Higher Engineering Mathematics

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Lecture-16

Lattices III

Hello friends. Welcome to my lecture on Lattices. We define first lattice as an algebraic system. By the algebraic system, what do we mean by an algebraic system?

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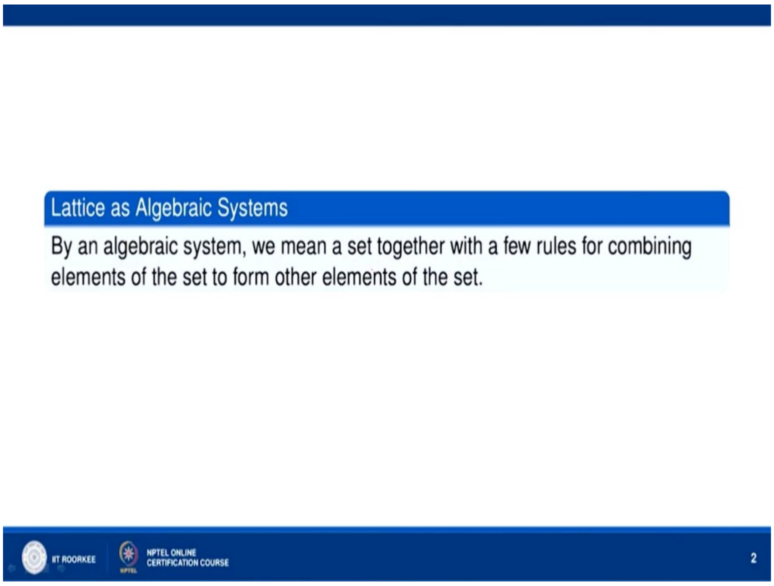

By an algebraic system we mean a set together with the few rules for combining elements of the set to form other elements of the set.

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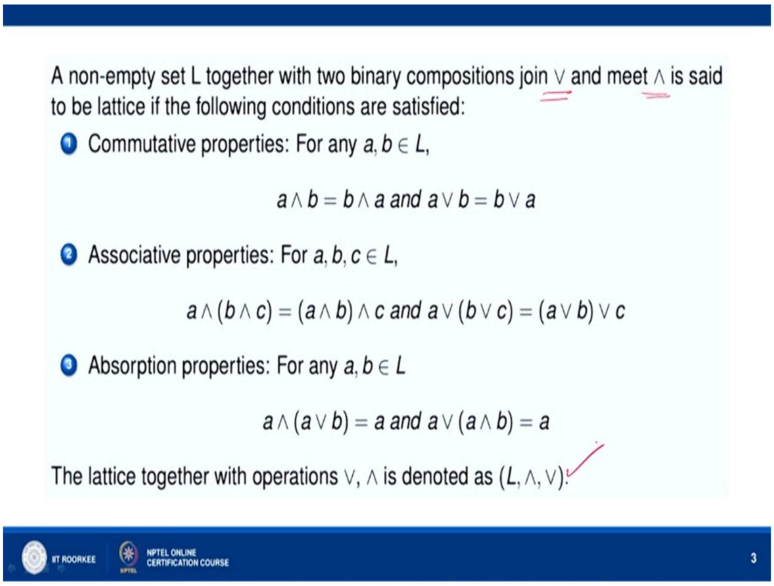

⟺≤∈∧∨A non-empty set L together with two binary compositions, ∨ and ∧, Ok is said to be a lattice if the following conditions are satisfied. So this is the definition for lattice as an algebraic system. We consider two binary operations, ∨ and ∧, Ok. So if a set contains these two binary compositions, it is closed with respect to these two binary compositions and satisfies the following properties then it will be called lattice as an algebraic structure.

So commutative property, for any a, b ∈ L, $a \wedge b = b \wedge a$. And $a \vee b$ equal to $b \vee a$. So commutative laws hold, then associative property. For a, b, c ∈ L, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$, Ok. Then we have absorption properties. If you take any a, b ∈ L then $a \wedge (a \vee b) = a$, and $a \vee (a \wedge b) = a$. So if commutative properties, associative property and absorption properties hold on a set L which is closed with respect to two compositions ∨ and ∧, we will call it as a lattice, as an algebraic structure. And such a lattice is denoted by this notation.

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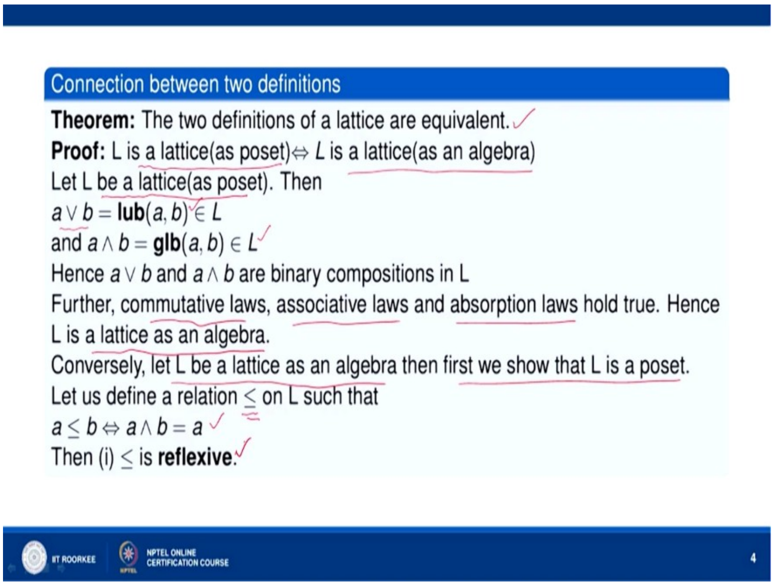

Now there is a connection between the two definitions, lattice as a poset and lattice as an algebraic structure. The two definitions of lattice are equivalent. So let us prove how the two definitions of lattice are equivalent. Let us first assume that L is a lattice as a poset, Ok. We are going to show that L is a lattice as poset $\Longleftrightarrow$L is a lattice as in algebraic structure, Ok.

