Higher Engineering Mathematics
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Lecture - 20
Boolean Algebra - II

Hello friends, welcome to my second lecture on Boolean Algebra. Let us consider this problem and see the Boolean algebra with D_{70}which consists of the divisors of 70, ok.

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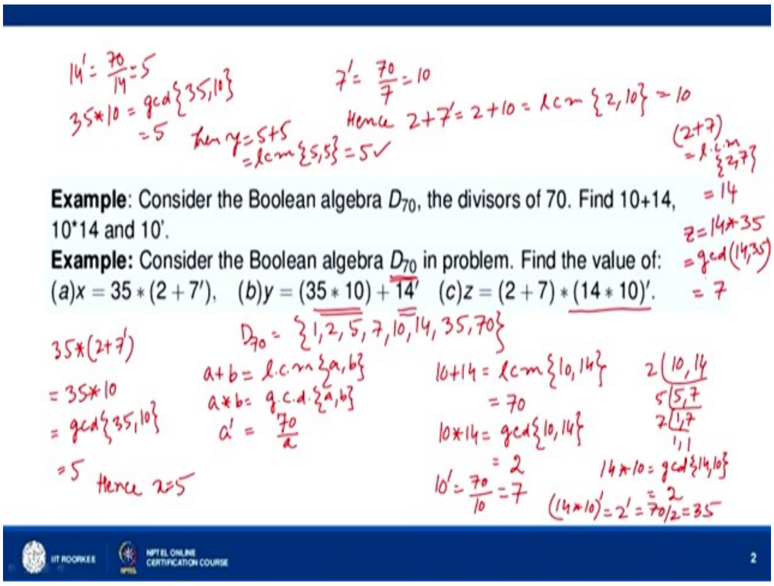

So, let us list the divisors of 70, D_{70} = {1, 2, 5, 7, 10, 14, 35, 70}, ok so they are the divisors of 70. Now, let us find the value of 10+14, ok remember that let us recall that $a+b$, $a+b$ the sum of a and b be defined as lcm(a, b) in $D_{70}$and $a*b$ be defined as gcd(a, b) and then complement of an element a in D_{70} we defined as $\frac{70}{a}$, ok.

So, 10 +14, let us look at 10+14 then it is lcm(10, 14), ok and lcm(10, 14) let us find, so 2, 5, 7, ok so we have 5, so this means $7 \times 5 \times 2$ that is 70, ok so lcm is 70, so 10+14 =70, $10*14$ is gcd(10, 14), ok so gcd(10, 14)=2 because the divisors of 10 are the, I mean 10 is divided by 2 and 5 and 14 is divided by 2 and 7, so gcd(10, 14)=2, ok and then $10'$, $10'$ means $\frac{70}{10}$, so that is equal to 7, ok. So the a value is of 10+14=70, $10*14$=2, $10'$=7. *Type equation here*.

Now, let us can you consider another example let us consider the Boolean algebra D_{70}, ok the the elements of $D_{70}$are {1, 2, 5, 7, 10, 14, 35, 70} we have to find the value of $x=35*(2+7')$, ok so $35*(2+7')$, so let us first find $7'$, $7'$ is equal to $\frac{70}{7}$, so this is equal to 10, ok hence 2+7' =2+10, ok.

Now, 2+10 means lcm(2,10), ok lcm(2,10) is 10, ok so we have so $35*(2+7')=$ $35*10$, ok. Now, this is equal to gcd(35,10) and we know the gcd(35, 10) =5, so the value of $x=5$, ok. Now, let us consider the other part (b) of this example $y=35*10+14'$, so let us first calculate $14'$, $14'=\frac{70}{14}=¿5$, ok and then for, $35*10=gcd\,(35,10)$, ok $gcd\,(35,10)$ is 5, ok.

So, we have the value of this as 5, ok so we have 5 plus 5. So then y=5+5, so there is a lcm (5, 5)=5, so value of y is also 5, ok y is also 5.

