Higher Engineering Mathematics
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Lecture 28
Paths and Connectivity

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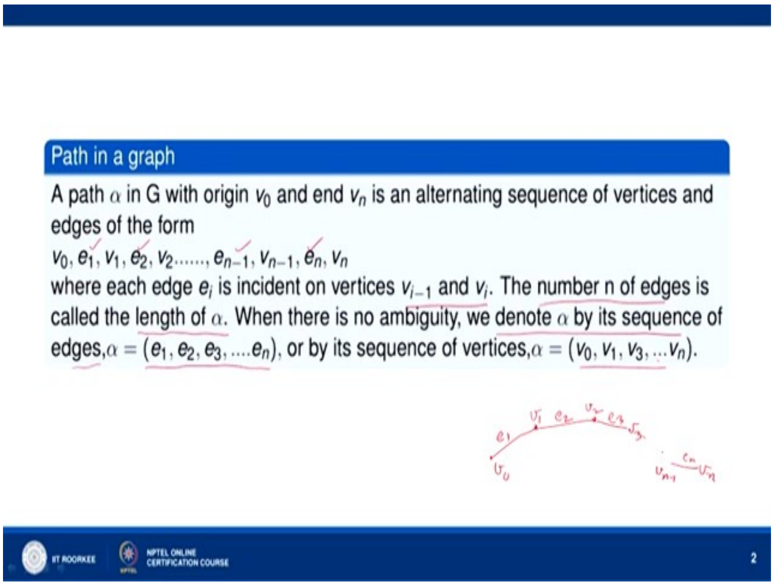

Hello friends, welcome to my lecture on Paths and Connectivity. A path alpha in a graph G with origin v_0 and end v_n is an alternating sequence of vertices and edges of the form v_0, e_1, v_1,e_2 , v_2, e_n- 1, v_n - 1,e_n, v_n say this is v_0 , then e_1 ,v_1,e_2 , v_2 , and so on, okay and we have this is say e_3 , v_3 and so on we have then v_n - 1, $e_n$and v_n , okay so a path α in G in a group G in a graph G with origin v_0 and end v_n is an alternating sequence of vertices and edges of the form v_0 , e_1,v_1, v_2, e_2 , and so on e_n - 1 , v_n - 1 , $e_n$and v_n, where each edge e_i is incident like for example this is e_1, e_1 is incident on v_1 and v_0 , so e_i is incident on vertices v_i - 1 and v_i.

The number n of edges now if initial vertex is v_0 the end vertex is v_n then there are n edges e_1 , e_2, e_3, e_n - 1, and e_n, so the number n of edges is called the length of α. Now, if there is no ambiguity α can also be denoted by its sequence of edges that is we can write $\alpha = e_1 , e_2 , e_3 \ldots e_n$ or by its sequence of vertices $\alpha = v_0 , v_1, v_2, \ldots v_n$.

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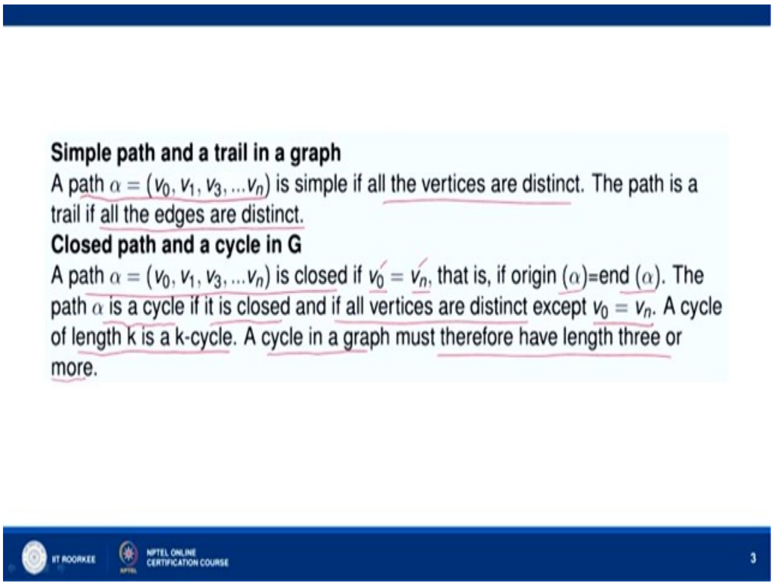

Now, let us define a simple path. A path α = (v_0 ,v_1, v_2, … v_n) is called simple if all the vertices are distinct, okay so the path is called a trail if all the edges are distinct, okay so there is a difference between a path and simple path and trail in the case of simple path all the vertices are distinct and in the case of trail all the edges are distinct.

Now, let us define a closed path. A path α is called closed if the initial vertex and the end vertex they coincide that is $v_0 = v_n$ that is origin of alpha is same as end of α and the path α is a cycle if it is closed and all vertices are distinct except v_0 and v_n. So, in the case of a cycle the path must be closed and all vertices must be distinct except the initial vertex v_0 and the end vertex v_n.

A cycle of length k, a cycle of length k is called k-cycle, okay. So a cycle in a graph must therefore have length three or more, okay in a graph cycle cannot be having cannot have length less than 3, okay.

