Higher Engineering Mathematics
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Lecture - 21
Boolean Algebra - III

Hello friends, welcome to my lecture on Boolean Algebra, this is third lecture on Boolean Algebra. Let us first prove some properties of a Boolean Algebra, let a, b, c be any elements in a Boolean Algebra B, then we have idempotent laws $a+a=a, a*a=a$.

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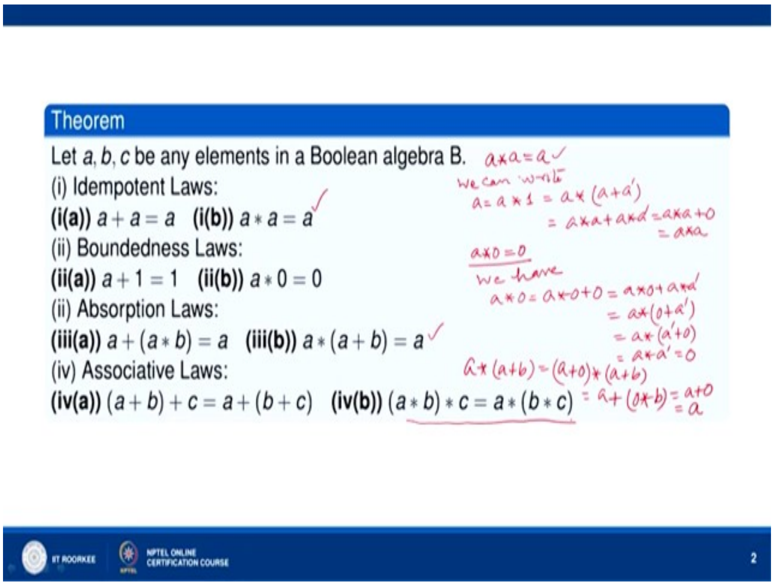

Let us prove one of them, the other one will follow by duality, so we prove $a*a=a$, ok we can write $a=a*1$ by identity law, ok so this is $a*(a+a')$, ok $a+a'=¿$ Now, this is equal to by distributive law $¿a+a*a'$, we know equal to 0, so we get $a*a+0$ which is equal to $a*a$, ok so we have $a=a*a$, so this is the proof for $a*a=a$.

The proof of $a+a=a$ can be written by writing the dual statement of every step here in this proof, ok we will get $a+a=a$.

Now, we have boundedness laws, ok boundedness laws means we have to show $a+1=1, a*0=0$. So, let us prove $a*0=0$, ok we shall prove $a*0=0$. So, $a*0$ we can write as $a*0+0=0$, now this is also equal to $a*0+a*a'$, ok $0=a*a'$.

Now, we can apply the distributive law, so this is equal to $a*0+a'$ which is same as

$a¿a'+0$ by commutative law, ok and this is equal to $a¿a'+0$ is a' and $a¿a'=0$, so we have $a*0=0$, it follows, the other one $a+a=1$ which is the other boundedness law follows by the duality theorem, ok.

Now we have (sec) absorption laws, ok. So, absorption laws as there are two absorption laws $a+(a*b)=a$ and $a*(a+b)=a$, ok so let us prove $a*(a+b)=a$, ok. So

$a*(a+b)=(a+0)*(a+b)$ ok which is same as $a+(0*b)$, ok so this is $a+(0*b)$ by distributive law and $a+(0*b)=a+0$which is equal to a, ok so this the absorption law.

Now, let us prove Associative law, ok. So we are going to prove the associative law this one, be there are two associative laws, one corresponding to the $+¿$ operation, the other one corresponding to $¿$ operation, we are going to prove the law corresponding to $¿$ operation $(a*b)*c=a*(b*c)$, the other one corresponding to $+¿$ operation will then follow by duality, so let us prove $(a*b)*c=a*(b*c)$.

