Higher Engineering Mathematics
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Lecture - 18
Lattices - V

Hello friends, welcome to my lecture on Lattices. Let us define again complemented lattice because we will be showing that the dual of a complemented lattice is again complemented.

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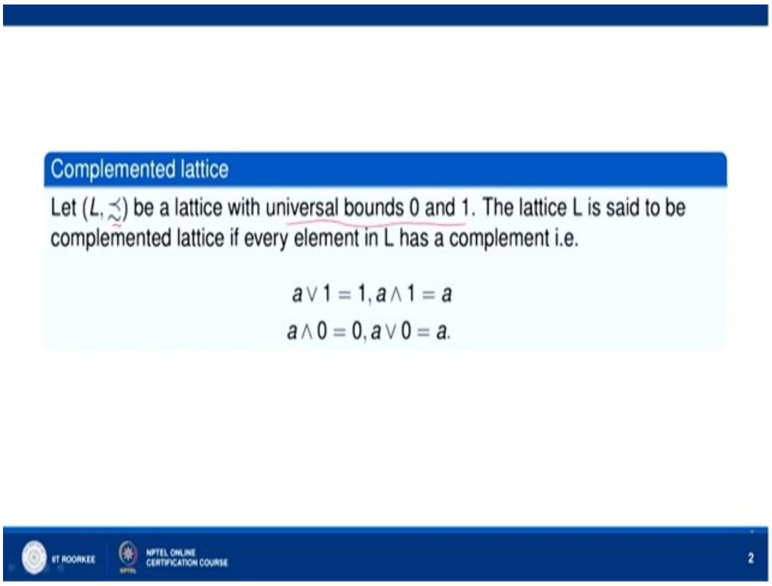

So, let L be a lattice with the operation $\leq$ and it has a universal bounds 0 and 1, 0 is the least element of the lattice, 1 is the greatest element, then the lattice L is called complemented if every element in L has a complement.

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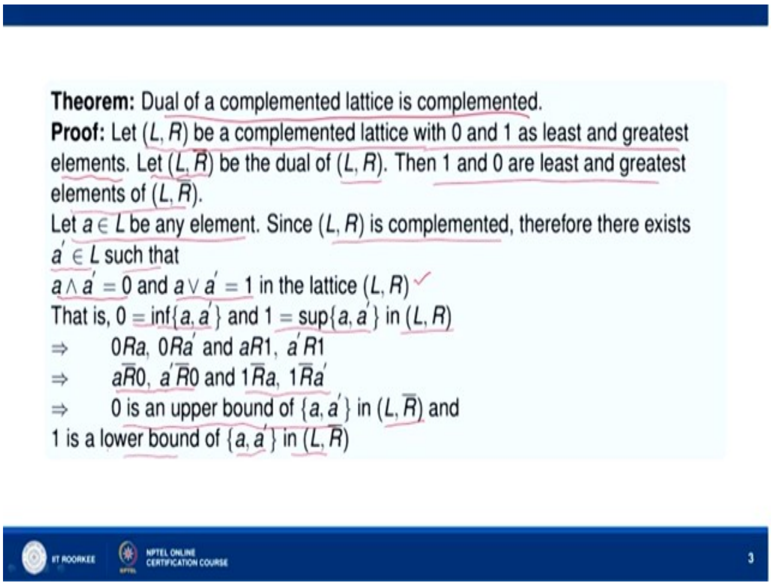

Now, let us proved that dual of a complemented lattice is complemented. Ok, so let L with the operation R, ok be a complemented lattice with 0 and 1 as least and greatest elements, ok. Let (L, R) be the (L, $\acute{R}$), ok (L, $\acute{R}$) these bar here, (L, $\acute{R}$) be the dual of (L, R) then 1 and 0 are the least and greatest elements of (L, $\acute{R}$), ok. Let, a ∊ L, we are going to show that (L, $\acute{R}$) is also complemented.