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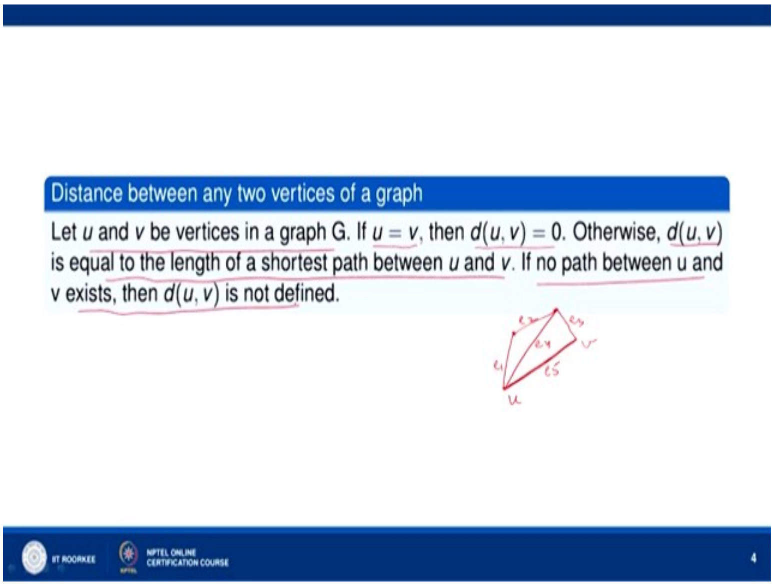

Now, let us say u and v are two vertices in a graph let us say u and v are two vertices in a graph. If u = v, then distance between u and v = 0, otherwise d (u, v) is equal to the length of the shortest path between u and v, if u and v are any two vertices in a graph say for example u is here, v is here, okay then the length between u and v is defined as the length of the shortest path between u and v, from u to v we can reach directly, okay we can also reach like this, like this or we can reach we can reach like this, okay.

So, suppose this is e_1, e_2, this ise_3, okay and this is e_4 and this is e_5 okay then length between u and v will be equal to 1 because this is the shortest path, okay. So and there are three paths this one we can go (from e_2) via e_1, e_2 and e_3, we can also go via e_4, and then e_3, but the path which has the shortest I mean which has the minimum length is this edge which joins u to v directly and so its length is 1, okay. Now, if no path between u and v exists, then u, v is d (u, v) is not defined.

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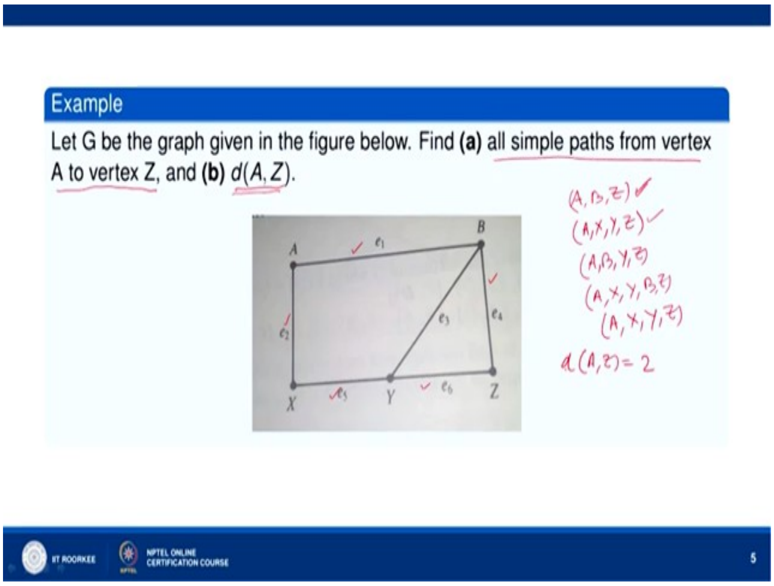

(Let G be the vertex given) Let G be the graph given in this figure, okay find all simple paths from vertex A to Z, okay. So, let us write all simple paths from vertex A to Z, okay so we have A, B, Z, okay A, B, Z, okay so we have taken this one A, B, Z this one okay then A, X, Y, Z we have taken this one, this one, this one A, X, Y, Z then we can take A, B, Y, Z, okay so A, B, Y, Z then we can take A, X, Y, B, Z, okay A, X, Y, B, Z, okay and then we can take so we have taken A, B, Z we have taken A, B, Z, we have taken A, B, Y, Z, we have taken A, X, Y, Z, we have taken A, X, Y, B, Z and we can also take A, B, Z, okay A, B, Y, Z, okay A, X, Y, Z, A, X, Y, Z we also have A, X, Y, Z, okay we have so that we have five simple paths from vertex A to vertex Z.

Now, (we have to find determinant) we have to find distance between A and Z, okay. Now this path A, B, Z it has distance 2 we have two edges to connect A to Z, in the case of A, X, Y, Z we have A, X, X, Y, Y, Z three edges to connect A to Z, in the case of A, B, Y, Z we have 1 edge, 2, 3, three edges to connect A to Z, and in the case of A, X, Y, B, Z we have 1, 2, 3, 4, okay four edges and in the case of A, X, Y, Z we have three, so A, B, Z if you connect it so the distance between A and Z will be = 2, okay because this is given by A, B, Z, okay this if we follow this path A, B, Z then this is the shortest path, so distance between A and Z is 2.

