Higher Engineering Mathematics
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Lecture - 19
Boolean Algebra - I

Hello friends, welcome to my lecture on Boolean Algebra, this is first lecture on this topic. Boolean Algebra is an algebra of logic, one of the earliest investigators of symbolic logic was George Boole who invented a systematic way of manipulating logic symbols which became known as Boolean Algebra. We are going to discuss this Boolean Algebra, this is going to be very useful to Scientist and Engineers.

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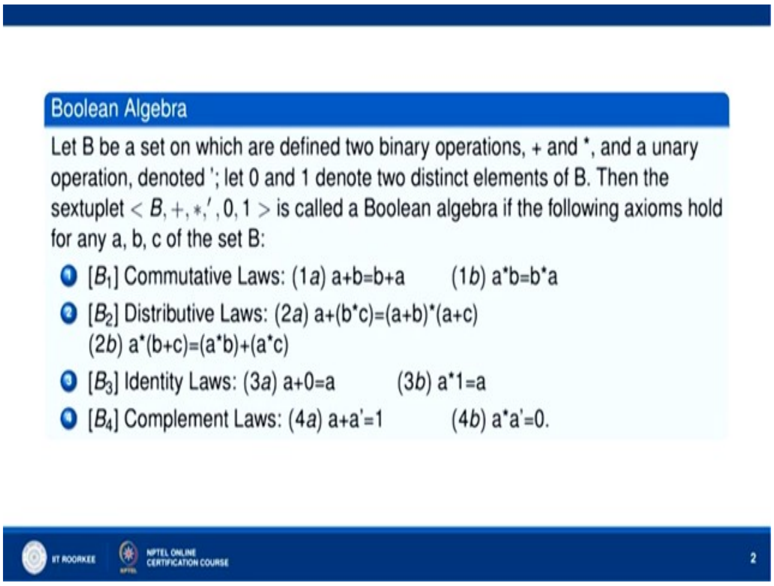

So, let us see, let B be a set on which are defined two binary compositions, + and ¿>¿´ and a unary operation denoted ´, let 0 and 1 denote two distinct elements of B. Then the sextuplet ¿ B, +, ¿ ,´, 0, 1 ¿is called a Boolean algebra if the following axioms hold for any a, b, c for any elements a, b, c of the set B. Commutative laws, a+b = b+a, a ¿ b =b ¿ a.

Distributive laws, a+(b ¿c) =(a +b) ¿ ¿a +c), a ¿ (b+c) =(a¿b)+ (a ¿ c). Now, this axiom is not common this is assumed here only in Boolean algebra, it is not there in other algebra axioms. So identity laws, a+0 =a, a ¿ 1= a. Complement laws, a+a´ =1, a¿a´ =0.

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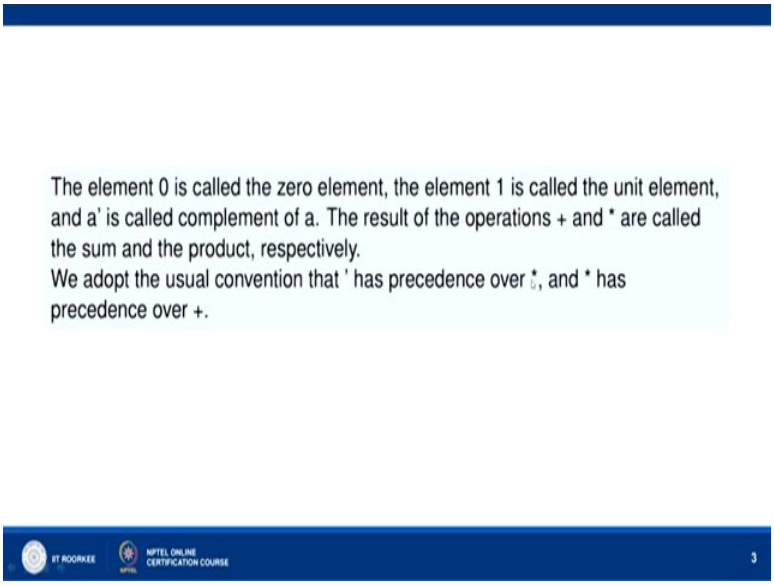

The element 0 is called the zero element, the element 1 is called the unit element, and a´ is called the complement of a. The result of the operations + and ¿ are called the sum and the product, respectively. We adopt the usual convention that ´, ok complement ´ has precedence over ¿ and ¿has precedence over +.