So, let a ∊ L be any element. Since (L, R) is complemented, therefore there exists a dash belonging to L, such that a ∧ a´ = 0 and a ∨ a´ = 1 in the lattice L, R, ok that is 0 is the greatest lower bound of a and a´ from here it follows that 0 is the greatest lower bound of a and a´, and from here it follows that 1 =sup(a, a´¿ in L, R, ok.

So 0 is 0Ra zero related to a, zero related to a´, ok and a related to 1, and a´ related to 1, ok because 1 is the least upper bound of a and a´, so aR1 we have a´ R1 we have. Now, 0Ra means a$\acute{R}$ 0, ok a$\acute{R}$ 0 because $\acute{R}$ is the complement of R, ok and $\acute{R}$ is the dual of R, ok. So, 0Ra´ means a´$\acute{R}$ 0, and aR1 means 1$\acute{R}$ a, a´ $\acute{R}$1 means 1$\acute{R}$ a´, and from here it follows that 0 is an upper bound of a and a´ in (L, $\acute{R}$) and from here it follows that 1 is a lower bound of a and a´ in (L, $\acute{R}$) ok.

Now, what we are going to show? We are showing that 0 is the least upper bound of (L, $\acute{R}$) and 1 is the greatest lower bound of (L, $\acute{R}$) so what we will do?

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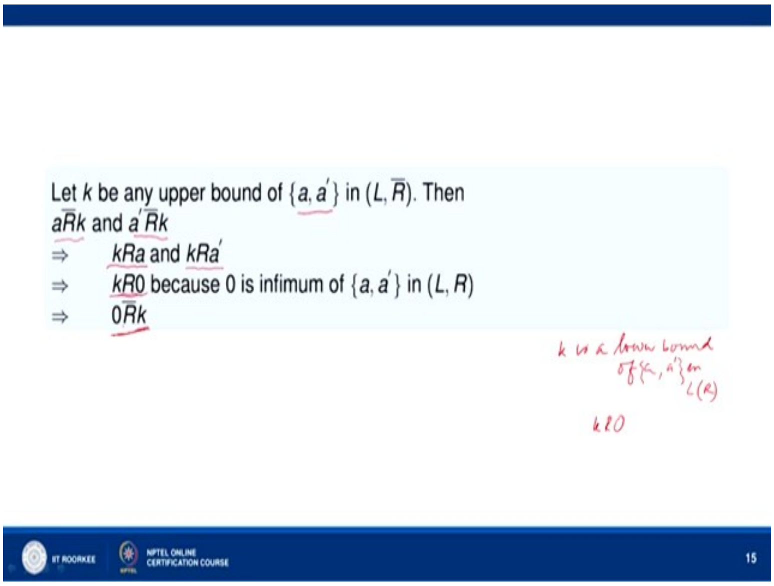