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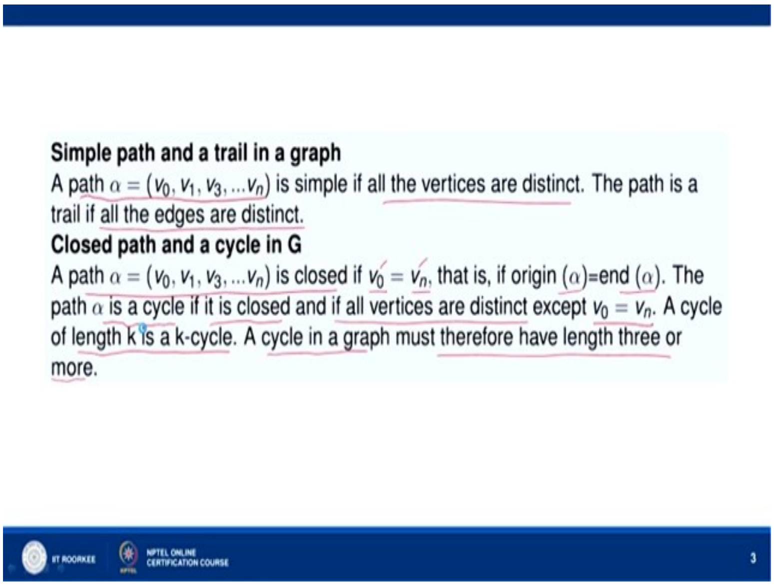

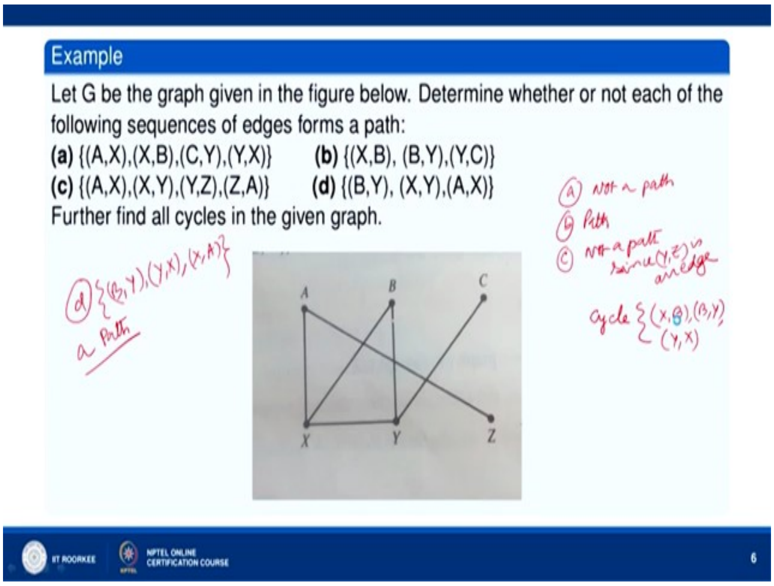

Now, let G be the graph given in the figure below, okay. Determine whether or not each of the following sequences of edges forms a path. So A, X, X, B, okay A, X, X, B, okay then C, Y, okay C, Y, C, Y and we have Y, X, okay between Y, X, okay. So A, X, X, B, okay now after b, X, B, C, Y, okay we have C, Y, okay so between B and C there is no edge, okay between B and C there is no edge, okay so therefore (a) is not a path, okay and then we take X, B, B, Y so X, B, B, Y and Y C okay (b) is a path, (c) A, X, okay X, Y, Y, Z, now between Y and Z there is no edge, okay Y, Z, Y, Z is not edge since Y, Z is not edge it is not a path, Y, Z is not an edge, okay so (c) is not a path, then (d) B, Y, B, Y, okay Y, X, Y, X, okay yeah so B, Y, X, Y it is written B, Y, X, Y, A, X can be written as (d) can be written as B, Y, B, Y then Y, X and then X, A, okay so B, Y, Y, X, X, A so (d) is a path it is a path.

Now, find all cycles in the given graph, okay so cycle is a closed path, cycle is a closed path it is a closed path and if all vertices are distinct except v_0 and v_n , okay so X, B, B, Y, Y, X, okay a cycle could be X, B, B, Y, Y, X, okay all the vertices are distinct, okay and it is a closed path that is one (cyc) so this is cycle, okay. Now another cycle could be okay so cycle is cycle is given by X, B, B, Y, Y, X, okay in this graph.