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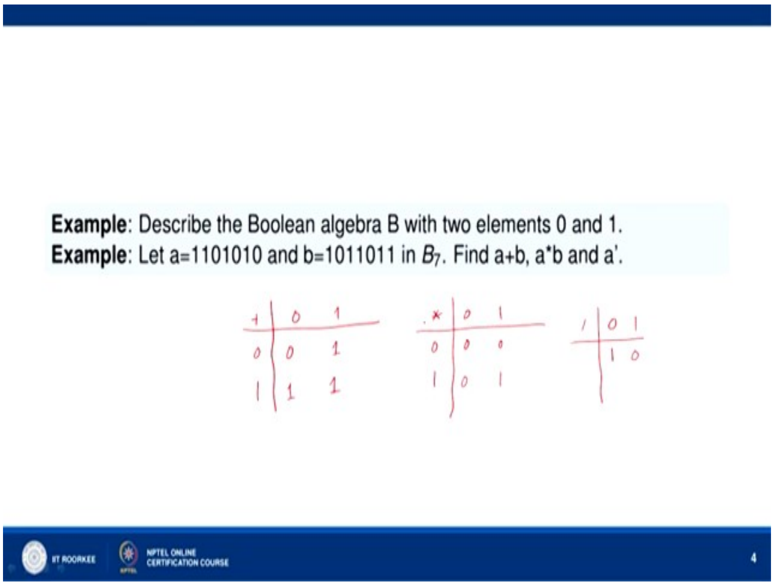

Let us describe the Boolean algebra B with two elements 0 and 1. So, let us say we have two elements 0 and 1, ok so then +, ok 0 + 0 0, 0 + 1 1, 1 + 0 1 and then 1 + 1 1, ok and then corresponding to ¿, ok let us write 0 1, 0 1, ok 0 ¿ 0 is 0, 0 ¿ 1 is 0, 1¿0 is 0, 1¿ 1 is 1 and corresponding to ´ 0 1 we have 1 0.

Now, let us see whether the set B with two elements 0 and 1 along with these tables, ok where we have defined the + operation, ¿ operation and ´ operation on B, ok becomes a Boolean algebra or not.

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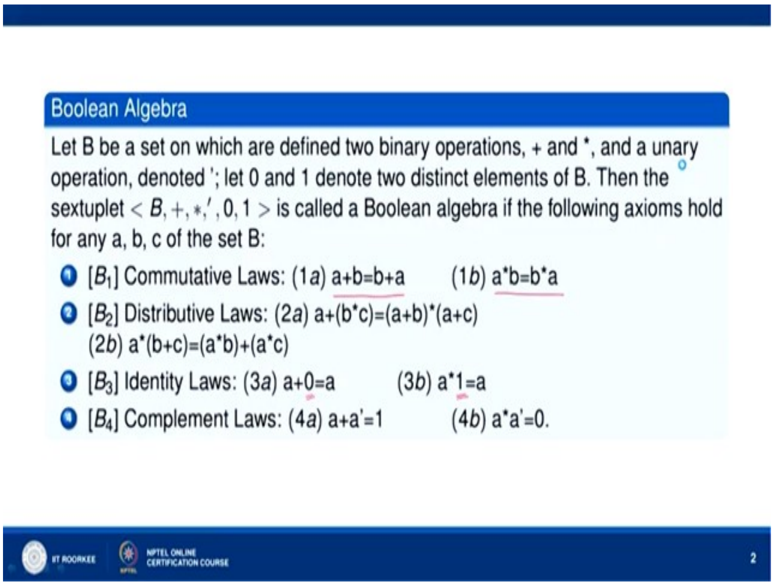