So let us say, let L be a lattice as poset. Then a $\vee$ b, as we know, a $\vee$ b is least upper bound of a, b which is there in L and a $\wedge$ b =g l b (a, b) which is there in L. So a $\vee$ b and a $\wedge$ b are two binary compositions in L. We are going to prove that L is an, L is a lattice as in algebraic structure. So for that we need to show that a $\vee$ b and a $\wedge$ b are two binary compositions in L. So that follows, now let us say, the commutative laws, associative laws and absorption laws.

When we consider lattice as a poset there we had shown that commutative laws, associative laws and absorption laws all hold true so L is a lattice as an algebra. Conversely let L be a lattice as an algebra then first we will prove that L is a poset. So let us define a relation $\leq$ such that a $\leq$ b $\Longleftrightarrow a \wedge b = a$.Now we have to show L is a lattice; L is a lattice as a poset so first we show that it is ordered. That is, it is a partial order set. So we show that this $\leq$ relation defined on L is reflexive. So let us see how we prove that it is reflexive?

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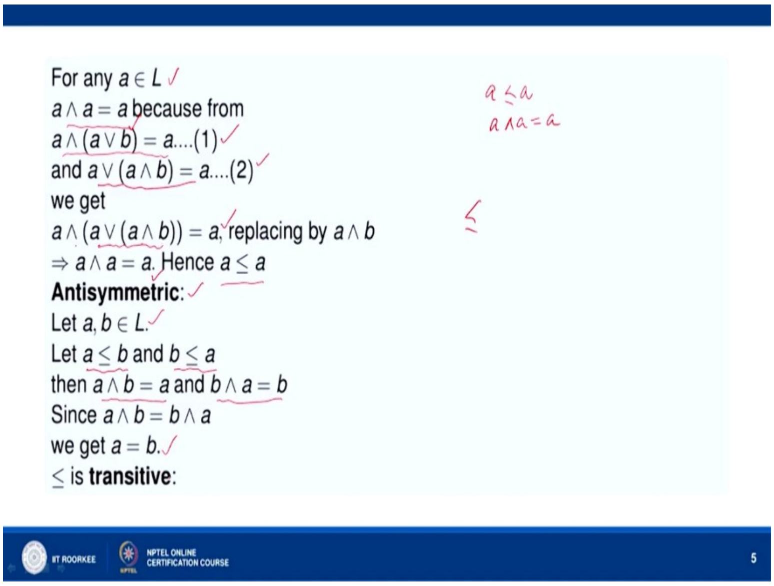

In order to prove that , L is reflexive. We have to show that $a \leq a$, Ok. $a \leq a$ means we have to show that, $a \leq a$ means we have to show that $a \wedge a = a$, Ok. We have to show that $a \wedge a = a$. Now what we do? Take any L, $a \in L$ then $a \wedge a = a$ because from this, absorption law, Ok. The absorption laws hold, because L is a lattice, as a poset. So $a \wedge a \vee b = a$. absorption law 1 and absorption law 2, $a \vee (a \wedge b) = a$.

Now in 1 what you do, replace b by, by $a \wedge b$, Ok. Then what we will we have? $a \wedge (a \vee b)$, we are replacing b by $a \wedge b$, so $a \wedge b$ Ok equal to a, $a \wedge (a \vee a \wedge b) = a$. And now let us use the second absorption law. $a \vee (a \wedge b) = a$, so put that value here, Ok then we have $a \wedge a$ equal to a. And $a \wedge a$ equal to a, by our definition implies $a \leq a$, Ok. So the operation, this $\leq$ is a reflexive operation.

Now antisymmetric, let us say take any two a, $b \in L$. Then assume that $a \leq b$ and $b \leq a$. Now $a \leq b$ means by our definition, $a \wedge b = a$. And $b \leq a$ means $b \wedge a = b$, Ok. $a \wedge b = b \wedge a$ because, because of the commutative law. So $a \wedge b = b \wedge a$ and therefore $a = b$. And so we say that this operation defined on L is symmetric. Now let us show that it is transitive, Ok.

