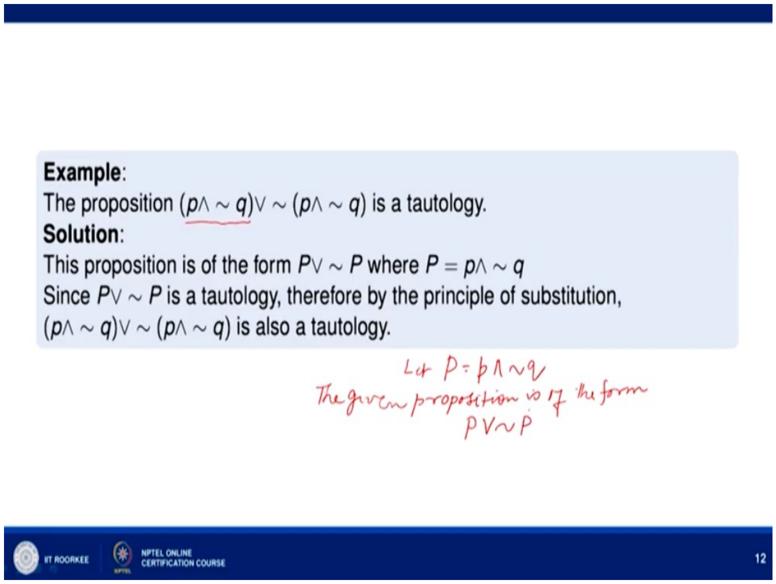

Let us say for example this proposition $(p\wedge\sim p)\vee\sim(p\wedge\sim q)$ we shall see that it is a tautology so you can see here I call this as P. Let us define P to be, let us take P to be $p\wedge\sim q$ then the given proposition is of the form $P\vee\sim P$. Now we have just now seen $P\vee\sim P$ is a tautology, let us go to that place where we proved that, $P\vee\sim P$, you can see here we have proved $P\vee\sim P$ is a tautology, so we have just now shown that $P\vee\sim P$ is a tautology therefore, by the principle of substitution.

Let us put the value of P, the value of P is $p\wedge\sim q$, so $(p\wedge\sim p)\vee\sim(p\wedge\sim q)$ is also a tautology, so we can see that the principle of the substitution is very useful here just by using the fact that $P\vee\sim P$ is a tautology we can decide about the given proposition that it is a tautology.

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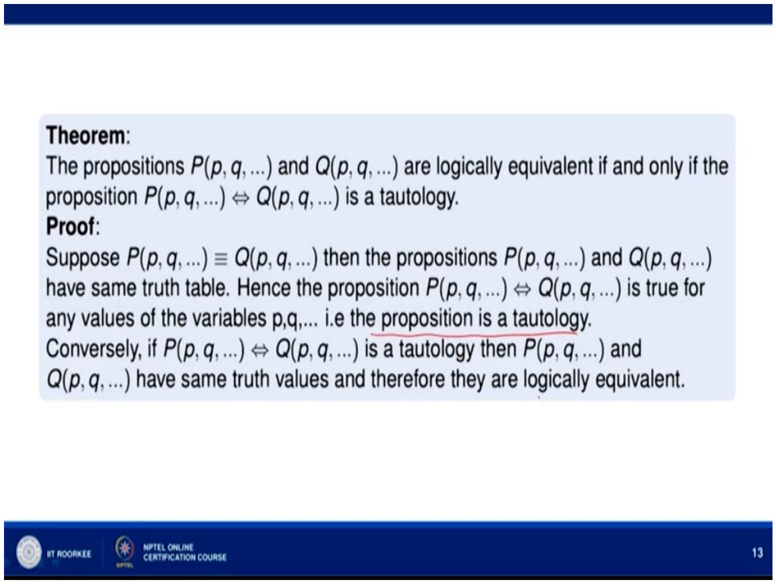

The propositions P(p,q,…) and so on and Q(p,q,…) are logically equivalent if and only if the propositions, if the propositions P(p,q,…) ⇔ Q(p,q,…) is a tautology so that is why we said that there are infinite number of tautologies, all logically equivalent propositions give us tautologies, so suppose P(p,q,…) ≡ Q(p,q,…) then the propositions P(p,q,..) and Q(p,q…) have same truth table, this we have seen when we define the logically equivalent proposition.

So, when two propositions are logically equivalent their truth tables are same hence the proposition P(p,q,…) if and only if Q(p,q,…) is true or any values of the variable p,q because when P(p,q,..) and Q(p,q,..) have the same truth table, then the proposition P(p,q,…) ⇔ Q(p,q,…) will have T, all Ts in it is column, so the proposition is a tautology and conversely if P(p,q,..) ⇔ Q(p,q,..) is a tautology then P(p,q,..) and Q(p,q,…) will have the same truth values because P(p,q,..) and Q(p,q,..), if they do not have the same truth values then P(p,q,..) and Q(p,q,..) cannot be P(p,q,…) will not implies and implied by Q(p,q,…), sorry if P(p,q,…) ⇔ Q(p,q,..) is a tautology then P(p,q,…) and Q(p,q,..) will have same truth values because this is a tautology.