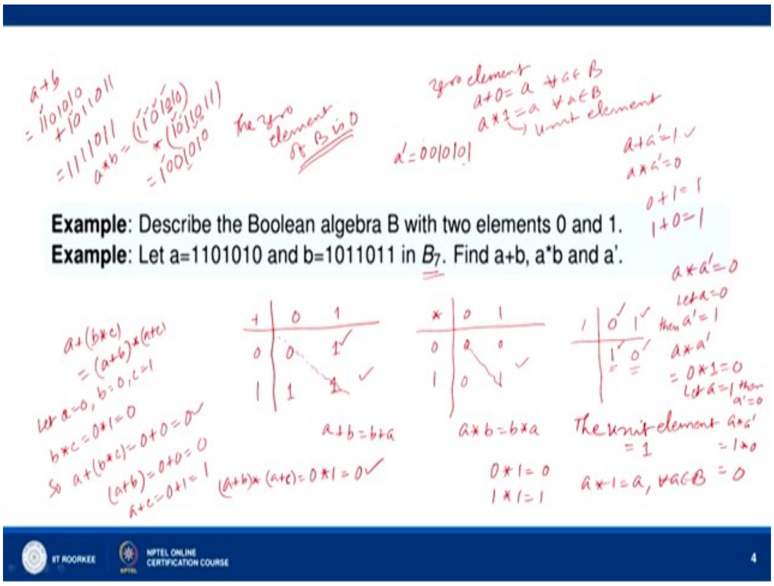

So, let us go to the axioms of Boolean algebra, commutative laws a+b=b+a, a ¿ b = b ¿ a. Now, let us see in this table, ok you see the main diagonal, ok this is main diagonal, the elements are on both sides of these table both sides of the main diagonal, ok are symmetric 1 1 here, here again 0 0 and here they are, so here what is happening is that a+b = b+a because in the about the main diagonal the elements the terms are symmetric, here also about the main diagonal they are symmetric, so a+b =b+a here, for any elements a, b and then here also a ¿b = b ¿a because these elements are elements in the table are same about the main diagonal.

Now, let us see we go to distributive laws , a + (b ¿c) = (a +b) ¿ (a +c) ,a ¿ (b+c)= (a¿b) +(a ¿c) these all distributive laws can also be verified, ok from these tables where we have defined the sum, product of any two elements and also the ´, you can see, you can verify say for example let us say suppose you want to vary verify a +(b ¿ c) = (a+b ¿∗¿ (a +c), ok.

So, let us take a= 0, b = 0 and c = 1, ok then what will happen b¿c, ok 0 ¿ 1= 0, ok and then so a +(b ¿ c) is equal to 0 + (b¿ c) is 0, now 0 +0 equal to 0, ok. Now, let us let look the right look at the right hand side a+b, a+b equal to 0+0, 0+0= 0, a+c=0+1, 0+1= 1, ok you can see 0+1 = 1.

Now, (a+b ¿∗¿ (a+c) = 0 ¿ 1, ok and 0¿ 1, 0¿ 1= 0, ok. So (a+b)¿ (a+c) =a+ (b ¿ c), ok so we can take any values of a, b, c, ok the elements of b are 0 and 1 and we can verify that a + (b¿ c) equal to (a¿ b¿∗¿ (a+c). Similarly, we can verify this one, the other one a¿ (b+c) = (a ¿ b)+(a¿ c).

Identity laws a we have to see what are the identity here, with the what is the zero element and what is the unit element, this is called as zero element, and this is called as unit element. So, let us see in our case what is the unit element and what is the zero element, unit element means a ¿1= a, and zero element means a+0= a. *Type equation here* .

So, zero element means a+0 =a for all a ∊this given set B, ok and a ¿ 1= a, ∀ a ∈ B, this then this B is called this 1 is called the unit element, ok.

So, what is the unit element in our algebra and what is the zero element? The zero element here is 0, ok the zero element of B it is 0, ok why? Because if I add 0 to 0, ok I add 0 to 0 I get 0, if I add 0 to 1 I get 1, ok. So, whether I take 0 ∊B or I take 1 ∊B when I when I consider the sum of 0+0 element are from with 1 element I always get the element 0 and the element 1, so a+0=a for all a ∊B and therefore 0 element is 0, ok.