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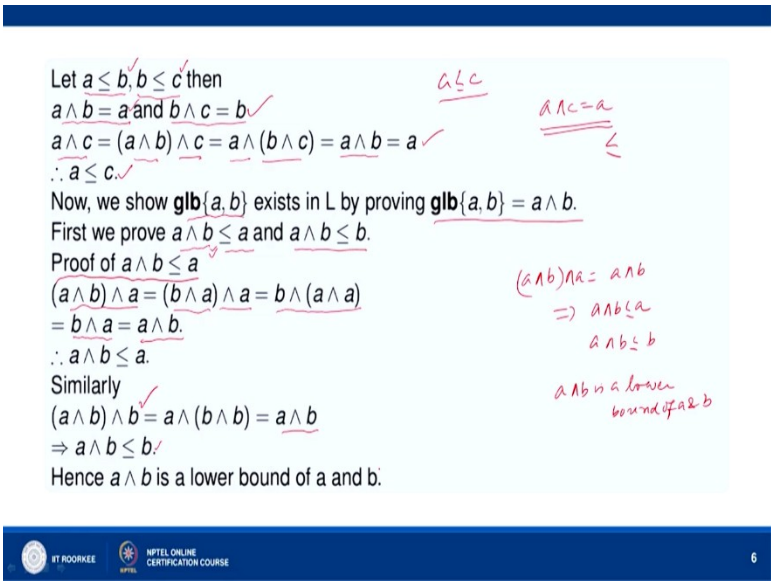

So take $a \leq b$. Assume $a \leq b$. $a \leq b$ and $b \leq c$, Ok then we have to show that $a \leq c$, Ok. So in order to prove that $a \leq c$, from $a \leq b$ it follows that $a \wedge b = a$ and from $b \leq c$ it follows that $b \wedge c = b$ Then in order to prove that $a \leq c$, we have to show that $a \wedge c = a$.

So consider $a \wedge c$, Ok. Put the value of a here, a is equal to $a \wedge b$. So $a \wedge (b \wedge c)$.

Now use the associative law, so we can write it as $a \wedge (b \wedge c)$. $b \wedge c = c$ so we get $a \wedge b$, Ok and $a \wedge b = a$, so we get a here. So $a \wedge c = a$ implies that $a \leq c$. And therefore the relation $\leq$ is a transitive relation on this set. Now let us show that the glb(a, b) exists and the lub(a, b) where a and b are any two elements in L. So we will first show that g l b(a, b) $= a \wedge b$, Ok, and $a \wedge b$, $a \wedge b$ is there in the set L. So g l b(a, b) will exist and it will$\in$L.

Now let us, so for this we will first show that $a \wedge b \leq a$, $a \wedge b \leq b$, Ok. Now proof of $a \wedge b \leq a$. First we are proving this. Now why are proving $a \wedge b \leq a$, and $a \wedge b \leq b$, because we want to show that $a \wedge b$ is a lower bound of a and b, Ok

So $a \wedge b \leq a$ because $(a \wedge b) \wedge a$, $a \wedge b$ by commutative law can be written as $b \wedge a$, Ok, so this is again by associative law, $b \wedge (a \wedge a)$, Ok. And $a \wedge a = a$ so we get $b \wedge a$, Ok. $b \wedge a = a \wedge b$. So $(a \wedge b) \wedge a = a \wedge b$. This means that $a \wedge b \leq a$, Ok. $a \wedge b \leq a$.

Now similarly $(a \wedge b) \wedge b$, Ok. Now we are going to show that $a \wedge b \leq b$, Ok. So $(a \wedge b) \wedge b = a \wedge (b \wedge b)$. $b \wedge b = b$, so we get $a \wedge b$, Ok.

And thus a ∧ b is, ≤ b, Ok. So a ∧ b is a lower bound of a and b, Ok. So therefore a ∧ b is a lower bound of a and b, Ok Now we will show that if c is any other lower bound of a and b, then c ≤ a ∧ b. So that a ∧ b is the glb (a, b).

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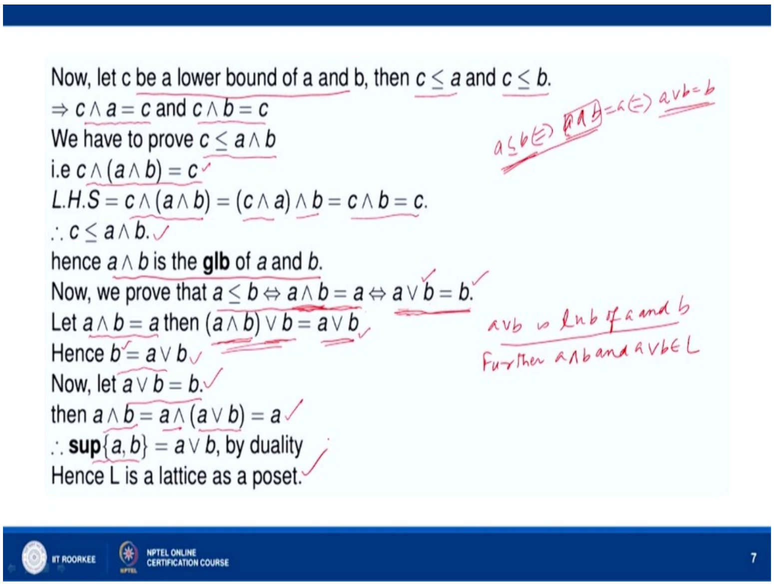

Now let us say, let c be any other lower bound of a and b. Then c ≤ a and c ≤ b, Ok. Now c ≤ a by our definition gives c ∧ a = c. And c ≤ b by our definition gives c ∧ b =c. We have to show that c ≤ a ∧ b so that a ∧ b becomes the greatest lower bound. Now in order to show that, c ≤ a ∧ b, let us, we have to show c ∧ (a ∧ b) =c, Ok. So c ∧ (a ∧ b) =c we have to show, Ok.

c ∧ (a ∧ b) equal to, associative law gives (c ∧ a) ∧ b, Ok. And c ∧ a=c. So we get c, Ok. So (c ∧ a) ∧ b = c, therefore, c ≤ a ∧ b. And hence a ∧ b =glb (a, b). Now we assumed that a ≤ b ⟺a ∧ b =a. We shall show that this is also equivalent to a ∨ b = b.