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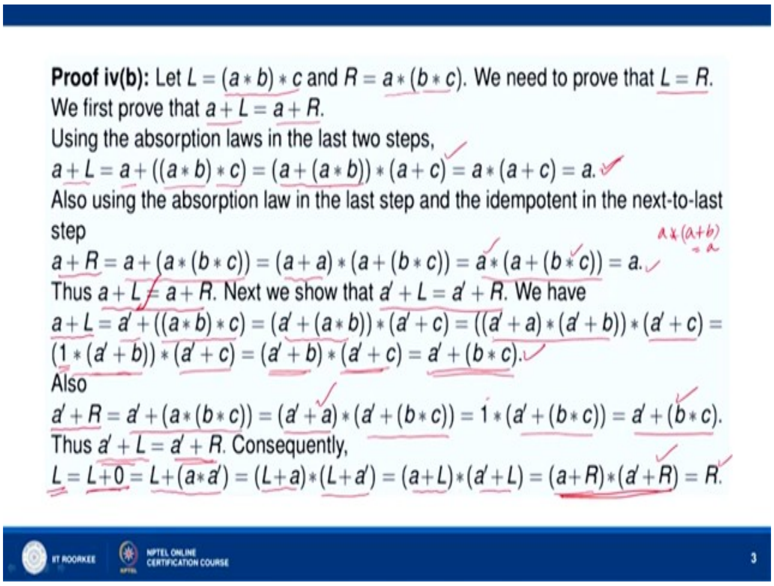

Let us assume that $(a*b)*c=L$ and $a*(bLc)=R$, so then we have to show that $L=R$. First we prove that if you take any (elem) if you consider $a+L$ then it is equal to $a+R$ ok. Now, let us see $a+L$ is how much? $a+L=a+((a*b)*c)$, ok and what we will do, this can be written as $a+((a*b)*c)$ by using the distributive law, this is $a+(a*b)$ then we have $¿a+c$, ok and $a+(a*b)$ by using absorption law is equal to a, so we get $a+(a*c)$ ok and then again we can use absorption law $a*(a+c)=a$, ok so we are using absorption law in the last two steps, this step and this step and we get $a+L=a$.

Now, let us consider $a+R$equal, $a+R$, ok so $a+R=a+(a*(b*c))$ and if you use the distributive law it is $(a+a)*(a+(b*c))$ ok and $a+a$ we know $a+a=a$ we have shown $a+a=a$, so $a+a=a*(a+(b*c))$ and again now here we are using absorption law, so $a*(a+(b*c))$, ok we know that $a*(a+b)$ for any b belonging to the Boolean algebra, this is equal to a so in place of b here we have $b*c$, ok so $a*(a+(b*c))=a$ and thus we have $a+L=a$ and $a+R=a$, so $a+L=a+R$. Now, let us prove that $a'+L=a'+R$, ok. So, a plus, $a'+L=a'+((a*b)*c)$ ok that is the value of L, ok so this is the equal to $a'+(a*b)$¿$*(a'+c)$ by using distributive law and $a'+(a*b)$ then will be equal to

$a'+a*$, ok $(a$¿¿$'+a)(a'+b)$¿ ok and we have ¿$(a'+c)$

Now, $a'+a$, ok $a'+a=1$ ok so we have $1*(a'+b)$ and we have $a'+c$.

Now, $1*(a$¿¿$'+b)=a'+b$¿ so $(a'+b)*(a'+c)$ we have and this can be written as

$a'+(b*c)$ using distributive law. Now, $a'+R$, let us consider $a'+R$ this is

$a'+(a*(b*c))$ and by using distributive law we can write it as $(a'+a)*(a'+(b*c))$ Now, $a'+a=1$,so we get 1¿$(a'+(b*c))$, ok.

Now, 1¿$(a'+(b*c))=a'+(b*c)$ so we see that the value of $a'+L$ is same as $a'+R, a'+(b*c)$ ok thus $a'+L$ is same as $a'+R$. Now, we have to prove that $L=R$.

So, we can write L as $L+0$ element, ok $L+0=L+(a'*a)$, ok and by using distributive law we can write it as $(L+a)*(L+a')$, $L+a$is same as $a+L$ by commutative law, so $(a+L)*(a'+L)$, $(L+a')$ can also be written as $a'+L$. Now, $a+L$ we have just now proved

$a+L=a+R$ this one, ok $a+L=a+R$, so let us put here and $a'+L$ we have shown equal to $a'+R$ so let us put here, ok.