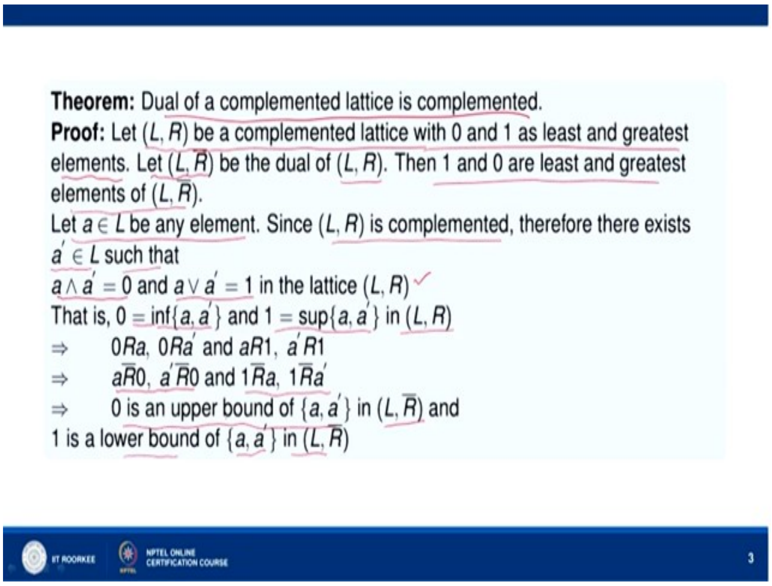

Let us consider k to be any upper bound of (a, a´¿, ok k then we shall show that 0 is the least upper bound, so we shall show that 0Ŕ k, ok, yeah. So, let k be any upper bound of (a, a´¿ in (L Ŕ**)** then aŔ k and a´Ŕ k hold true, ok. Now, aŔ k and a´ Ŕ k implies kRa and kR a´, ok. Now, yes so from here, ok k is the lower bound, k is a lower bound of a and a´ in (L, R), k is a lower bound of (a, a´¿ in (L, R) ok in (L, R) now , 0 is the greatest lower bound in (L, R) ok.

So, what will happen? k will be kRo will occur, ok because 0 is the greatest lower bound, so kR will be k related to o, kRo but kR0 means 0Ŕ k, ok kR0 means 0Ŕ k and therefore it follows that 0 is the least upper bound in (L, Ŕ) ok 0 is the because ḱ any upper bound and it follows that 0Ŕk, so 0 is the least upper bound in (L,Ŕ).

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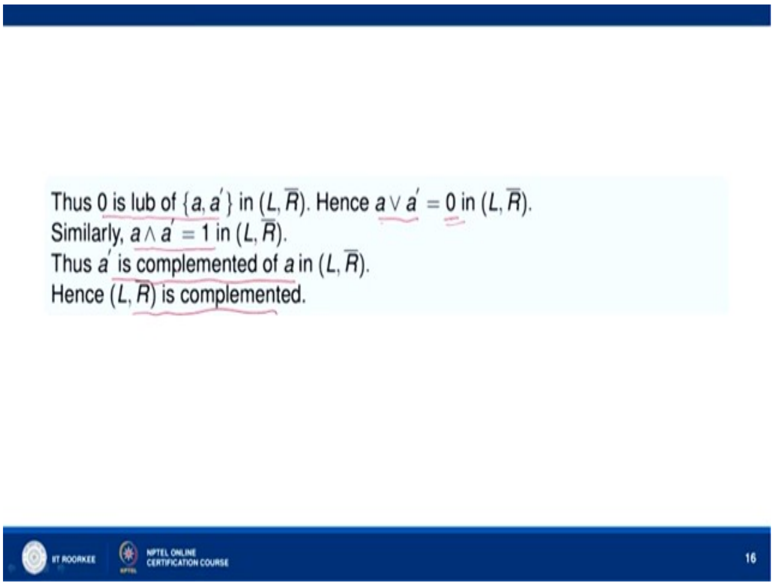

Now, hence a ∪ a´, ok a ∪, because 0 is the least upper bound in (L,$\acute{R}$)., ok. So a ∪ a´ = 0, ok because a ∨ a´ is nothing but least upper bound of a and a´, so a and a´ ∨ a´ is = 0 in (L, $\acute{R}$) Similarly, we have to we can show that a ∧ a´ = 1 in (L,$\acute{R}$) by following by proving on along the similar lines, so thus a´ is complement of a in (L, $\acute{R}$) ok hence (L,$\acute{R}$) is complemented. Because we have shown that a´ satisfies the property that a ∨ a´ =0 in (L, $\acute{R}$)and a ∧ a´ = 1 in (L, $\acute{R}$) ok. So a´ is the complement of a in (L, $\acute{R}$)hence therefore (L $\acute{R}$)is complemented lattice.