The unit element is 1 here, why? Because if I take element 0 ∊B, ok then 0 ¿ 1, ok we have to see a¿ 1=a, 0 ¿1 = 0, ok and the other element 1 you take, so 1 ¿ 1 is also 1, ok and therefore a ¿1= a for all a ∊B, ok and so identity laws hold in this set B, ok where we have defined the +, ¿ and ´ operations (like by the step) by the step these tables.

Now, let us go to complement laws, let us see what are the complements of the elements of a and they satisfy this properties, a+a´ =1, a¿a´=0. So from this table we see that complement of the element 0 is 1, complement of the element 1 is 0, ok both 1, 0 ∈ B and therefore 1 and

0 are complements of 0 and 1. Now, let us see whether they satisfy these properties a+a´ = 1 and a¿a´=0, ok.

So, when you consider the sum of a with a´ you should be getting 1, ok so you take the element 0 its complement is 1, so 0+1, ok 0 is complement is 1 and 0 +1 is how much? 0 +1 you can see 0 +1 =1, ok if I consider the element 1, ok if I consider the element 1 it is complement is 0 and 1+0 from here is equal to 1, ok. So the complement of each element satisfies a+a´ = 1.

Now, let us look at the other axiom that is a¿a´=0. So, let us take a= 0, ok let a be equal to 0, then from our definition a´ = 1, ok and so a¿ a´ = 0 ¿1 and 0 ¿1 ¿ 0, ok so a¿ a´ is 0 and now the other case is, let us take a= 1 then by our definition a´ =0, ok and a ¿ a´ = 1 ¿ 0 and 1¿0 = 0.

So, a+a´ = 1, a¿ a´ = 0 for all a ∊B and therefore we satisfies all the axioms of the Boolean algebra and so we can say that B together with the elements I mean B containing the elements 0 and 1 where the sum product and complement are defined by this tables, satisfy all the axioms of the Boolean algebra therefore B is a Boolean algebra.

Now, let us go to the other example let a be equal to this sequence¿ 1, 2, 3, 4, 5, 6, 7;¿ 7 bit sequence and this B is another 7 bit sequence in B_7, B_7 means it contains all 7 bit sequences containing 0 and 1s, ok. So then we have to find a+b, a+b means (1 1 0 1, ok 0 1 0) 1 1 0 1 0 1 0 and B is 1 0 1 1 then 0 1 1, ok.

Now, when we add you can see when we add 1 and 1 we get 1, ok so 1 this 1 is added to this 1 and we get 1, then 1, second bit 1 here and second bit 0 here when added gives you 1, ok then third bit is 0, third bit is 1 here, so you get 1, fourth bit is 1 here, fourth bit is 1 here, so you get 1 and fifth bit is 0 here, here also fifth bit is 0 so you get 0, sixth bit is 1 here, here it is 1, so you get 1, seventh bit here is 1, so we get 1, ok.

So a+ b equal to 1 1 1 1 0 1 1, ok that means when you find the sum you get 0 only when both a and b has 0 at the same position, ok otherwise the sum will always be 1, ok.

Now, let us say a¿ b, so 1 1 0 1 0 1 0 ¿ b is 1 0 1 1 0 1 1, ok. Now, let us do this product bitwise, so 1 ¿ 1, 1 ¿ 1 is equal to 1, ok then second bit 1 here ¿ 0 is 0 here, 1 ¿ 0 equal to 0, and then third bit is 0 here it is 1, so 0 ¿ 1 is 0, then fourth is 1 here, here fourth is 1 so 1 ¿ 1 equal to 1, fifth is 1 here, fifth is 0 here, so we get 0, sixth is 0 here and sixth here is no 1 ¿ 1 gives you 1, 1 ¿ 0 gives you 0, 0 ¿ 1 gives you 0, 1 ¿ 1 gives you 1 and then 0 ¿ 0 gives you 0, then 1 ¿ 1 gives you 1, 0 ¿ 1 gives you 0, ok so this is a¿ b, ok.

Now, a´, so a is given to be 1 a´ a is given 1 1 0 1 0 1 0, so a´ will be, ok for 1 the complement is 0, for 0 the complement is 1, so for 1 we will write 0 here, for this 1 we again write 0, for 0 we write 1 and then for 1 we write 0 and then for 0 we write 1 for this 1 we write 0 and for this 0 we write 1, so that is the complement of a.