Now, $(a+R)*(a'+R)=R$ ok is equal to this is equal to R, ok. So $L=R$ like just now we have proved, ok $(a+L)*(a$¿¿$'+L)$¿ was equal to $L+(a*a')$ here also, ok $a*(a$¿¿$'+R)$¿ this is nothing but this expression is nothing but $a*(a$¿¿$'+R)$,¿ ok $a*(a$¿¿$'+R)$,¿ so $a*a'=0$ so we have $0+R=R$, so we get $L=R$. So this proves that $(a*b)*c=a*(b*c)$and the other one, other associative law $(a+b)+c=a+(b+c)$follows by duality, ok.

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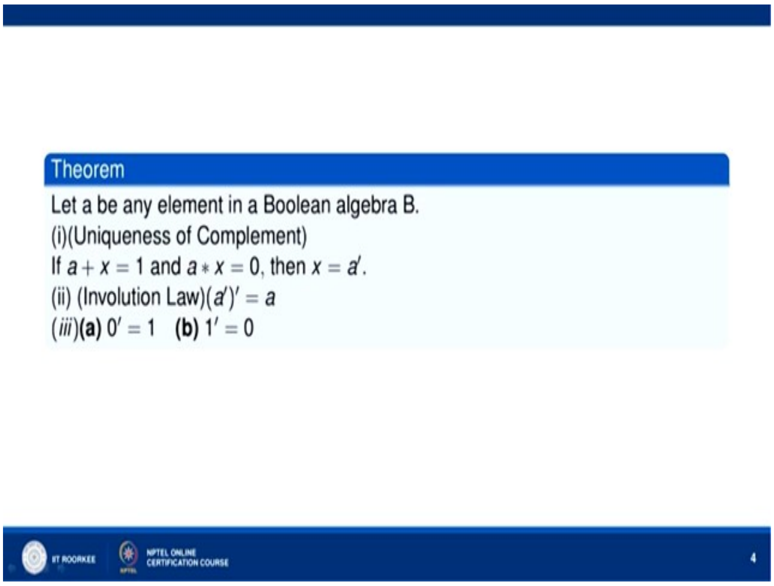

Now, let us a let a be any element in a Boolean algebra B, let us prove the uniqueness of complement, if $a+x=1$ and $a*x=0$ then $x=a'$ and we then prove Involution law $(a')'=a, 0'$ then we next $1=0'$ dash the $0'=1$ and then $1=0'$.

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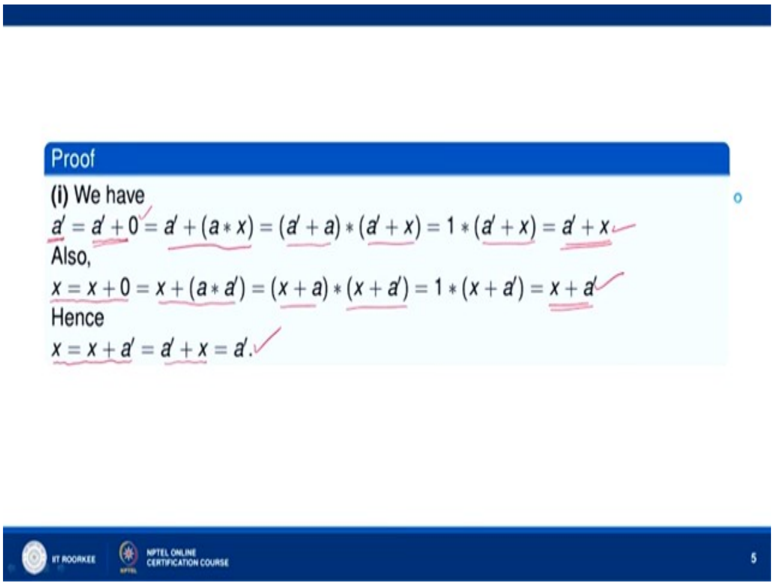