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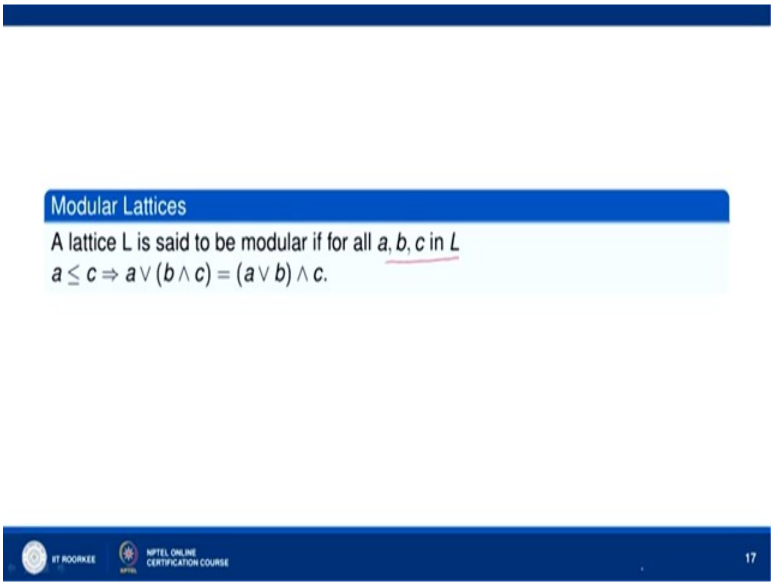

Now, let us define modular lattice, a lattice L is called modular if you take any a, b, c in L then $a \leq c$ implies that $a \vee (b \wedge c) = (a \vee b) \wedge c$.

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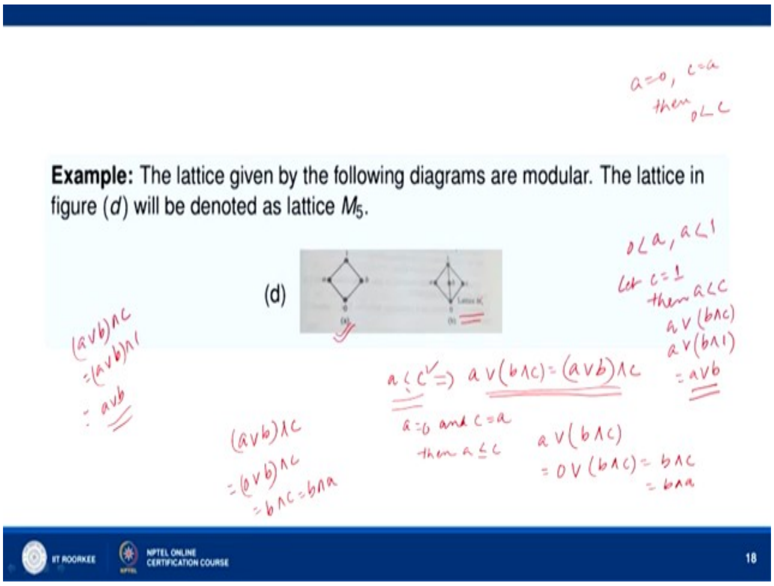