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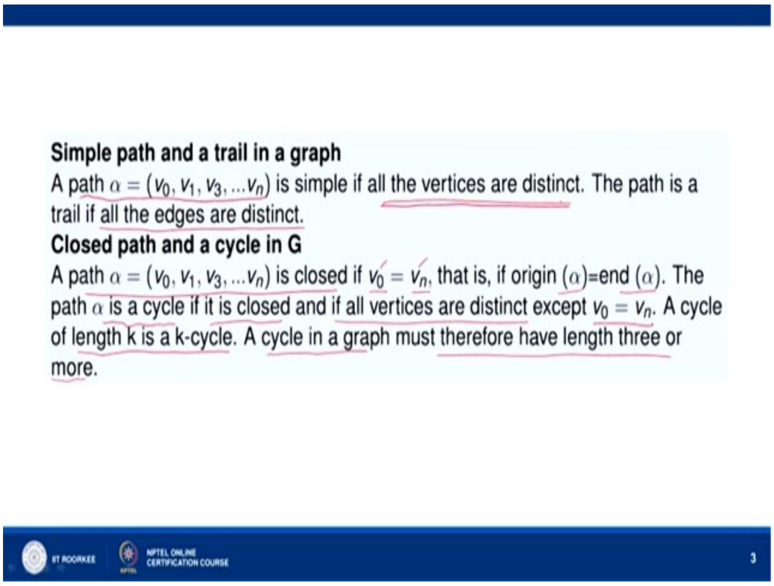

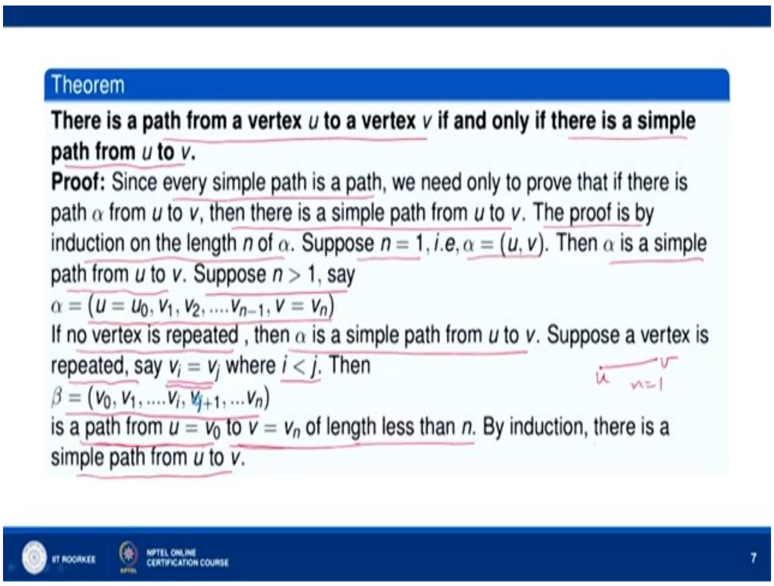

Now, let us go to this theorem, there is a path from a vertex u to a vertex v if and only if there is a simple path from u to v, okay so for a path from u to v if means there will be a path from the vertex u to the vertex v if and only if there is a simple path from u to v and we have defined simple path as a path whose all vertices are distinct a path is called simple if all the vertices are distinct there, okay.

So, let us see how we proof this theorem. Now, let us assume that let us assume that there is a simple path, okay we assume that there is a simple path from u to v and we want to proof that there is a path from a vertex u to a vertex v, okay. So let us assume that there is a simple path from u to v and we want to prove that there is a path from vertex u to vertex v, so if since every simple path is a path, okay therefore, if there is a simple path from u to v, then there is a path from vertex u to the vertex v because every simple path is a path, okay.

Now, conversely let us assume that, there is a path from vertex u to vertex v and we want to prove that there is a simple path from the vertex u to the vertex v, okay. Suppose there is a path from a vertex u to a vertex v and then we want to prove that there is a simple path from the vertex u to the vertex v, this is what we have to prove, then we can prove this by induction on the length n of the path alpha, okay.

So, suppose n = 1, suppose n = 1, then α will be equal to u, v, okay u here and v here, okay they are connected, okay there is a path between u and v, the edge joining u to v. So in this case α = 1, because this length of α, length n = 1, n is the length of α, okay n = 1 means α = (u, v), now, so clearly α is a simple path from u to v, okay because a path is called simple if all its vertices are distinct, okay.

Now, suppose n > 1 , suppose n > 1 , and α $=(u=u_0, v_{1,} v_2, \ldots v_{n-1}, v=v_n ¿$, okay. So, u the vertex u we have called denoted by u_0 , the vertex v we have denoted by v_n, okay and the other vertices are $v_{1,} v_2$, v_n - 1 . Now, if no vertex is repeated, okay no vertex is repeated then alpha is a simple path from u to v.

If a vertex is repeated say for example, okay say a vertex is repeated that is v_i, v_j are same, where i < j, okay then what we can do? We can consider beta, this beta, okay as v_0,v_1 ,v_i, v_{j+1} , v_n, okay. Now this defines a path, okay this is a path which has length less than n, okay. See by induction, since we want to prove it for n and so we are assuming that result holds for all values of less than n, so the result for this holds true and therefore, this must be a simple path, okay.

So, since this is a path from u = v_0, to v = v_n of length less than n, okay it is a path and it is a, this must be a simple path and therefore, there is a simple path from u to v, what we have said here? That if the verticesv_0, $v_{1,} v_2$, v_{n-1} ,v_n all are distinct, okay no vertex is repeated, okay then α is a simple path from u to v and if a vertex is repeated say v_i = v_j then u consider this, β = (v_0, v_1, v_2,… v_i, v_{j+1} …,v_n) then it is a path from u = v_0 to v = v_n of length less than n, okay. We are assuming that the result holds for length less than n, so there is a simple path from u to v. So this way the result holds for all n.