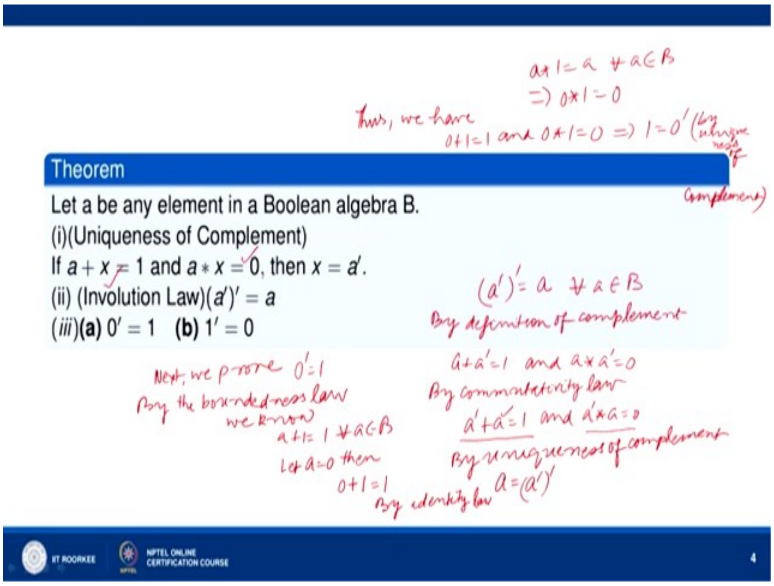

So, let us prove this one, first one if $a+x=1$ and $a*x=0$, then $x=a'$,ok. We have

$a'=a+0$,ok $a'+0$, 0 is given to be equal to $a*x$ ok 0 is given to be $a*x$, so we put $a*x$ for 0 here and we get $a'+(a*x)$, ok now use distributive law we can write it as $a'+a$ and then star $(a¿¿'+x),¿$ok $a'+a=1$, ok by identity laws, so this is $1*(a+x)$and $1*(a+x)$ again by identity law is equal to $a'+x$.

Now, let us take x, x can be written as $x+0$ and $0=a*a'$, so we get $x+(a*a')$ use distributive law here, we will have $(x+a)*(a¿¿'+x),¿$ ok. We are given that

$x+a=1$, $x+a$or $a+x$, ok $a+x$ or $a+x=1$ so let us put it there, so we get

$1*(x+a')=x+a'$. So what we have? $x=x+a'$, ok and $x+a'$ is same as $a'+x$ but

$a'+x=a'$, so we get $x=a'$, so this is the proof of uniqueness of complement, ok.

Now, let us prove¿ , ok. By definition of complement $a+a'=1$ and $a*a'=0$, ok by commutativity law we can write $a'+a=1$ and $a'*a=1$. Now, let us apply the uniqueness of complement which we have just now proved. So, if $a+x=1$ and $a*x=0$ then $x=a'$ here we see that $a'+a=1$ and $a'*a=0$, so we can say that a is the complement of a', ok so by uniqueness of complement $a=$¿ ok.

Now, let us prove $0'=1$, ok next we prove $0'=1$, ok by boundedness law, by the (boundess) boundedness law, we know $a+1=1, \forall a \in B$, ok. So taking $a=0$ ok let a be taken equal to 0, then $0+1=1$ ok by identity law, 0 we know that by identity law $a*1=1, \forall a \in B$, ok so this implies $0*1=0$, ok.

Now, what you see here again let us see $a+x=1$,so we have $0+1=1$ thus we have

$0+1=1$ and $0*1=0$, now apply the uniqueness of complement, so here $0'=1$, ok so this $\Longrightarrow 1=0'$, by uniqueness of complement. The other one, $1'=0$ can we proved similarly by duality, ok.