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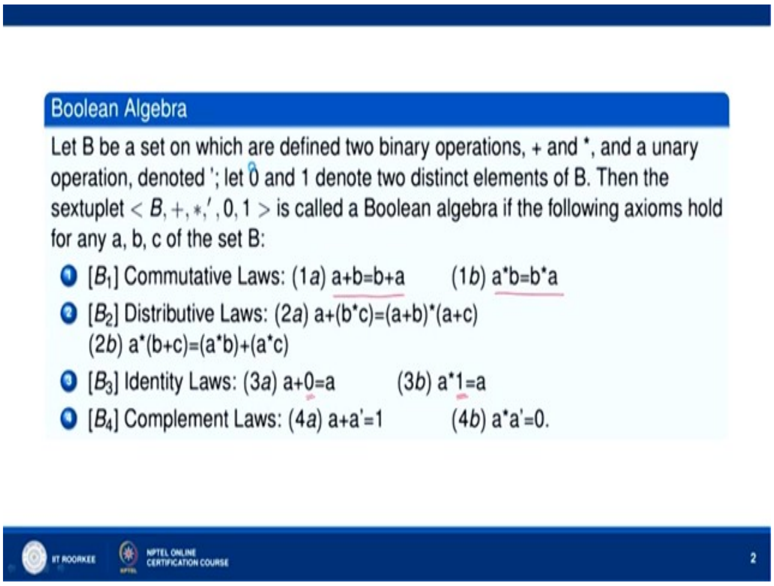

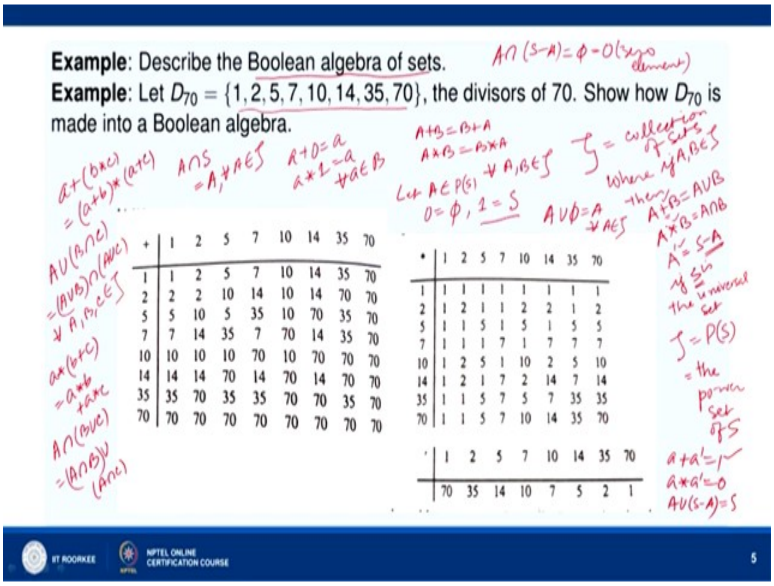

Now, let us consider the Boolean algebra of sets, ok. Suppose you have a ζ algebra of sets, suppose you consider the this collection of sets, ok where the ∪, intersection and complement, ok they are the when the they correspond to the binary operations sum of two sets, and then product of two sets, and then complement of two sets. So let us say it is a collection of sets where if A, B ∈ζ, ok then we have this is ζ.

So A, B ∈ζ then A + B by A + B mean A ∪ B, ok so + operation is defined as ∪ of A and B and then A ¿ B, B defined as A ∩ B and then complement of A´, ok complement of A is if you take A ∈ζ , then complement of A means the complement of so it is contents universal set S- A, if S is the universal set, if S denotes the universal set then it is complement S -A it denotes the complement of element A here, so that is how we define the binary composition + the binary composition star and the unary operation ´, unary operation ´ is defined as complement of A where S is the universal set.