If we can show that it is equal to a ∨ b = b then by using duality we shall be able to say that, a ∨ b =lub(a, b), Ok. So we have to show that a ∧ b ⟺ a ∨ b equal to b, Ok. Now let us first assume that a ∧ b = a, Ok. a ∧ b = a, and we want to show that a ∨ b =b. So then a ∨ b, a ∨ b becomes a ∧ (b ∨ b), Ok. a ∧ (b ∨ b), Ok, a ∧ (b ∨ b) becomes a ∨ b.

Ok so from here what happens, yeah, now a ∧ (b ∨ b), a ∧ (b ∨ b) = b, Ok. a ∧ b, because a ∧(b ∨ b) =b by absorption law, so this is equal to b and then this is a ∨ b, Ok. So b equal to a ∨ b. Now let us consider a ∨ b =b, Ok. Let us prove the converse. a ∨ b =b, then we have to show that $a \wedge b = a$. So $a \wedge b = a \wedge (a \vee b)$, Ok $a \vee b$. But again we use absorption law and this becomes a, Ok.

Ok so what happens? $a \wedge b = a \Longleftrightarrow a \vee b = b$, Ok and also $a \leq b \Longleftrightarrow$ this. Ok so when we assume that, $a \leq b \Longleftrightarrow a \wedge b = a$, Ok and it turned out that $a \wedge b = \text{glb}(a, b)$, Ok. By duality theory it follows that $a \leq b \Longleftrightarrow a \vee b = b$ will imply that $a \vee b$ is the lub(a, b), Ok.

So $a \vee b$, Ok, $a \vee b = b$, $a \vee b$ is the least upper bound of a and b, and $a \vee b = b$ means $a \vee b$ implies, $a \vee b \in L$ and $a \wedge b$ equal to a means that $a \wedge b \in L$, Ok. So further $a \wedge b$ and $a \vee b \in L$, Ok. So L contains the greatest lower bound and the least upper bound, Ok. Hence L is a lattice as a poset. So this proves the theorem.

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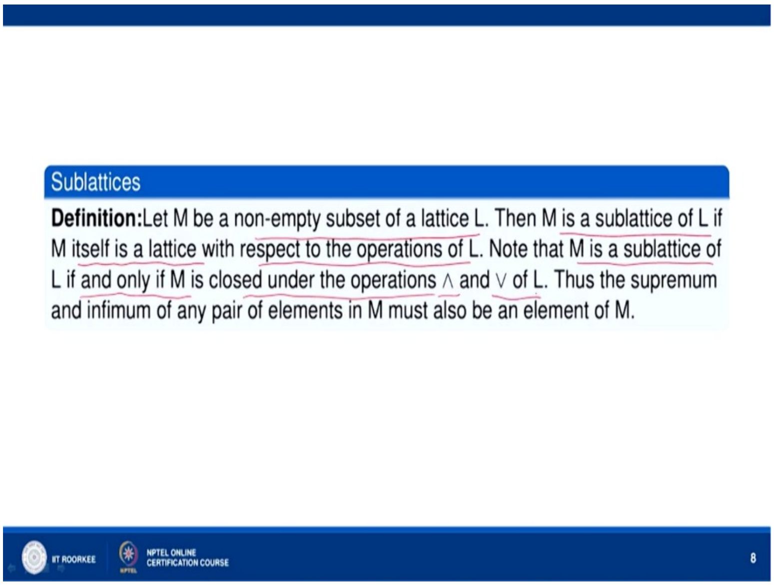

Now let us find some lattices. Let M be a non-empty subset of a lattice L. Then M is called sub lattice of L if M itself is a lattice with respect to the operations of L, Ok. Note that M is a sub lattice of L $\Longleftrightarrow$M is closed under the operations $\wedge$ and $\vee$ of L, Ok so that means if a and b are any two elements of M, Ok then $a \wedge b \in M$ and $a \vee b \in M$, Ok. Thus the supremum and infimum of any pair of elements in M must also be an element in M.

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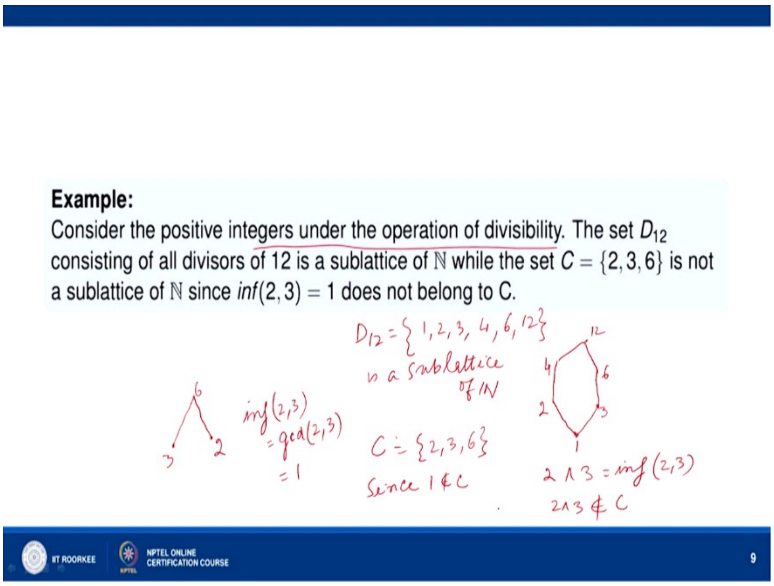