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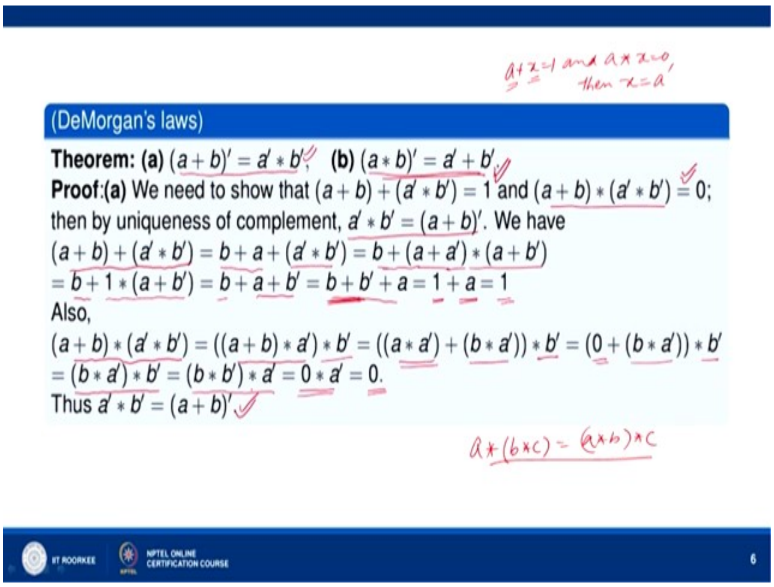

Now, DeMorgan's laws, ok. We have two DeMorgan's laws $(a+b)'=a'*b'$ and

$(a*b)'=a'+b'$, we shall prove the $(a*b)'$, we shall, we are going to prove that $(a+b)'=a'*b'$, so this one we are proving this will follow by duality, ok.

So for in order to prove that $(a+b)'=a'*b'$we have to show that $(a+b)+(a'*b')=1$ because we have just now seen that if $a+x=1$ and $a*x=0$, then $x=a'$, ok. So, we are going to show that you can take $x=a+b$, here, ok so we have we are taking x as yeah x' we are taking yeah we are taking a as $a+1, a+b$ we are taking a as $a+b$ and x as $a'*b'$ then $a+x=1$is to be shown and $a*x=0$ is to be shown, ok.

Now, so yeah so by uniqueness of complement then it will follow that $a'*b'=(a+b)'$, ok. Now, what we will do is? $a+b$, let us consider this $(a+b)+(a'*b'), a+b$ we can write by commutative law $b+a$ ok and then we have $+(a'*b')$ and this can be written as $+b$, now $a+(a'*b')$ can be written by distributive law as $a+a'$, ok ¿$(a+b'), a+a'=1$. So, we have $b+1*(a+b')$, now $1*(a+b')=a+b'$, so we get $b+a+b'$, now using $(a1)$ commutative law $b', a+b'$ can be written as $b'+a$ and then we can make use of the associativity, so $b+b', b+b'=1$ and we get $1+a=a$. So, we get this one the proof of $(a+b)+(a'+b')=1$.

Now, let us prove that $(a+b)*(a'*b')=0$, so $(a+b)*(a'*b')$is equal to this can be written as by using the distributive law $((a+b)*a')$, ok we are using here, ok we are using the associative law we are using the associative law $a*(b*c)=(a*b)*c$ we are using that, so $(a+b)*(a'*b')=((a+b)*a')*b'$ by using this law, ok then we use distributive law so $((a+b)*a')$ is $((a*a')+(b*a'))*b'$, now $a*a'=0$, so $0+(b*a')*b'$.So,

$0+(b*a')*b'$ so we get $(b*a')*b'$ and this then can be written as $(b*b')*a'$, ok. So, now $b*b'=0$, so we get $0*a'$and $0*a'=0$,so we get the proof of this other one other statement $(a+b)*(a'+b')=0$. Now, let us make use of uniqueness of complement so here we see that $a+x=1, a*x=0$, so $x=a'$ that is $a'*b'=(a+b)'$ and we get the result, ok. The other one can then be followed can then be proved by making use of duality.