Now, let us see whether with these definitions of sum, product and complement this ζ becomes a Boolean algebra or not, so first of all we have to see whether these two are binary operations on this set, ok. So it let us see if A and B are any two elements ∈this ζ then A + B is A ∪ B, so it ∈ζ and then A, B ∈ ζ , so A ¿B means A ∩ B ∈ ζ , we are assuming that this so this is ζ is the collection of all sets of the set S, collection of all subsets are you can say ζ is the let us say P (S), ok.

So, if ζ is P (S)with power set of S, ok then it will contain A ∪ B, it will contain A∩ B, it will also contain because S -A is the subset of S and so it is a it is these two are binary operations + and ¿are binary operations in ζ and then the other properties like commutative law, ok commutative laws are A + B = B + A, ok and then A¿B= B ¿ A, they are hold a true for all elements A, B ∈ ζ for all A, B ∈ ζ because A + B is A ∪ B and B + A is B ∪ A and we know that A ∪ B is a same as B ∪ A inside theory and then A¿ B is A ∩B, B¿A is B ∩A, so A ∩ B is equal to B ∩A therefore A¿ B = B ¿ A. ∩∪

So commutative laws hold and then we have the distributive laws, so we have a + (b ¿c) = (a + b)¿ (a + c). So, let us see this also holds here because A + means A ∪, B ¿ C means B ∩ C, ok and what we get here A + B means A ∪ B, ¿ means A + C means A ∪ C and we know that A ∪ (B ∩C) =(A ∪ B ¿ ∩(A ∪ C) for all A, B, C ∈ ζ, ok.

So, if ζ is denotes the power set of S, ok then this distributive law holds true here and then we have a¿ (b + c) =(a ¿ b) + (a ¿ c), ok. So, a ¿ b equal to a ¿ (b + c) if you I mean write I mean by our definition a¿ (b + c) will be A ∩(B ∪ C) there is equal to A ∩ B and then you have ∪ A∩C, ok so that also is true in for any sets a, b, c therefore it satisfies all the associative laws.

And then the third one is identity laws, ok identity laws means a + 0 = a and a ¿ 1 = a for all a ∈ the Boolean the set S, it should in a set B, if B is the set here the set is S a power set S is a power set of S we have, so let us if A, B ∈ P (S) or rather just A ∈ P (S) let us take, ok then what is the zero element here? Zero element here is equal to ϕ set, ok and what is the unit element 1? Unit element 1 is S here, ok S here, why? Now you can see a + 0 = a, a + 0 = a means A ∪ ϕ, ok we know that A ∪ ϕ = A, so A ∪ ϕ = A holds for every A ∈ ζ.

And then we have a ¿ 1, a ¿ 1 means A ∩ 1 is S since S is the universal set, ok every set A ∈ ζ is a subset of S, so A ∩ S is equal to A and this is valid for any A ∈ ζ, ok. So the ϕ set, empty set, and the universal set are denote the are the zero element and unit element for this P(S) and then we have the last one that is the complement.

So, complements let us see complement laws. So a + a´ = 1, a ¿ a´ =0 we have to see, a + a´ =1 a ¿ a´ =0. Now what are the complements let us see if A take any A ∈ ζ then it is complement is S-A, ok and you can see a + a´, a + a´ will be then equal to A ∪ (S –A) which will be equal to S, ok A ∪ (S –A) = S and S is the unit element, ok, so this satisfies.

And a star a dash, so a star a dash means A∩ S-A, ok in A∩ of A with the it is complement S -A is equal to ϕ, ok which is the zero element here, ok. So all the axioms of the Boolean algebra are satisfied if this ζ is equal to the power set of set S, ok, so this is a the ζ is a Boolean algebra. Now, let D_{70}, let D_{70} be equal to {1, 2, 5, 7, 10, 14, 35, 70} the divisor of 70 you can see these are the divisors of 70 let us see how D_{70} is made into a Boolean algebra.

So let us first define the sum product and complement operations the two binary operations and the unary operation complement and then let us see a weather all the axioms of Boolean algebra are satisfied here.