So let us now take an example. Consider the set of positive integers under the operation of divisibility. We know that it is a lattice, Ok. Let us consider its subset D_{12}. D_{12} consists of {1, 2, 3, 4, 6, 12}. They are all divisors of 12, Ok. So let us draw Hasse diagram to show that D_{12}

is a sub lattice, Ok. So 1, 2 then we have 4, then we have 6, Ok and 1 divides 3, so we have 3, then we have 4 divides 12, so this 3 divides 6, and 6 divides 12, Ok. So from this Hasse diagram it follows that D_{12} is a sub lattice.

Now let us consider a subset C of N. C equal to { 2, 3, 6} Ok,{ 2, 3 6}. So let us draw diagram 2, then we have 6, then we have 3, Ok, then we have... Ok so from this Hasse diagram it follows that 2, 3, 6 is not a sub lattice Ok of N because the infimum of 2, 3; inf (2, 3) =1. Infimum of 2, 3 means g.c.d. of, under the operation of divisibility, infimum becomes g.c.d. and the supremum becomes l.c.m. So infimum of 2, 3 is equal to g c d of 2, 3 and g.c.d. (2, 3) =1. So this means that, and since 1 $\notin$ C, 1 $\notin$ C it follows that C is not a sub-lattice. Because inf(2, 3); means 2 $\wedge$ 3. 2 $\wedge$ 3 is inf(2, 3) Ok. So 2 $\wedge$ 3 $\notin$ C , Ok. So C is not closed under the operation $\wedge$.

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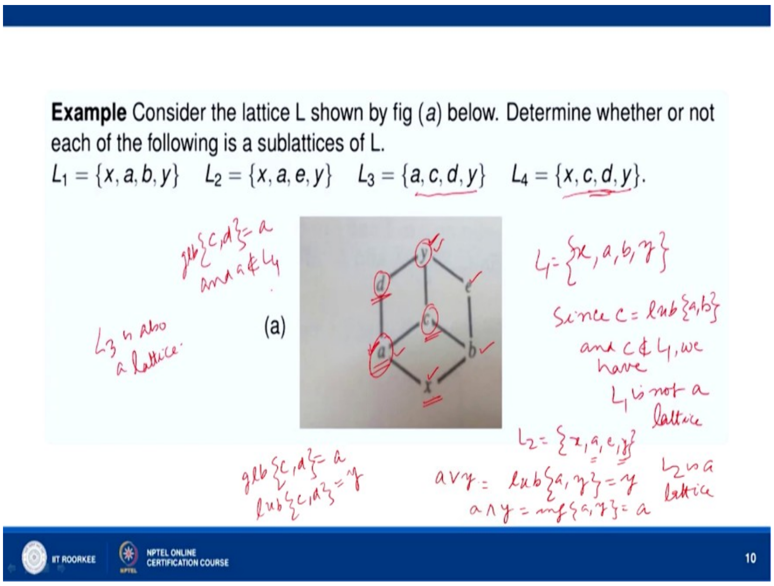

Now let us go to another example. consider the lattice L shown in figure a below, Ok. Determine whether or not each of the following is a sub-lattice of L, Ok. So let us take $L_{1=¿¿}$ {x a, b, y}. Let us consider, Ok. If we take a, b, Ok. Take a, b here Ok then we can see that c is the least upper bound for the pair a, b. x is the greatest lower bound, Ok. Since c is the least upper bound of a, b Ok and c does not$\in L_1$, it follows that L_1 is not a lattice. Now if you take L_2, L_2 ={x, a, e, y}. So you take x here, a here, e is here, y is here. You take any pair. Say, for example you take a and e, Ok.

For a and e, Ok x is the greatest lower bound, y is the least upper bound, Ok. And x and y both$\in L_2$. If you take any other pair, say for example you consider a and x, then x is the greatest lower bound. a is the least upper bound, Ok. If you consider a and y, let us consider a and y. If we take a and y then the least upper bound is y which is there and the greatest lower bound is, greatest lower bound is a itself. Because a $\leq$ a, a $\leq$ y. So a is the greatest lower bound, Ok.

So , lub(a, y) = y. Ok. So a $\vee$ y, Ok and a $\wedge$ y will be equal to infimum of (a, y). If you find infimum of a, y, infimum of (a, y) will be a, Ok. So you take any two elements $\in L_2$, Ok. Their infimum and supremum$\in L_2$. So L_2 is a lattice. Now let us take {a, c, d, y,} Ok. a, this is a, c and then d and then y, Ok so {a, c, d, y}. Here you take any pair, say you take {a, y} for example. For {a, y} a is the least, greatest lower bound, y is the least upper bound. If you take a and c, Ok, then a and c, a is the greatest lower bound, c is the least upper bound.