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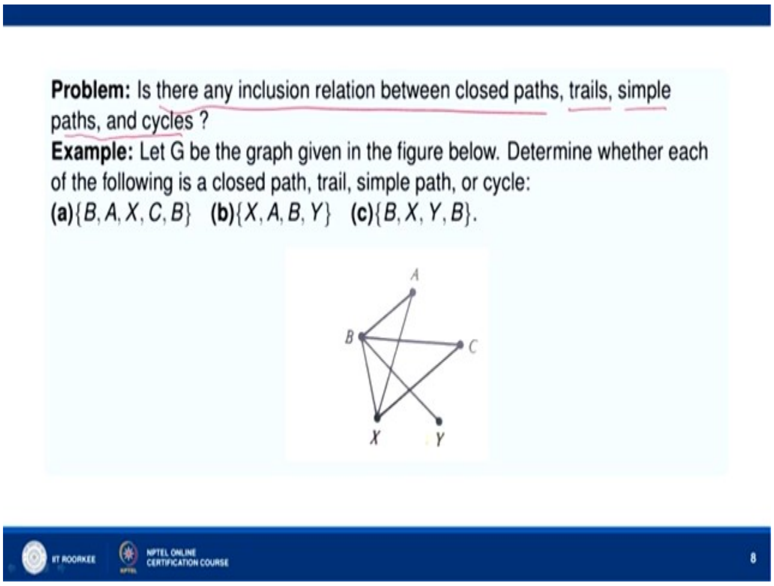

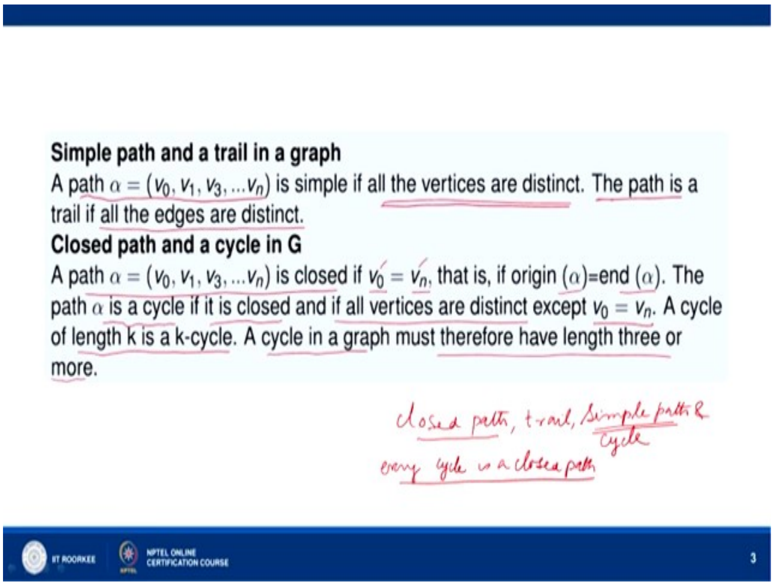

Now, let us proof, let us consider this problem is there any inclusion relation between closed path, trails, okay simple paths, and cycles? Okay, let us see what is a closed path, what is a trail, simple path and cycle, okay. Closed path, okay trail, simple path and cycle, okay, okay. So how did we design a closed path? A path is closed, if v_0 = v_n , okay that is closed path, the path is called cycle if it is a closed path, okay. So cycle is always the closed path, okay cycle is, every cycle we can take any cycle it is a closed path, so cycle, every cycle is a closed path by definition, okay so every cycle is a closed path.

Moreover, let us see what we call trail as? A path is called simple if all vertices are distinct, okay. So in the case all vertices are distinct, the path is called simple, okay. Now the path is called trail if all the edges are distinct, okay. So, if u consider a simple path all its vertices are distinct, okay and the path is called trail if all the edges are distinct. In the case of a cycle it is closed, all the vertices are distinct, except $v_0 = v_n$, so closed path, in the case of closed path the end vertices the initial v_0 and end which is a v_n join each other, I mean they are they coincide each other, they coincide so every cycle is a closed path, correct.

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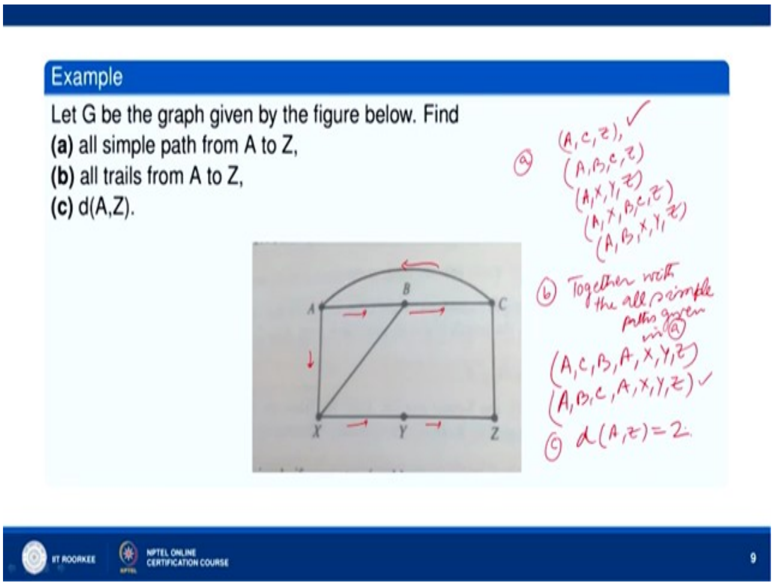