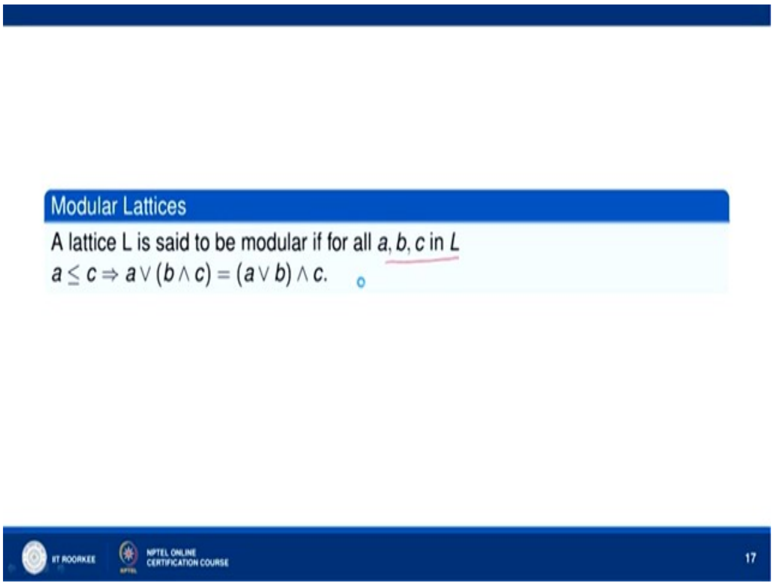

Lattice given by the following diagrams are modular, ok the lattice in figure d, this lattice is denoted by M_5, there are you can see 5 elements here {0, a, b, c, 1,} ok so we call it lattice M_5here we have {0, a, 1, b}ok you take any three elements here in the in case in the first case, ok {a, b, c} if a ≤ c, a ≤ c then we have this property, ok a ≤ c implies a ∨(b ∧ c) = (a ∨ b) ∧ c, ok.

For example, let us say I take a =0, ok a =0 and then c = a, ok here in this figure, ok let us take a =0 and c =a then what happens a is then a ≤ c, ok a ≤ c and what is this here? Third element b we can take as let us say any other element, ok any other element then what will happen if I take b to be any other element, then a, a is what? a is 0, ok 0 ∨ b ∧ c = b ∧ c , ok c = a, so we get b ∧ a, ok b ∧ a and what is the right hand side? a ∨ b ∧ c, ok a is 0, ok 0 ∨ b ∧ c we have, 0 ∨ b = b, ok then ∧ c we have, ok and c =a, so we get b ∧ a, so the both sides are equal.

So, whenever a ≤ c, ok from this figure we can see that this result holds, ok and then therefore this one is a this lattice is a modular. Now, here but what happens is we have {0, 1, a, b, c}, a, b, c are symmetric, ok here so we can take say, so we will have 0 ¿ a, and a ¿ 1, ok. So, what we will do? We let us take a let us a ≤ c, ok let us take a ≤ c, so let us take c to be equal to 1, ok then a ≤ c and ok.

Now, what will happen? Let us see, a ∨ (b ∧ c), ok b is any other element, ok now a is a, then we have a∨ b ∧ c, c=1, ok right so this is equal to a ∨b ∧ 1 equal to 1, ok so we get a ∨ b, ok a ∨ b we will get, right. Now, let us say the right hand side, right hand side is how much?

(a ∨ b) ∧ c, ok c is 1, ok so (a ∨ b) ∧ 1, ok and (a ∨ b)∧ 1 = a ∨ b, ok so both sides are equal, so this is again modular, ok.

Now, next choice could be that you take a as 0, ok a = 0 and c =a, then this will occur this a ≤ c, ok 0 ¿ c will occur, and again we can show that this result holds, ok so they are both lattices this is called lattice 5 denoted by lattice M_5.

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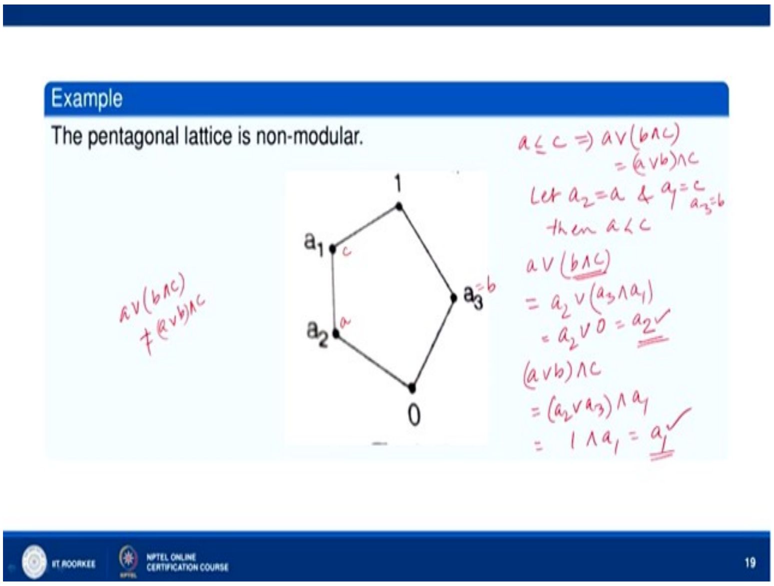