If you take, say for example you take c and d, Ok if you take c and d, then for (c, d) the greatest lower bound is a. Ok and the least upper bound is y. Ok so you take any pair of points here in L_2 which is L_3which is {a, c, d, y}. We find that there a greatest lower bound and least upper bound exist. So L_3 is also a lattice. L_3 Now let us take L_4={x, c,d, y} so x here, c, d and y. Ok now for c, d. if you take c, d, a is the greatest lower bound, Ok. For c, d, Ok a is the greatest lower bound, Ok and a does not belong to L_4. Ok so $L_4$is not a sub lattice. Ok so L_1, L_1 is not a sub-lattice, and L_4 is not a sub-lattice. The other two, $L_2$and L_3 are sub-lattices

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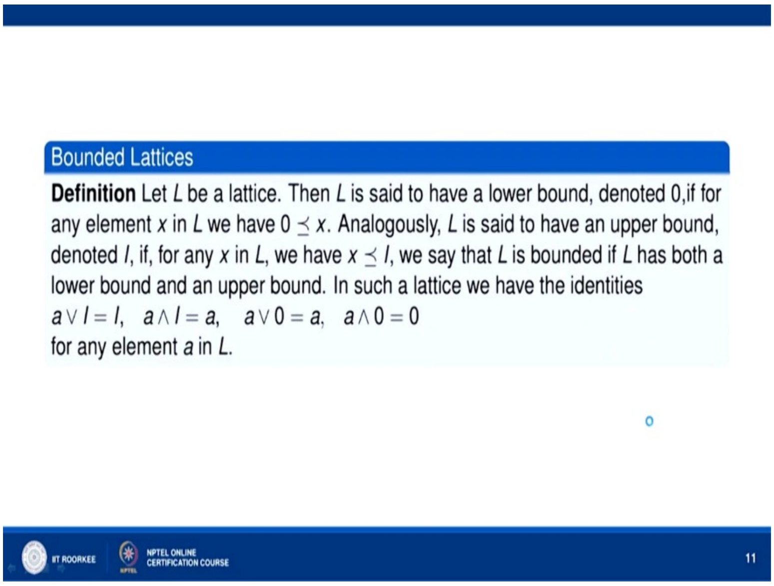

Now let us define bounded lattice. Let L be a lattice. Then L is , called to have a lower bound denoted by this 0 for any element x in L, we have $0 \leq x$, Ok. Analogously L is said to have an upper bound denoted 1 if for any x in L we have $x \leq 1$, Ok.

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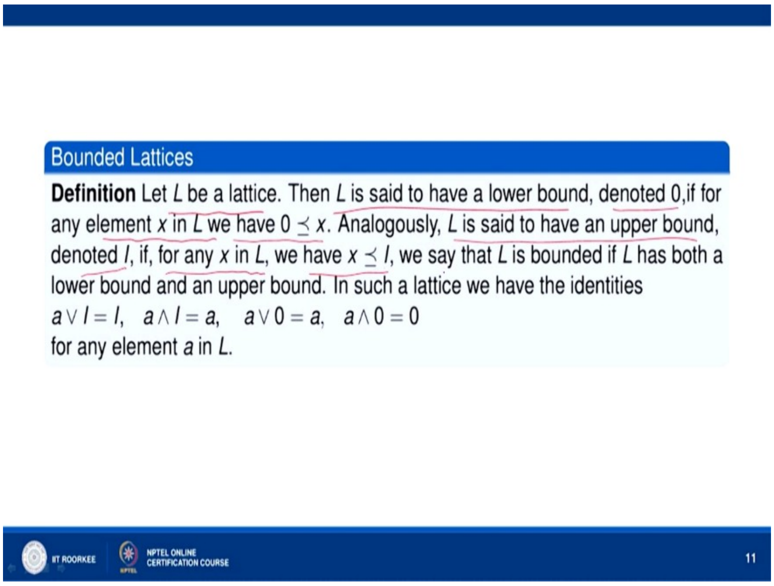

We say that L is bounded if L has both lower bound and an upper bound, Ok. In such a lattice we have the identities, $a \vee 1 = 1$, $a \wedge 1 = a$, $a \vee 0 = a$. $a \wedge 0 = 0$. These elements, denoted by 0 and by 1, they are the least and greatest elements of L, Ok. So 0 is the least element of L. and 1 is the greatest element of L. So in other words we shall, we say that if L is a lattice then it is called bounded if it has least element as well as the greatest element. Least element is denoted by 0 and greatest element is denoted by 1.

Let us show these identities. We have to show that a ∨ 1 = 1, Ok. So to prove this a ∨ 1 = 1, what we notice? Since 1 is the greatest element and a ∨ 1 belongs to L we have a ∨ 1 ≤ 1, Ok. a ∨ 1 ≤ 1. Now a ∨ 1 by definition, a ∨ 1 is the , supremum, least upper bound, Ok, least upper bound that is supremum of a and 1. So 1 ≤ a ∨ 1, Ok.