Now, let us take another one, let G be the graph given by the figure below. We have to consider all simple paths from A to Z, okay so let us list all simple paths from A to Z, so we have A to Z, okay so we have A, C, Z, okay then we have A, B, C, Z, okay and then we have A, X, Y, Z, okay then we have A, X, B, C, Z, we then have A, B, X, Y, Z, so there are five simple paths, okay from A to Z.

Now, all trails from A to Z, okay so together with these five simple paths so this is solution to part A and then we have together with the all simple paths given in (a), okay which are trails, okay these are also trails A, C, Z, A, B, C, Z, A, X, Y, Z, A, X, B, C, Z, A, B, C, Z, okay the trails are A, B, X (we could) we can have A, C, B, A, okay A, C, B, A, X, Y, Z, okay we can also have we can also have so we have A, C, B, A, X, Y, Z. Now, we can have A, B, C, A, okay A, B, C, A, X, Y, Z, okay so we go like this A, B, C, A, X, Y, Z, okay here what we did considered was (A, C, okay B,) A, C, B, A, X, Y, Z, okay and then what we can do so we have together with these paths we have these two more paths from A to Z, okay.

So together with these paths we have these two so they are all trails, okay and then (c) part, c part d A, Z distance between A and Z, distance between A and Z is 2 that is A, C, Z, okay given by this one, okay so distance between A and Z is 2.

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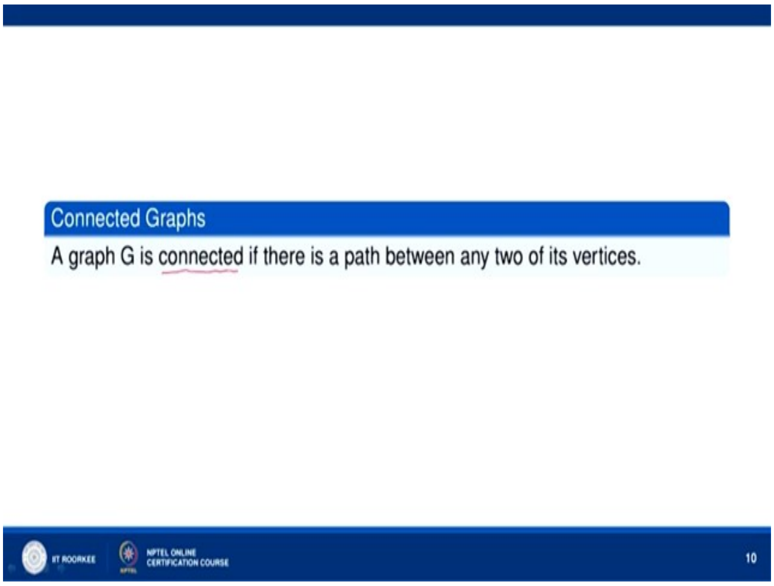

A graph G is connected if there is a path between any two of its vertices, let us define now a connected graph. A graph G is called connected if there is a path between any two of its vertices.

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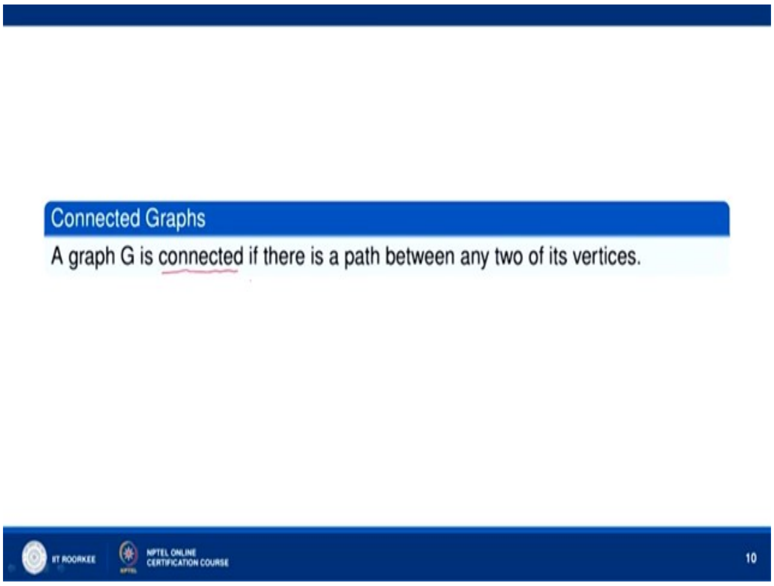