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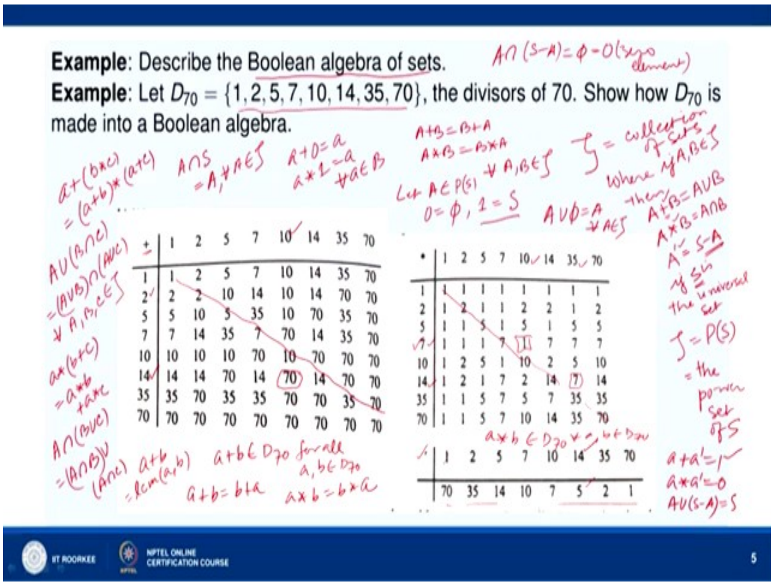

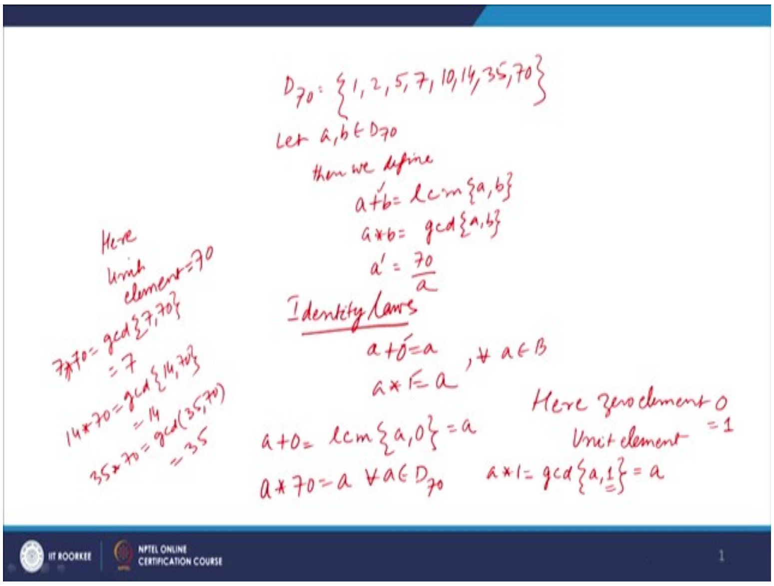

So what we do is? Let us we have D_{70} equal to, so D_{70} equal to D_{70} consist of all divisors of 70, so 1, 2 all divisors of 70; 1, 2; 70 means 5 then we have 1, 2, 5 let me write from here 1, 2, 5, 7, 10, 14, 35, 70, ok so 7, 5, 7, 10, 14, 35, 70, ok. Now, let us define let a, b belong to D_{70}, then we define A+B equal to lcm(a, b) and a ¿ b, ok equal to gcd(a, b) greatest common divisor of a and b and then a´equal to $\frac{70}{a}$,, ok.

So, let us define the sum of a and b as lcm of a and b least common multiple of a and b, a ¿ b has greatest common divisor of a and b and complement of a as 70 divides a, ok then you can see, then when you take this composition +, ok then 1 + 1 will be 1, 1 + 2 will be 2, ok because we are taking least common multiple, ok so when you take 1 , 1 least common multiple is 1, when you take 1, 2 the least common multiple lcm is 2, when you take 1, 5 lcm is 5, when you take 2 let us say suppose you take 7, 5,ok least common multiple is 35, ok when you take 14 14 and 10, ok 14 and 10 let us see this 14 and this 10 then this least common multiple is 70.