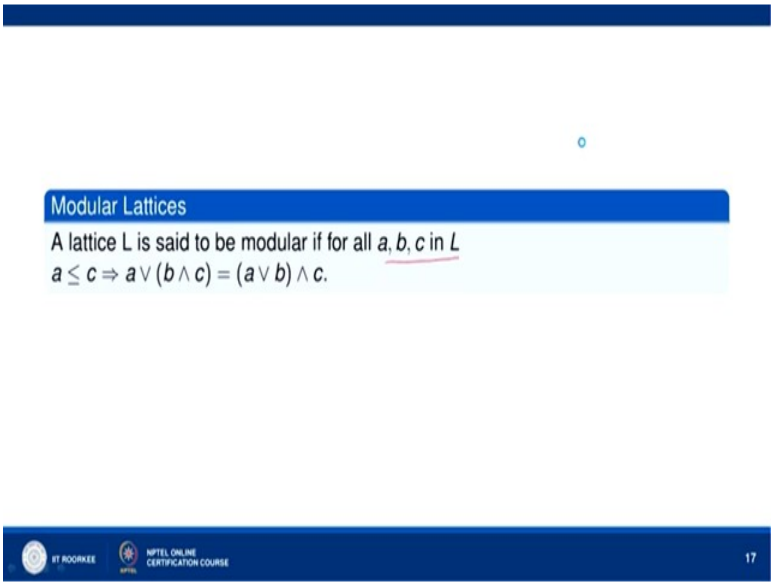

Now, let us show that pentagonal lattice is non-modular. Let us consider this pentagonal lattice, we want to show that it is not modular, ok so we have to show that this equation does not hold a ≤ c, ok we have to show that a ≤ c whenever a ≤ c we should have a ∨ (b ∧ c) = (a

∨ b) ∧ c, ok. Let us take here, let a_2= a, ok and a_1=c, ok then a ¿ c, ok a_2 ¿ c , as a_2 ¿ a_1, so a ≤ c.

Now, let us look at the values of these expression, so a ∨(b ∧ c), ok b ∧ c, b let us take to be a, this is my a, this is c that take the this a_3 to be b, ok a_3 = b, then what we will have here? So a is a_2, ok so a is a_2, so a_3 ∨ b is what? b is a_3, ok ∨ c is a 1, ok. So, $a_3 \wedge a_1$ means greatest lower bound of a_3 and a_1, ok this is a_3, this is a_1 greatest lower bound of a_3 and a_1 is 0.

So, we get a 2 ∨ 0, ok and 0 is the greatest lower bound, ok so a 2 ∨ 0 is the it is the least element a 2 ∨ 0 equal to a_2, ok $a_2 \vee 0 = a_2$. Now, let us see right side , a ∨(b ∧ c), ok a ∨ b ∧ c means a_2 ∨ b means a_3 or ∧ c, c is a_1, ok. So $a_2 \vee a_3$ means least upper bound of a_2 and a_3 least upper bound of a_2 and a_3 is 1, ok so 1 ∧ a_1, ok.

Now, here we are getting 1 ∧ a_1, and 1 ∧ a_1,, yeah 1 ∧ a_1, means greatest lower bound of 1 and a_1,, so be this is equal to a_1,, ok. So, here we get a_1,, here we get a 2 and you see that a_1, is a strictly succeeding a 2, ok a 1 strictly succeeds a_2,, so the two are not equal, a_2 here, a_1 here, so they are not equal. a ∨ b ∧ c is not equal to a ∨ b ∧ c when we choose a_2=a, a_1 =c and a_3=b, so this is lattice is not modular.