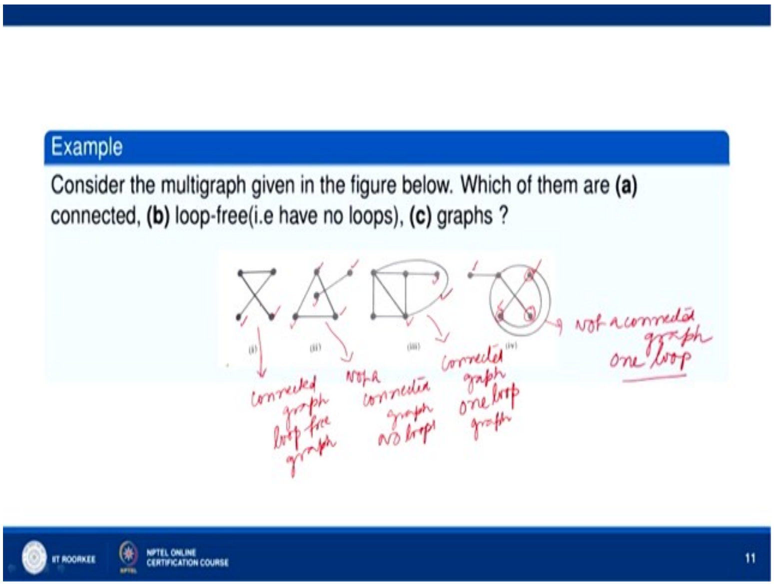

Now, consider the multi graph given in the figure below. Which of them are connected, loop-free (have no loops), graphs? Now you can see here, we have defined a graph to be connected if there is a path between any two of its vertices. So, we can see here if you take any two vertices here. Say for example you take this one and this one, they are connected like this, this way, this way, this way, they are connected and if you take these two they are connected, if you take these two they are connected, so this is a connected graph, it is connected graph. Moreover, it is loop-free, it is loop-free, okay and it is a graph, okay it is a graph.

Now, this one here, okay this one. Now there is no way to join this vertex with this vertex or this vertex or this vertex. Similarly, we cannot join this vertex with any of these vertices, okay this one, this one, this one, so it is not a connected graph. It is a graph, but it is not connected, okay. Now, further it has does not have loops no loops, okay. Now, let us see this one, in this graph we can see that if you take any two vertices, okay any two vertices are connected say if you want to connect this one to this one, you connect this to this, then you connect this to this, then this is connected so it is a connected graph and it has one loop, okay this one, this one, this loop, okay so it has one loop, okay. So it is a graph which is connected and has one loop.

Now, here we can see if you take yeah if you want to join this vertex to this vertex, then you join this to this and then you join this to this, yeah there is no way to connect yeah you cannot join this vertex with this or this, okay. So, similarly you cannot join this vertex, this vertex with this vertex and this vertex, so it is not a connected graph, okay it is not a connected graph. Moreover, it has one loop, okay this one loop it has, okay we agree two vertices same, okay if you have an edge, where the two vertices are same, okay then that is a loop. So here you can see for this for this edge the vertices are same so it is a loop.

So in the case of this graph it is not connected, but it has got one loop. In the case of this graph it is connected and has one loop, this is not a connected graph and have no loops, this is a connected graph, but it does not have any loop, it is loop-free graph.

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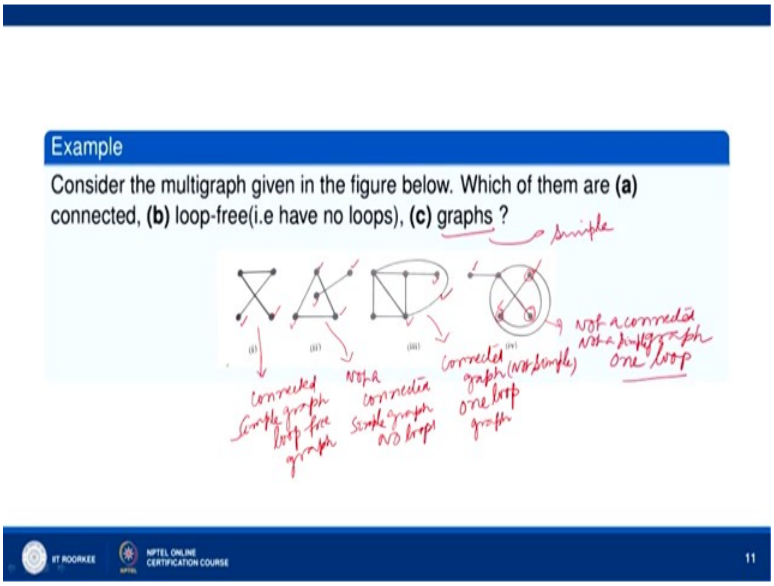