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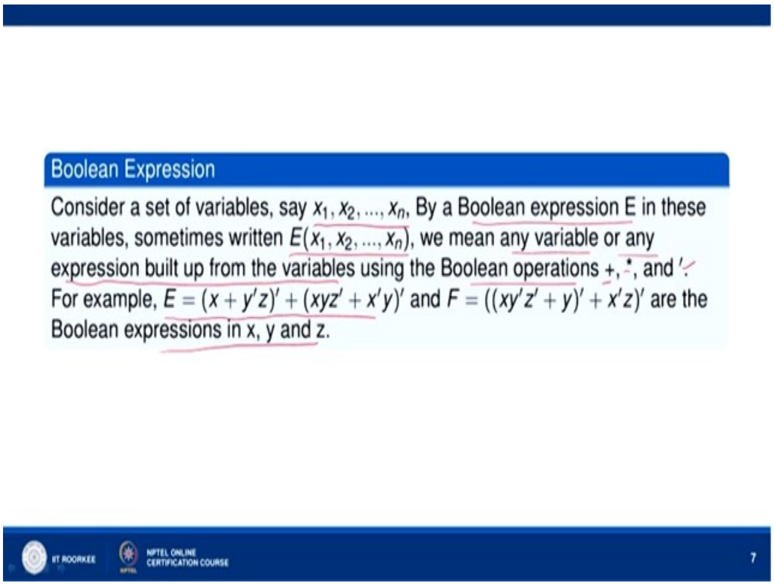

Now, let us come to a Boolean expression, let us consider a set of variables $x_1, x_2, \ldots, x_n$ by Boolean expression E in these variables sometimes we write it like this $E(x_1, x_2, \ldots, x_n)$we mean any variable are any expression built up from the that variables using the Boolean operations $+, *$¿ and then $'$. For example you can consider this

$E=(x+y'z)'+(xyz'+x'y)'$and $F=((xy'z'+y)'+x'z)'$, so they are all Boolean expressions in x, y, zok.

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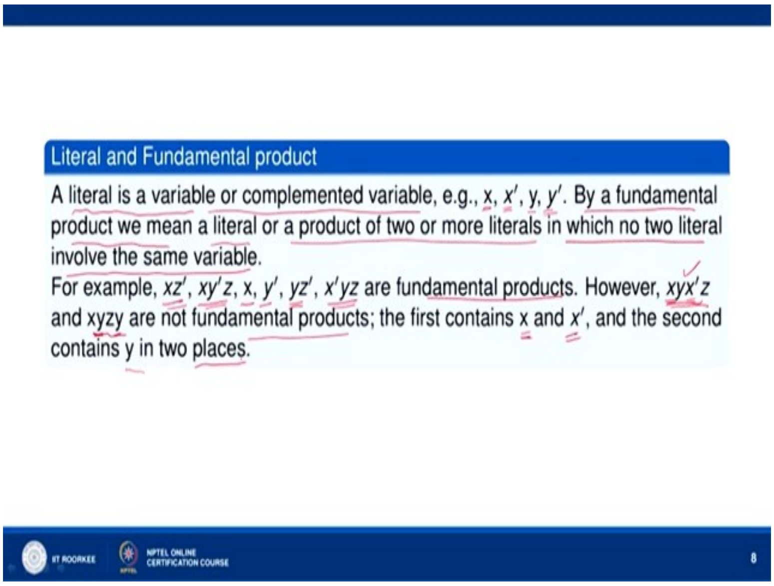

Now, what do we mean by a literal are fundamental (project) product? A literal is a variable or complemented variable say variable if the variables are x, y then x, y and their x', y' they are called literals, so literal is a variable or complemented variable by a fundamental product we mean a literal are a product of two or more literals in which no two literals involve the same variable, ok.

For example $xz', xy'z, xy', x'yz$ they are all fundamental products. However, if you consider $xyx'z$ ok. Then or you consider $xyzy$ they are not fundamental products because this (leat) this expression contains Boolean expression contains x and x', see x here, ok and x here, x' here, so it contains two x and x' you see here we are saying that no two literals involve the same variable, so two literals here x and x' involve the same variable x, so this is not a fundamental product here we have literal y occurring twice, ok. So, the second one contains y in two places therefore it is not a fundamental product.