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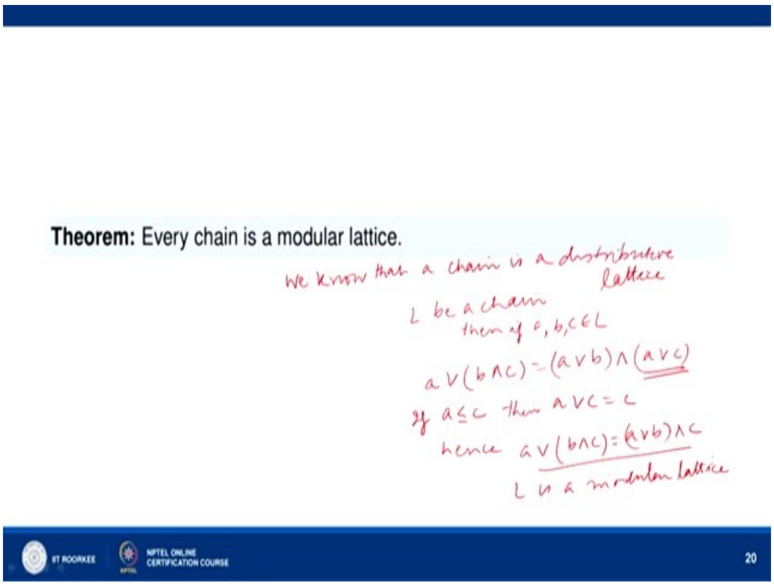

Now, every chain is a modular lattice, we have shown that every chain, chain is a distributive lattice, ok we know that every chain is a distributive lattice, ok. So if you take a let L be a chain then if $a, b, c \in L$ then $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, ok. Now if $a \leq c$, ok $a \leq c$, then $a \vee c = c$, ok hence $a \vee (b \wedge c) = (a \vee b) \wedge c$ because this is equal to c, ok so the L is a modular lattice.

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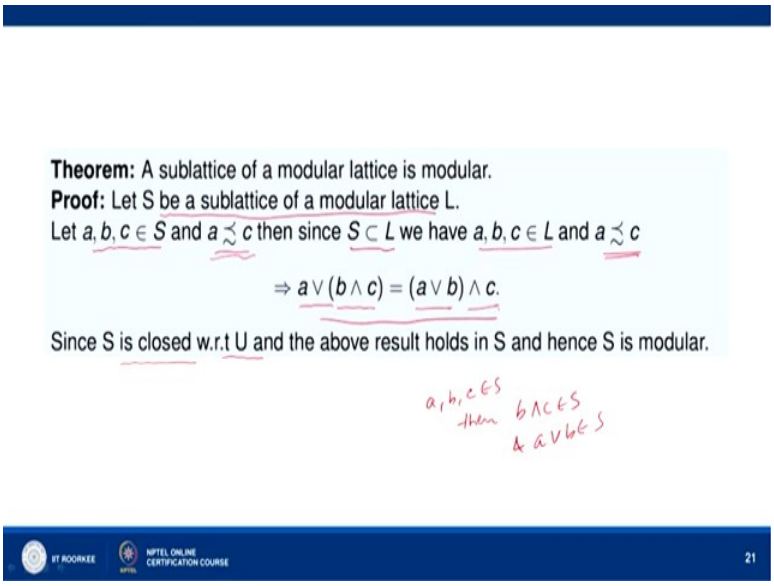

Now, a sublattice of a modular lattice is modular, let S be a sublattice of a modular lattice, ok you take $a, b, c \in S$, ok let $a, b, c \in S$ and $a \leq c$, ok then because S is a subset of L, ok so $a, b, c \in L$ and we have assume that $a \leq c$, ok therefore L being modular it follows that $a \vee (b \wedge c) = (a \vee b) \wedge c$ and therefore S is modular because S is closed with respect to S is closed with respect to the $\vee$ and $\wedge$ operations, ok.