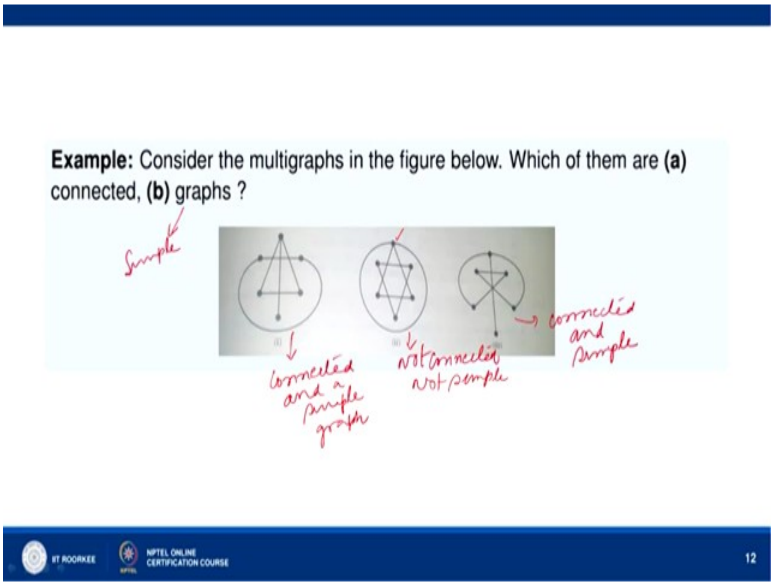

Now, consider the multi graph shown in the figure below. Which of them are connected and graphs? Okay, so we see here. Now, here by graph actually we mean simple graphs, okay simple graphs, so this is not a simple graph, you see it is not a graph because in the case of a simple graph we do not allow loops, okay so it is not a simple graph, not a, graph means this graph means simple graph, okay so not a simple graph, it is not a simple graph, here also not a simple graph, not simple, it is (multiple) multi graph, it is not a simple graph, okay. And here it is a simple graph, it is a simple graph, okay but it is not connected. And here we have a simple graph, which is connected.

Now, let us go to this one. Consider the multigraphs in the figure below. Which of them, now these are all multigraphs we see, okay and we want to see which one is a connected? Okay, so we see that if you want join this one to this one, we can go like this, usually, so this is a connected graph, this graph here again, by this graph we will simple graph, okay.

So connected it is you see, you take any two vertices, there is an edge joining them, okay and there is no loop here so it is connected and simple graph. Now, in this case here, you can see this is a loop, okay at this vertex you see that is a loop so this is a, and moreover, if you want to join this to this, how will you proceed? Okay, you join this to this, now it is not connected, b you cannot join this vertex to this vertex, there is no way to connect them, okay. So, or you cannot join this vertex to this vertex, this vertex to this vertex, okay so not connected, okay and it is not simple, not simple, okay.

And here what we have you see it is a connected graph, we can join, we are, we can join any two vertices here, okay it is a connected graph. See, if you want to join this to this, you come like this, okay if you want to join this to this, you come like this, if you want this to this, then what you do? You go like this and then you come like this, if you want to join to this to this, you come like this, and then this, okay okay. So it is connected graph and moreover it is a simple graph, okay. So connected and simple, okay.

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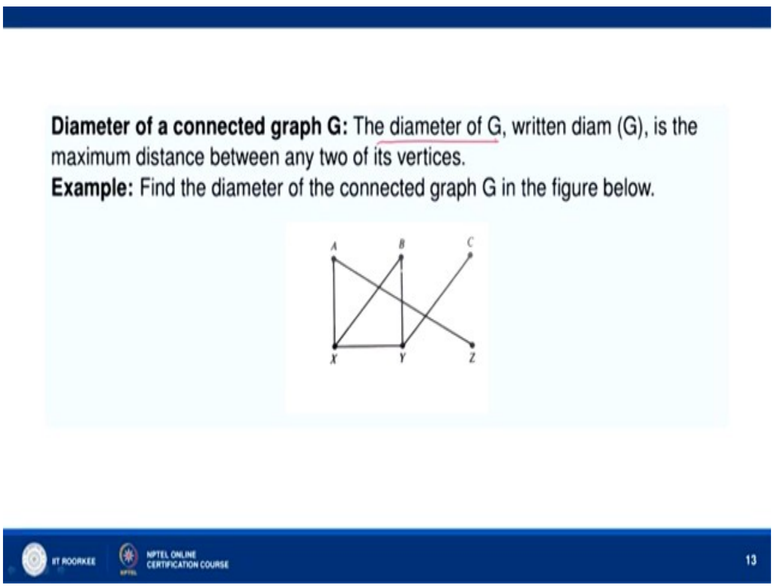

Now, diameter of a connected graph. The diameter of a graph G, which is connected written as diam (G) is the maximum distance between any two of its vertices. Find the diameter of the connected graph G in the figure below. So, now you take the vertex. Say for example we have to find the maximum distance between any two of its vertices. So maximum distance between A and C, okay between A and C, if we can find, maximum distance is how much? A, X, X, Y, C, Y, okay so that is 3, okay.

And then between A and B, so this is 2. Between A and Y, this is 2, okay. So then 1, 2, 3, 4, yeah so maximum distance is 4 here, you can see between A and C, if you take A and C, then A, X, X, B, B, Y, Y, C, okay so diam (G) here is equal to 4, okay. By taking the path A, X, B, Y, C, okay if you take this path, if you take this path, then the length is 4 and that is the maximum length. So the maximum distance, so the distance between A and distance diameter of this graph G is the maximum distance between A and C that is 4, so that is all in this lecture. Thank you very much for your attention.