Now, let us take third part (c) $z=(2+7)*(14*10)'$, ok so let us calculate , $14*10$and then we shall calculate its complement, ok. So $14*10$, ok $14*10=gcd\,(14,10)$, ok so gcd(14, 10) is, the factors of 14 are 2 and 7, the factors of 10 are 2 and 5, so gcd(14, 10) =2, ok. So $(14*10)'$, ok $(14*10)'=2'$, $2'=\frac{70}{2}=¿35$, ok so the value of $(14*10)'$ is 35.

Now, 2+7 let us calculate 2+7, 2+7 = lcm(2, 7) and clearly lcm(2, 7) is 14, ok so we have $z=14*35$, ok $14*35$ means gcd(14, 35), gcd(14, 35) = 7 because the factors of 14 are 2 and 4, factors of 35 are 5 and 7, so z =7.

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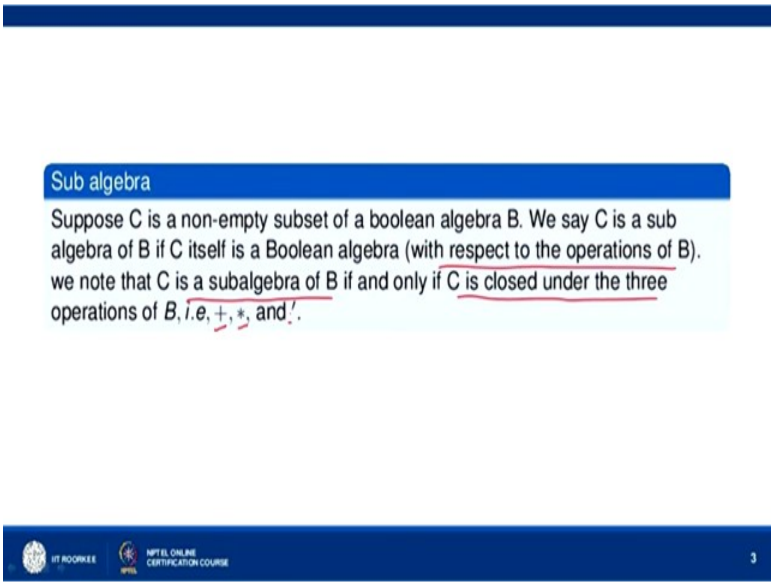

Now, let us go to Sub algebra. Suppose C is a non-empty subset of a Boolean algebra B. We say C is a sub algebra of B if c itself is a Boolean algebra with respect to the operations of B. We note that C is a sub algebra of B if and only if C is closed under the operations of B that is +, * and ′. So, if you take any two elements A, B ∈ C then A+B must belong to C, A¿B must belong to C and a′ must belong to C.

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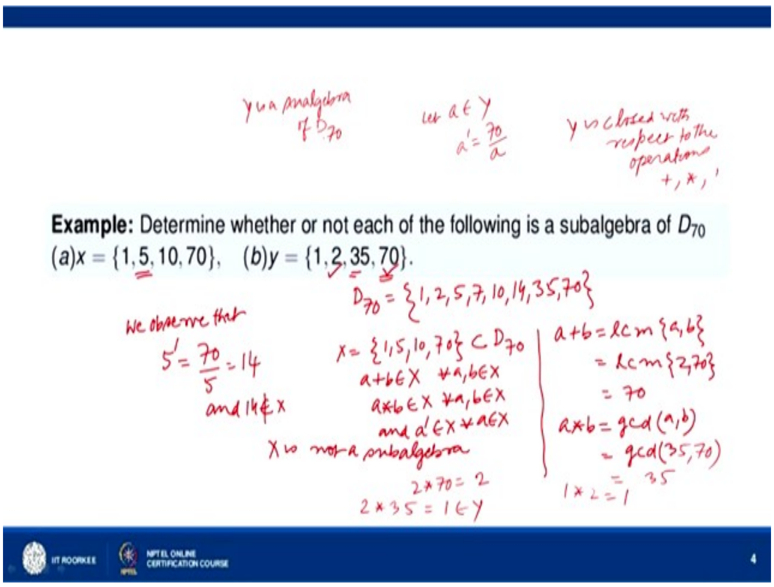