We are given that S is a sublattice, so if $a, b, c \in S$, ok then $b \wedge c \in S$, and $a \vee b$ it also $\in S$, ok. So, since S is closed with respect to $\vee$ and $\wedge$ operations, ok this result holds in S and hence S is modular.

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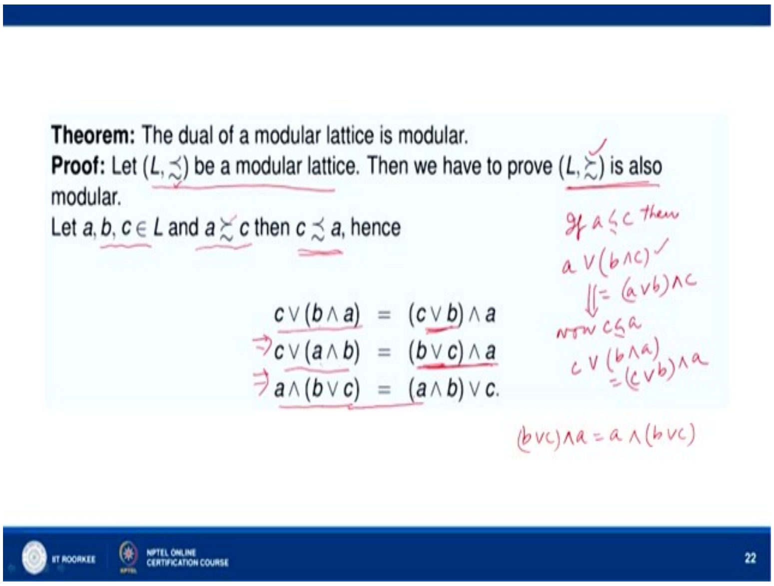

Now, dual of a modular lattice is modular, let us see let this be a modular lattice, ok then we have to prove that this is also modular because this is the dual of this operation, ok so let a, b, $c \in L$ and , $a \geq c$, you can let us say this is succeed c, ok $a \geq c$, then $c \leq a$, ok then $c \leq a$, and so $c \vee (b \wedge a)$ will be equal to we can write it as now this is equal to this, and this equal to $c \vee b$, ok by this we can write $c \vee b$, yeah by because L is a modular lattice because of that, yeah because of that $c \leq a$ when $c \leq a$ then yeah, ok because we had this if $a \leq c$, ok then $a \vee (b \wedge c)$ was equal to $(a \vee b) \wedge c$, ok.

Now, what is happening is that we are having $c \leq a$, ok so this will imply that $c \leq a$, so this will imply that $c \vee (b \wedge a) = (c \vee b) \wedge a$, ok we have to interchange. So this $c \vee (b \wedge a) = (c \vee b) \wedge a$, ok. Now, what? We know that complementary laws hold, this commutative law holds in a lattice, ok so what we have? $b \wedge a = a \wedge b$, so this implies $c \vee (a \wedge b)$ equal to and again $c \vee b = b \vee c$, so this is equal to this, ok so this is equal to this, and then $\wedge$ a, ok.

Now, what we do? Now this side, ok here again we used commutative law, so $b \vee (c \wedge a) = a \wedge (b \vee c)$, ok so we have $a \wedge b \vee c$ and here we have again commutative law, so this is $a \wedge b$ and then $\vee$ c, ok so this implies this, and this implies this. So this proved that this is modular because in order to prove that this is modular we have to show that if $a \geq c$, ok then a $\wedge$, because here we have a yeah this is dual case, so $a \wedge (b \vee c) = (a \wedge b) \vee c$, ok, so because of duality, ok so this duality this the operations have also been interchanged, in the case of L, I mean L being a modular lattice we had $a \vee (b \wedge c) = (a \vee b) \wedge c$.

Now, we have to prove that $a \wedge (b \vee c) = (a \wedge b) \vee c$, ok. So, thus we have shown that dual of a modular lattice is modular, with that I would like to end this lecture, thank you very much for your attention.