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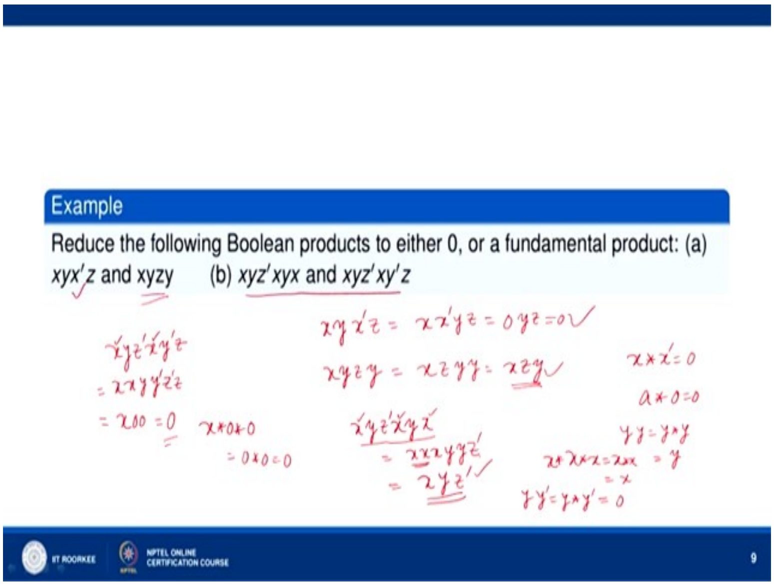

Let us reduce the following Boolean products to either 0 or a fundamental product, so we have $xy\,x'\,z$, ok by making use of associative law I can write it as $x\,x'\,yz$, ok by making use of commutative law and then associative law it is equal to $x\,x'\,yz$ and we know that $x ¿ x'$, ok $x ¿ x'=0$ ok. So, we get $x ¿ x'=0$, so $0\,yz$, ok $0\,yz=0$ because $a ¿ a'=0$, ok here $¿$ we have not written it $xy\,x'\,z'$means $x*y*x'*z'$, ok.

Now, then we consider $xyzy$, so this value is 0, xynow zy let us consider I can write it as $xzyy$ ok. Now, $yy=y*y$I mean what I mean is $yy=y*y$ and $y*y=y$ we know this idempotent law, this is idempotent law, so this is xzy, ok. Now, let us go to the second part, this one, ok so $xy\,z'\,xyx$ this is this can be written as by using commutative, associative laws I can write xxx I can write xxx and then I can write yy and we can write z', ok.

Now, $x*x=x$, ok so again so this is $x*x=x$ then $x*x=x$ so we get $x*x*x=x$ so we get x here and $y*y$ we have seen it is y, so xyz dash, ok.

Now, let us consider xyz dash, ok $x\,y'\,z$ so we can write it as xx, ok this x and this x and then we can write yy dash, ok $y\,y'$ and we can write $z\,z'$, ok we know that $xx=x$, $y\,y'$, $y\,y'=y\,y'=y*y'$, ok and $y\,y'=0$,so we get x$x=0$ and $z\,z'=0$,so we can write x is this is $x00$ means $x*0*0$ ok $x*0=0$, $0*0=0$ so we get value 0.

So this Boolean product $xy\,x'\,z=0$ is equal to 0, ok this Boolean product $xyzy$ can simplify too xzy or xyz you can say then $xy\,z'\,xyx$ is this one $xy\,z'\,xyx=xy\,z'$and then $xy\,z'\,x\,y'\,z=0$, and clearly you can see these are fundamental products these two because they involve the variables x, y, z and no two literals involve the same variable.

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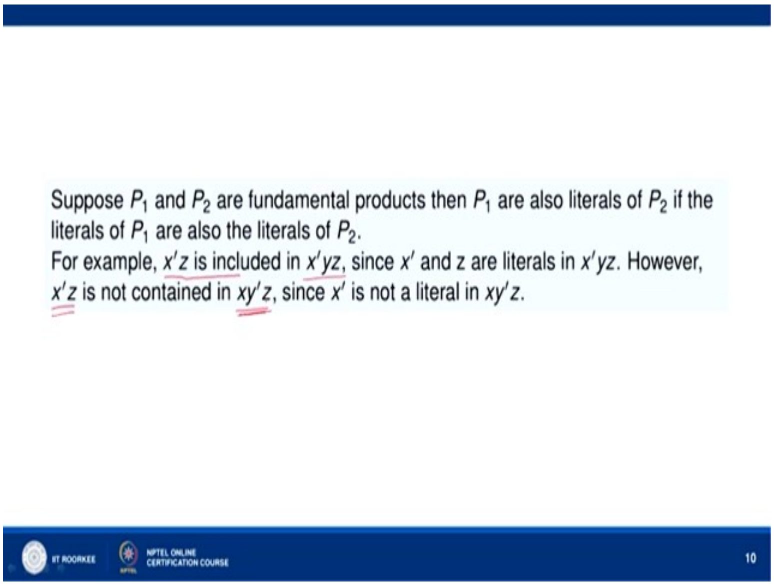