Let us look at this problem for example, determine whether or not each of the following is a sub algebra of D_{70}. Let us recall the elements of D_{70}they are {1, 2, 5, 7, 10, 14, 35, 70}, ok and we see that this set {1, 5, 10, 70}, ok is a subset of D_{70}, {1, 5, 10, 70} is a subset ofD_{70}. So, let us see whether it if it is a sub algebra of D_{70}, ok for that if you take any two elements belonging to this set x, ok this set x then $a+b$ must belong to x for all a, b ∈ x, a¿b ∈ x for all a, b ∈ x and a´, ok∈ x for all a∈ x.

Now, we see the this one, ok let us take this element 5, ok you see that the observe that the complement of 5, 5´, ok complement of 5, ok is equal to $\frac{70}{5}$ =14 and 14 does not belong to this set x, ok, so x is not a sub algebra, it is not a sub algebra, ok.

Now, let us take the set y, the set y if you take any two elements $a, b \in y$, ok then $a+b$, $a+b$ is the lcm(a, b), ok if you take any two elements for example let us say 2 and 70, ok so lcm of 2 and 70, ok lcm (2, 70)= 70, ok you take any pair of elements here their lcm is always there in the set y, ok.

So, if you take any two elements $a, b \in y$ then$a+b$ also is there in y and so y is close with respect to the operation of +, ok. And, then if you take any two elements $a, b \in y$ then a¿b, ok it is gcd(a, b), ok. So take any two elements say for example I take 35 and 70, ok the 35 and 70, then gcd is 35, ok. So and gcd belongs to this set y, if you take say for example 1 and 2, 1¿2, ok 1¿2 =1,gcd(1, 2)=1 and then if you take 2 and 70, ok 2¿70 gcd is 2, ok.

So, and you should take 2 and 35, ok then gcd is 1, ok because 2 and 35 both are divided by 1 each, ok and $1 \in y$, ok. So a¿b $\in y$, $\forall\, a, b \in y$.

Now, let us see whether complement of each element is there in y. So, let $a \in y$, ok then if you take a´ be defined as $\frac{70}{a}$, ok so if I take a = 1, $\frac{70}{1}$=70 and 70$\in y$,, if I take a=2 then 2´= $\frac{70}{2}$ which is 35, 35 is there in the set, if I take a =35, $\frac{70}{35}$=2 and 2 belongs to this set, if I take a =70 then $\frac{70}{70}$=1 ,which also belongs to the set.

So y is closed with respect to the operations +, ¿ and ´ and so y is a sub algebra, y is a sub algebra of D_{70}, ok.

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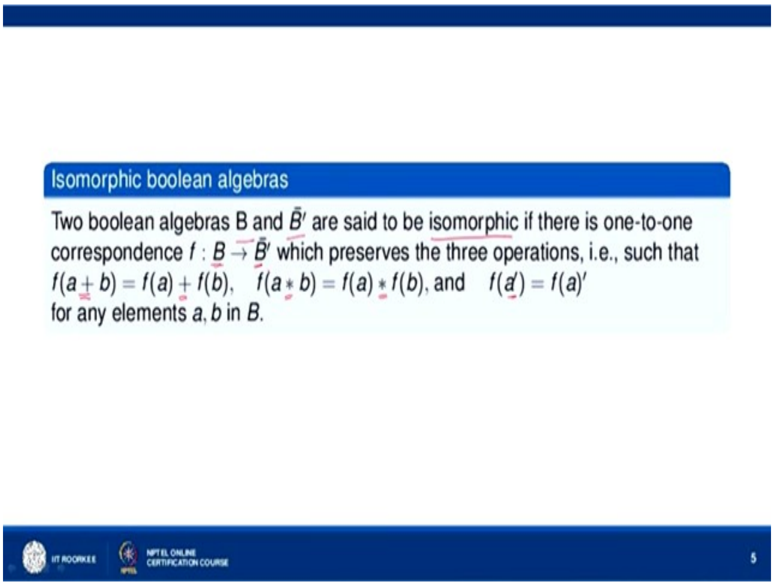

Now, let us go to Isomorphic Boolean algebras. Two Boolean algebras B and $\acute{B}'$ are called isomorphic if we can find a one to one correspondence f : B → $\acute{B}'$ which preserves the three operations such that f (a+b) = f (a)+f (b), f (a ¿ b)=f (a) ¿ f(b), f(a´ ¿ =f (a)´, ok.