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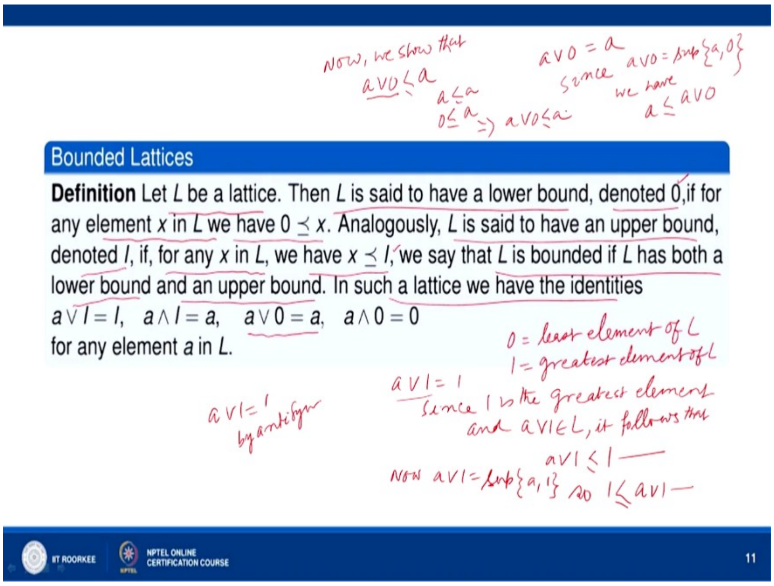

So this inequality, this condition a ∨ 1 ≤ 1 and this other one, 1 ≤ a ∨ 1. They together give us a ∨ 1 = 1, by antisymmetry, Ok. Similarly one can show that a ∧ 1 =a. Now let us show that a ∨ 0 = a. So a ∨ 0 = a. So a ∨ 0 = a. Let us prove this, Ok. Since a ∨ 0 is the, a ∨ 0 is the supremum of (a , 0), Ok we have 0 ≤ a ∨ 0, Ok.

Now we have to show that, now we show that a ∨ 0 ≤a. See a ∨ 0 is equal to supremum of a and 0. So a ≤ a ∨ 0. Now let us show that a ∨ 0 ≤ a, Ok. So we will have to show that a is an upper bound for a and 0. Then a will succeed a ∨ 0, because a ∨ 0 is the least upper bound, Ok.

So we know, so a ≤ a, by reflexive property and 0 is the least element, so 0 ≤ a, Ok. And therefore a is an upper bound for a and 0. So this implies that a ∨ 0 ≤ a, Ok. Now this condition and this condition together, they give you a ∨ 0 = a, by antisymmetry. And similarly we can show a ∧ 0 = 0. So if L is a bounded lattice Ok then it has least and greatest element denoted by 0 and 1, and it satisfies these properties

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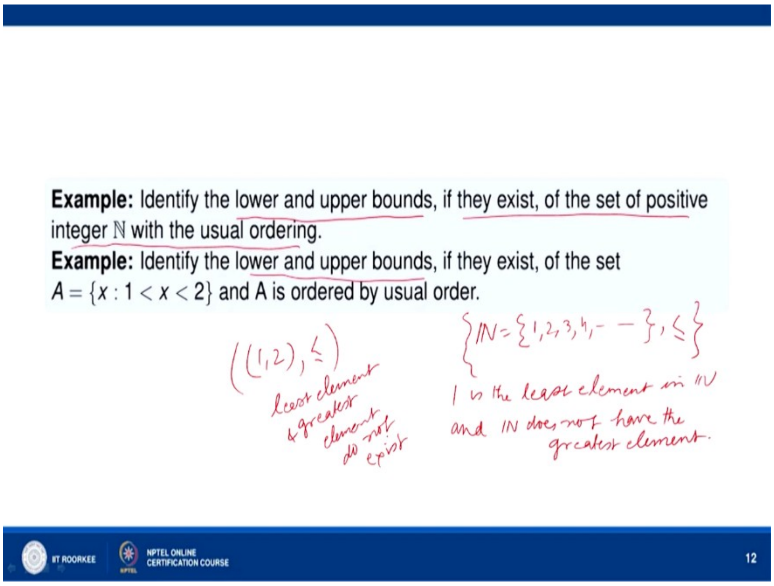

Let us consider this example. Identify the lower and upper bounds if they exist of the set of positive integer N and with the usual ordering. Now N contains {1, 2, 3, 4 ,….}, Ok and the ordering is given as usual ordering. Usual ordering means ≤, Ok. With ≤, 1 is the least element in N and N does not have a greatest element. There is no element in N which succeeds every other element of N.

Ok now identify the lower and upper bounds, if they exist of the set this, lower and upper bound means least and greatest elements. So you can see, (1, 2) is an interval here, open interval. (1, 2) is an open interval. So it does not have the least element and it does not have the greatest element if it is ordered by usual order, Ok. So in the second example the least element and the greatest element do not belong to, are not there, Ok. Least element and greatest element do not exist. Now let us go to this example. Identify the lower and upper bounds, least and greatest elements, if they exist of the power set P(A) of a set A under the operation ∩ and ∪. Ok P(A) is the power set of A, power set of A Ok.

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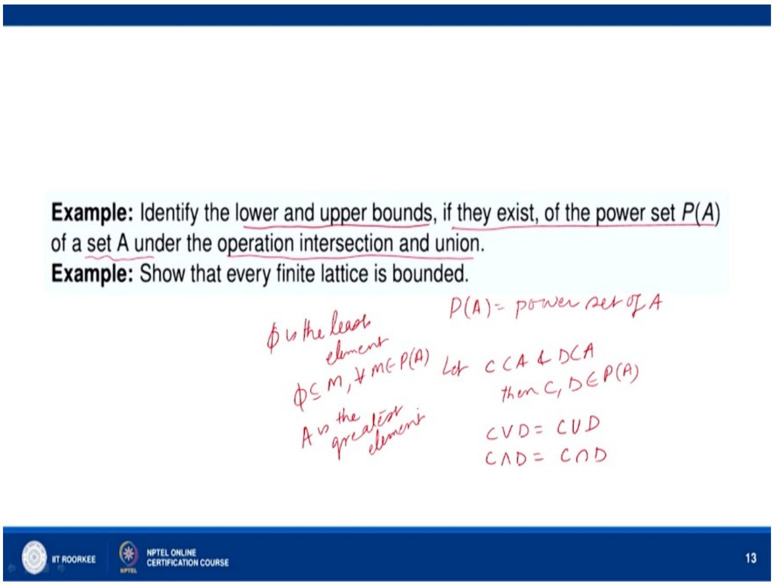