Now, let us go to, suppose P_1 and P_2 are fundamental products then P_1 is set to be included or contained in P_2, ok if the literals of P_1 are also the literals of P_2. For example let us consider $x^{'}yz$ ok and $x^{'}yz$ the literals of $x^{'}z$ are $x^{'}$ and z and the literals of $x^{'}yz$ are $x^{'}yz$, so literals of $x^{'}z$ are included in the literals of $x^{'}yz$ and therefore we can say that $x^{'}z$ is included in $x^{'}z$ or it is contained in $x^{'}yz$, ok.

Now, let us consider this one $x^{'}z$ and $xy^{'}z$ ok so $x^{'}z$ if you consider and $xyz^{'}$ you consider then literals of $x^{'}z$ are $x^{'}$ and z and here literals are $xy^{'}z$ ok so the literal $x^{'}$ is not contained here in $xy^{'}z$, so we can say that $x^{'}z$ is not contained or included in $xy^{'}z$.

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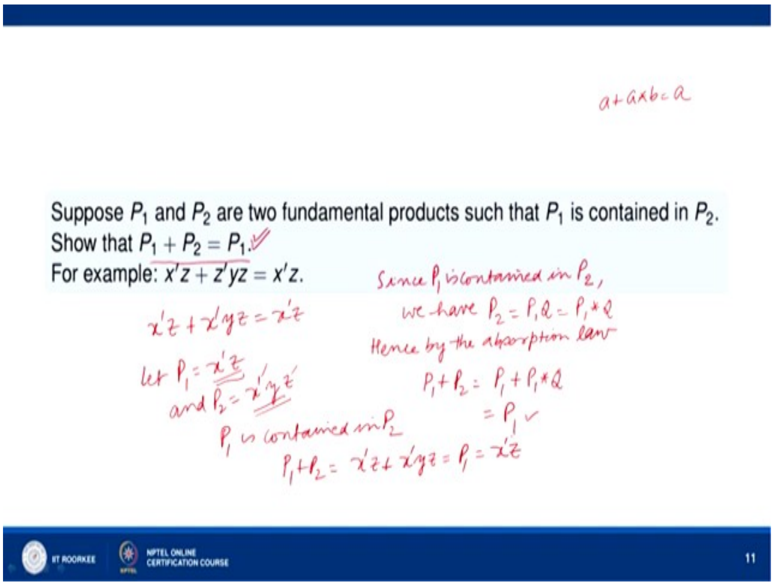

Now, let us suppose P_1 and P_2 are two fundamental products, ok such that P_1 is contained in P_2, let us show that $P_1+P_2=P_1$. Now, since P_1 is contained in P_2, P_1 is contained in P_2 we have $P_2=P_1 Q$ ok $P_1 Q$ or you can say $P_1 * Q$, alright. Now, by then by the absorption law $P_1+P_2=P_1+P_1*Q$, ok absorption law says that $a+(a*b)=a$, ok so $P_1+(P_1*Q)=P_1$, so $P_1+P_2=P_1$.

Now, let us show that $x' z+x' yz=x' z$, ok. So, you can see here that let us say let $P_1+P_2=P_1=x' z$ ok then the literals of P_1 are included in the literals of P_2, we have x' here, z here, ok so P_1 is contained in P_2 or included in P_2, ok and therefore by this theorem, ok

$P_1+P_2=x' z+x' yz=P_1$ that is $x' z$, ok.

So with that I would, (conc) I would like to end this lecture, thank you very much for your attention.