Now here this, this + is define is a sum of is operation of sum operation in the set B while this plus is the operation in the sum operation in the set $\acute{B}'$, here a¿ is the sum operation in B is product operation in B and this is product operation in $\acute{B}'$.

Similarly, a´, a´ is the complement of a as defined in the set B and f(a´ ¿ is the complement of f(a) as defined in the set $\acute{B}'$, ok. So, we are I mean giving the same notations here, but they are not same, ok so if we can find a function f : B → $\acute{B}'$ such that it is one one and one two and f (a+b) = f (a)+f (b), f (a ¿ b)=f (a) ¿ f(b) , that is it if and f(a´ ¿=f(a¿´ , that is it preserves the three operations then we say that B and $\acute{B}'$ Boolean algebras are isomorphic.

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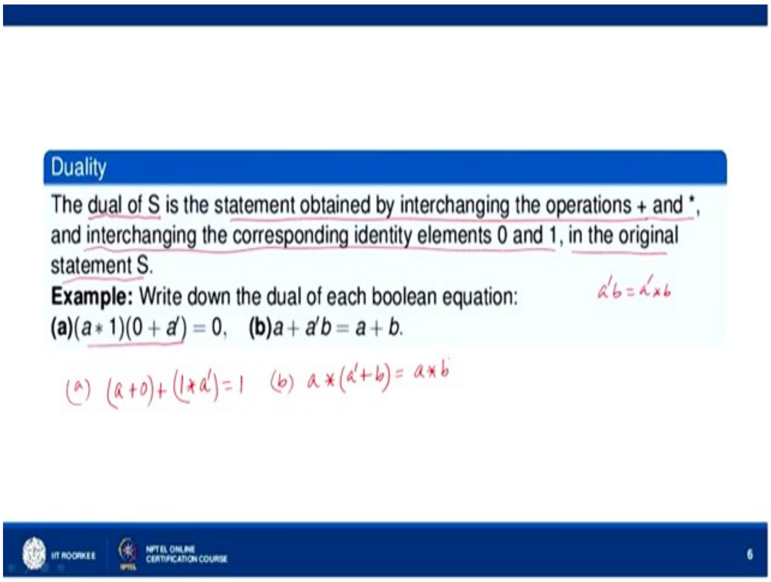

The dual of S is the statement, let us see how we define dual of a statement. Dual of S is the statement obtained by interchanging the operations +, ¿, ok and interchanging the corresponding identity elements, ok 0 and 1, ok. So, in the original statement S, ok. If we have to write down the dual of this equation a ¿ 1 then what we will do? a ¿ 1 let us say part a, ok.

So, a ¿ 1 ¿ will be replaced by +, so + identity element will be the unit element 1 will be replaced by the zero element 0, and then there is a ¿ here in between the two, ok so we replace that by +, ok and then we have 0+a´, 0 will be replace by 1,+ will be replaced by ¿, ok and a´ will be a´, ok equal to 1 that is part (a), the part (b) will be a ¿, + will be replace by ¿, ok a´∗¿b this is a´ ¿ b, ok a´ b is nothing but a´ ¿b, so we shall replace that by a´ + b, ok equal to a¿b, ok so they are the dual of the equations given in a and b.