So, here you can see we have found the A+B for any two elements a and b belong to the set D_{70} , A+B means the least common multiple of a and b, so this is the table and here you take any two elements a, b ∈ D_{70} then we have found a ¿ b, a ¿ b is the greatest common divisor of a and b. So, let us say for example 7 and 10, ok when you take 7 and 10 greatest common divisor is 1, ok when you take 14 and 35, ok 14 and 35 the greatest common divisor is 7.

So, they are the elements which are the greatest common divisors of the elements a and b ∈ B when ¿ is the operation.

Now, when you take this complement, so {1, 2, 5, 7, 10, 14, 35, 70} their complements are 70 divided by if a is the element 70 divided by a, so for 1 the complement is 70 divided 1 by 1 that is 70, for 2 the complement is 70 divided by 2 that us 35, for 14 the complement of 14 is 70 divided by 14 that is 5, ok.

Now, let us see we can see that when you take any two elements a, b ∈D_{70} a + b, ok A+B takes these values they are the elements of the set D_{70} , ok. So, A+B ∈D_{70} for all a, b ∈D_{70} and here all the elements here in this table, ok you can see they are elements of D_{70} , so a ¿ b ∈D_{70} for all a, b ∈ D_{70} .

So, D_{70} is closed with respect to the operations + and ¿, ok so they are binary operations on D_{70} and this is a unary operation, ok.

Now, you can see that commutative laws holds because if you look at the main diagonal, ok if you look at the diagonal, ok the element about it is main diagonal, ok they are symmetric about these diagonal the elements are symmetric, ok on the two on both the sides of the these diagonal, so commutative law follows A+B equal to b + a, here also if you look at the diagonal {1, 2, 5, 7, 10, 14, 35, 70} ok the elements are symmetric about these diagonal therefore a ¿ b equal to b ¿ a, ok. The associative laws also can be verified easily from here, ok.

The third one is identity laws, identity laws are a + 0, a + 0 equal to a and a ¿ 0, a ¿ 1 equal to 1, a ¿ a equal to a ¿ 1 equal to a, this should be true for all a ∈ the Boolean algebra B, ok the set B, ok. So, now here what is this is zero element, this is unit element, what is the zero element here and what is the unit element here for our case? Ok.

So, for that we have to see our definition, ok our definition for + is A+B =lcm(a, b), ok so if we want the zero element that is a + 0 = a, ok then what should be the zero element? The zero element should be such that the least common multiple of a and that zero element should be equal to a, ok a + 0, ok a + 0 means least a + 0 means lcm(a, 0) should be equal to a, ok.

So, let us see lcm (a, 0) should be a, ok. So, if we take say for example 2, ok we take for example 2 what is it is what is the zero element here? Zero element will be if you take lcm of yeah, so zero element will be 1 here because if you take the 1 + 1 it is the least common multiple of 1, 1 which is equal to 1, if you take 2 the and the zero element 1, the least common multiple will be 2, if you take 5 and 1 the least common multiple will be, 5 ok.

So here zero element is 1, ok and what is the unit element? Unit element means a ¿1, ok a ¿ 1 should be equal to a, so a ¿1 =gcd(a, 1), ok this gcd of a and 1 should be equal to a, ok that is we want, so what is the element 1 here in our case? Ok. S,o if you take any element here, ok let us say for example you take 7, ok then we want gcd of 7 and that element should be equal to 7, ok so that should be if you take 70, ok then the greatest common divisor of the element under considerations and 70 will always be that element.

So here, if you take greatest so here a here unit element equal to 70, ok you take any element say for example you take, ok 7, 70, ok 7¿70 will be equal to gcd of 7 and 70 which will be equal to 7, ok and then if you take say for example 14 ¿70, ok gcd(14, 70) will be equal to 14, if you take, 35 ¿70 will be equal to gcd of (35, 70) is this also equal to 35, ok. So you take any element, if you take its product with 70 then the gcd will be always that number, ok.

So a star 70 is equal to a for all a $\in D_{70}$, ok. So, identity laws hold, complement laws also hold here, complement of 1 is 70, complement of 2 is 35, ok complement of 5 is 14 they all belong to D 70 and they satisfies the complement laws also, so this is a Boolean algebra, ok. So with that I would like to end my lecture, thank you very much for your attention.