So here we will have T in, all values will be truth values will be T here and therefore, P(p,q,…) and Q(p,q,..) should have same truth values and therefore they are logically equivalent.

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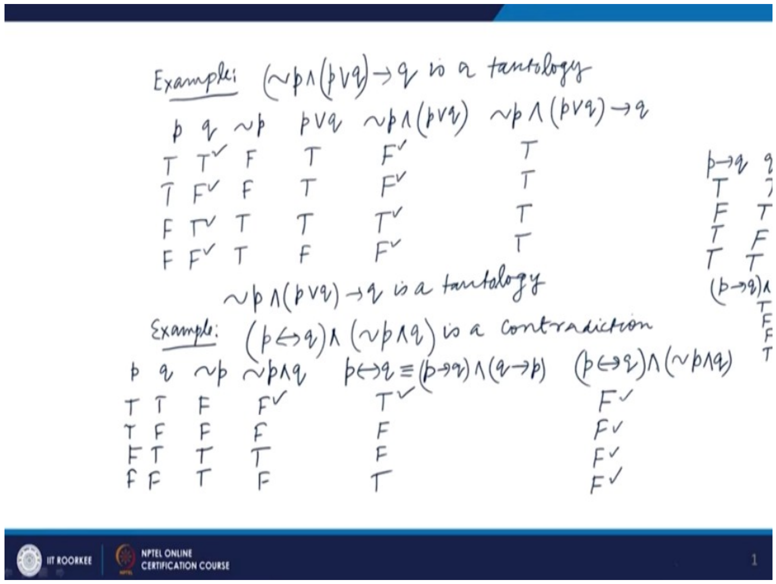

Let us consider the example $\sim p \wedge (p \vee q) \Rightarrow q$, so let us show that it is a tautology, so let us form the truth table for this, so p q have truth values TT TF FT FF, then ~p has truth values F F T T, now $p \vee q$, let us write the truth values $p \vee q$, when $p \vee q$ both have the value of T, $p \vee q$ has T, T and F give you T, FT give T, FF gives you F, then $\sim p \wedge (p \vee q)$, so $\sim p \wedge (p \vee q)$, so ~p has truth value F $p \vee q$ has truth value T, so we have F here and then it is F this is T, so again we have F then this is T this is T we have T here we have T here, here F so we have F her, now $\sim p \wedge (p \vee q) \Rightarrow q$.

So, let us look at this column here we have F and here we have T, so we have T here, here we have F here we have, so we have T here, here we have T here we have T, so this proposition gives the truth value T and then we have F here and here we have F, so we get T here, so you can see the column corresponding to the proposition, $\sim p \wedge (p \vee q) \Rightarrow q$ as truth value T for all combinations here and therefore $\sim p \wedge (p \vee q) \Rightarrow q$ is a tautology.

Now, let us take an example on contradiction, so let us consider the example $p \Leftrightarrow q$, $(p \Rightarrow q) \wedge (\sim p \wedge q)$, we have to show that it is a contradiction so we have to show it is a contradiction, let us write its truth table, so p, q, ~p, $\sim(p \wedge q)$ then $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$, so we have to write the truth values of $p \Leftrightarrow q$ by finding the truth values for $(p \Rightarrow q) \wedge (q \Rightarrow p)$ and then we consider lastly$(p \Rightarrow q) \wedge (\sim p \wedge q)$ we must have truth values F here for all combinations.

So, p q have truth values T T T F F T F F, then ~p has truth value F F we have T here we have T here now $\sim p \wedge q$ will have truth value F will have value F will have truth values T and will have truth value F, now $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$, so let us find the truth value for $p \Rightarrow q$, so $p \Rightarrow q$, T T gives T here, then T F TF gives F here, then FT gives T, then FF gives T, then $q \Rightarrow p$, so $q \Rightarrow p$ TT gives T, then FT gives T, then TF gives F, and then FF gives T, now $(p \Rightarrow q) \wedge (q \Rightarrow p)$ we have to consider,$(p \Rightarrow q) \wedge (q \Rightarrow p)$.

So, TT T FT F then TF F and then TT T, so we get T F F T, now $(p \Leftrightarrow q) \wedge (\sim p \wedge q)$, so let us consider this column and this column, so T here F here gives F here F, F gives F, F T gives F, and then TF gives F, so you can see in the column corresponding to the proposition $(p \Leftrightarrow q) \wedge (\sim p \wedge q)$ has all truth values F and therefore we can say that$(p \Leftrightarrow q) \wedge (\sim p \wedge q)$ is a contradiction, so that gives the solution of this example thank you very much.