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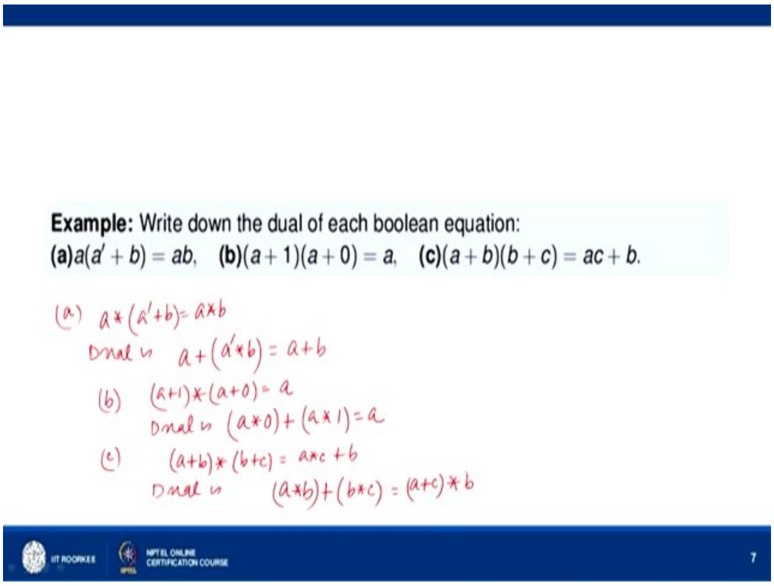

Write down the dual of each Boolean equation here, so this (a) part is a ¿¿a´+b) equal to a¿b, ab is a¿b, so this dual is a+(a´∗¿ b) =a+b, ok.

In part (b) we have (a+1) ¿ (a+0) = a, so dual will be (a ¿ 0) unit element 1 will be replace by 0 element (a ¿ 0) plus for this ¿ we write + and then (a ¿ 1), 0 will be replace by 1 equal to a, (a ¿ 0) +(a ¿ 1)=1, ok.

And (c) part, we have (a+b ¿∗¿(b +c) =a¿ c+b, ok so we have dual as a ¿ b, ok + b ¿c, ok and then here we have a ¿c will be replace by a+c and + will be replaced by ¿,b, ok so that is the dual in this case.

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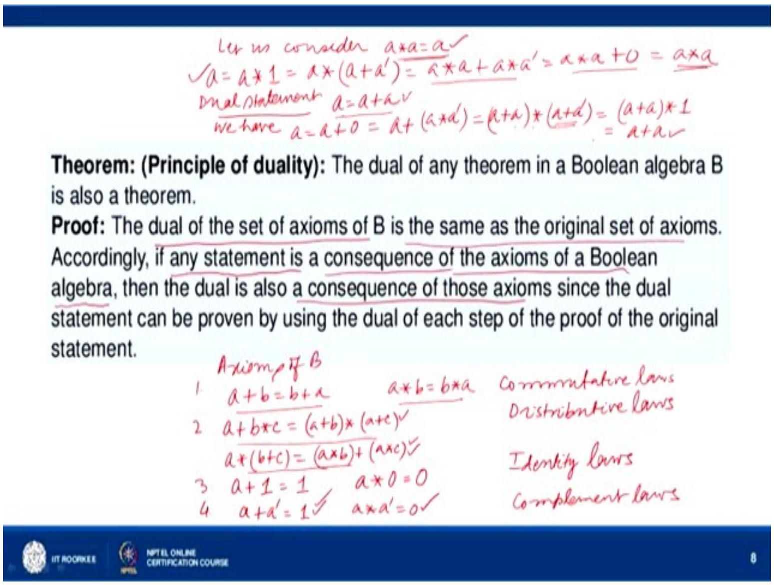

Now, let us go to principle of duality. The dual of any theorem in a Boolean algebra is also a theorem, why? Because the dual of the set of axioms of B is the same as the set original set of axioms. See what are the axioms of B? First is the commutative laws, so if you have any two elements a, b ∈ B then we have a+b= b+a and we have a ¿ b = b ¿a, ok you can see 1 is the dual of the other, ok.

Then, second one is associative laws, associative laws. So, we have a+(b +c) = (a+ b) + c we have district we have commutative laws, we have commutative laws, we have distributive laws, we have ok, we have commutative laws, distributive laws not associative laws, so we will write a commutative laws then distributive laws, so we have a +(b ¿ c) = (a + b)¿¿a +c), ok and then we have a ¿(b +c) = (a ¿ b)+ (a ¿ c), so they are commutative laws.

So, you can see one is the dual of the other, ok + is replace by ¿ here and ¿ is replace by + here, ok here + is replace by ¿, ¿ is replace by + and + is replace by ¿ so they are dual of each other.

And then third one is identity laws, identity laws a +1, ok if you take a +1, a+1=a, ok a+1=a and then its dual is a ¿ 0 =0, ok a+1=1, a ¿0 = 0 you can see a +1 =1 with dual of this is a ¿ 1 is replace by 0, 1 is replace by 0, ok.

Then, the fourth one is complement laws, ok for each element a ∈ B there is a´ ∈ B such that a+a´, ok a+a´=1 and a¿a´ = 0, ok. So you can see, if you look at the axioms here, ok then the dual of the axioms if you say the dual of this, ok dual of this is this and dual of this is this, so dual of these axioms is the same as the original set of axioms, if you look at this dual of this is this one and dual of this is this one.

So, you do not get any new equations they are they come in after you taking after you take the dual you have arrived at the same set of axioms, so the dual of the set of axioms of B is the same as the original set of axioms. Accordingly if any statement is a consequence of the axioms of the Boolean algebra then the dual is also a consequence of those axioms, ok since the dual statement can be proven by using the dual of each step of the proof of the original statement.

So, let us consider a¿a =a, let us consider this statement, ok we shall so first show that a¿ a =a, ok so a= a ¿ 1, ok a = a ¿ 1 and this is equal to a ¿, 1 =a+ a´, a + a´ and if I make use of these axioms then a ¿ a we get and we get a ¿ a´, ok a ¿ a =a ¿ a and a+a´, a ¿ a´, a ¿ a´ =0 so we get 0, ok and a ¿ (a+0) = a ¿ a, so we get a = a ¿ a, so we have proved that a¿a = a.

Now, let us consider its dual statement. The dual statement is obtained by writing by replacing a + by the ¿, ¿by + and then interchanging the unit and the zero elements, ok. So, we have a=a ¿1, we shall show that dual statement is a =a+a, ok a +a = a that is the dual statement, ok let us make write the proof of this we have a+a=a , writing the dual of each step here we shall arrive at the proof of this dual statement.

So, a=a +, ¿ is replace by +, 1 is replace by 0, a =a+0 then we have a+ we have a+a´, a´, so we write a ¿a´, ok the dual of a +a´ is a¿a´. Now, let us make use of this property a+(b ¿ c) =(a+b¿∗¿ (a+c), so this is a+a and we get ¿ a+ a´, ok a+ a´ is = 1, ok so we have (a+a) ¿1 and (a+a) ¿ 1 =a+a, ok so we have a= a+ a.

So, so you can see at each step here we are taking the dual of each statement corresponding step here in the proof of a ¿a = a, first step was a=a¿1 which we wrote as a =a + 0 here, next step was a ¿1= a ¿ (a+a´) which we wrote as the corresponding dual statement dual of that this step, so a **+**0= a **+**¿ (a¿a´), and then we make use of this, so yeah so here we are not using the property we are writing dual statement.

So a ¿a we wrote as a+a and then + is replaced by ¿, a ¿ *a´* is replaced by a+a´, ok and then we come to this, so a¿a becomes a+a, + becomes a ¿, 0 becomes 1, ok and then we have a ¿a which becomes a+a, so we are just writing dual of each statement while proving this dual statement, dual of each step here when we write we arrive at the proof of the dual statement, so this means that dual of any theorem in a Boolean algebra is also a theorem, ok.

So, that can be a I mean dual of any statement can be obtained by taking the dual of each step of the proof of the original statement. So that we have shown here by considering a the statement a ¿ a= a, we proved a¿a=a, ok this is the proof of a¿a =a, then we showed that the dual statement of the this statement a ¿a=a which is a +a=a can be obtained the proof of this dual statement can be obtained by writing the dual of each step here, ok just write the dual of each step here you will reach the when you will get the proof corresponding proof of a+a=a . So that is all in this lecture, thank you very much for your attention.