And ∩, ∧ is the ∩ operation. So let C be subset of A and D be subset of A, Ok. C, D ∈ P A, Ok. So we define ∩, ∪ means we define ∧ and ∨, ∨ we define as C ∪ D, Ok. And ∧ we define as C ∩ D, Ok. Now if you take this P(A), so P(A) contains all subsets of A, so ϕ is there, also A is there. ϕ, ϕ is the least element and therefore, because ϕ is contained in every other element of P(A). ϕ is contained in m, say m for every m ∈ P(A), Ok. So it is the least element, Ok. And a is the greatest element. every subset of, every set ∈ P (A) is a subset of A, Ok. Now let us show that every finite lattice is bounded. Let us assume that L contains elements { $a_1, a_2, \ldots\ldots a_n\}$.

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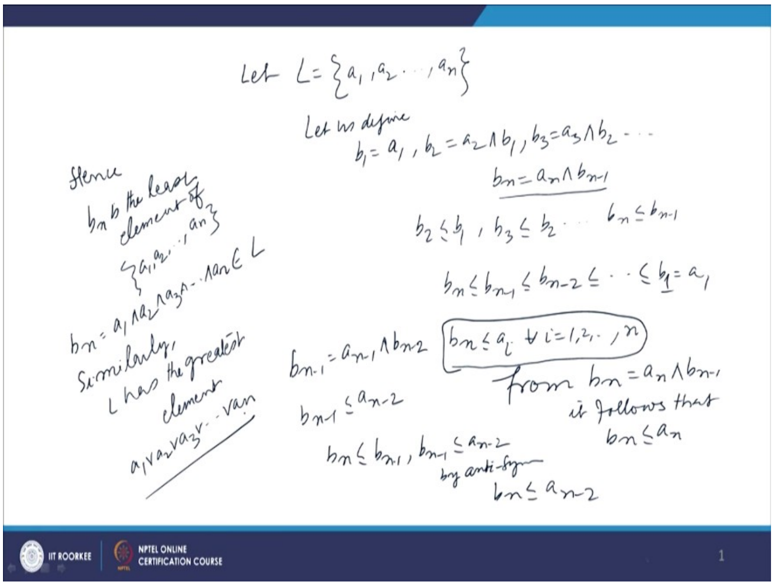

L contains finite number of elements. Let us say it contains n elements, $\{a_1, a_2, \ldots\ldots a_n\}$. So it is a finite lattice. We want to prove that it is bounded. That means it has least element as well as the greatest element. Let us define $b_1 = a_1$, $b_{2=¿a_2¿} \wedge b_1$, $b_3 = a_3 \wedge\ b_2$, and so on. $b_n = a_n \wedge b_{n-1}$, Ok. Then what do we notice?

Ok. $b_{2=¿a_2¿} \wedge b_1$, so $b_{2=¿¿}$ is the greatest lower bound of $a_2$and b_1. So b 2 ≤ b 1, b 3 ≤ b 2, and so on. $b_n \le b_{n-1}$, Ok. So we can see $b_n \le b_{n-1}$, $b_{n-1} \le b_{n-2}$ and so on, so we get $b_n \le b_{n-1}$. $b_{n-1} \le b_{n-2}$and so on, $\le b_1$. b_1 means a_1.In this process you can see that $b_n \le a_i$ for every i, i = 1, 2 ,…. n, because from here you can see, $b_n = a_n \wedge\ b_{n-1}$. So b_n is the greatest lower bound of a_n and b_{n-1} . So $b_n \le a_n$ and $b_n \le b_{n-1}$.

Now $b_n \le a_n$ and from the previous equation $b_{n-1} \le a_{n-1}$, but $b_n \le a_n$ and $a_n \le$, $b_n \le$ we can say, if you want to prove that, that $b_n \le a_i$for every i, so then from b_n ≤, b_n =equal to $a_n \wedge$ b_{n-1}. It follows that $b_n \le a_n$, Ok. $b_{n-1} \le a_{n-1} \wedge b_{n-2}$, Ok. So, $b_{n-1} \le a_{n-2}$, Ok.

Now $b_n \le b_{n-1}$, and $b_{n-1} \le a_{n-2}$, so by antisymmetry, antisymmetry $b_n \le a_{n-2}$. Continuing this process we can show that $b_n \le a_{n-3}$, and so on. And then it follows that $b_n \le a_i$for every i, Ok. So b_n is a, least element of, hence b_n is the least element of $\{a_1, a_2, \ldots\ldots a_n\}$. And $b_n = a_n$; b_{n-1} is equal to $b_{n-2} \wedge b_{n-2}$, so what we get? b_n becomes $a_1 \wedge a_2 \wedge a_3 \ldots\ldots \wedge a_n$, Ok.

Now clearly L is a lattice so $a_1 \wedge a_2 \wedge a_3 \ldots.. \wedge a_n$ is an element of L. By associativity, so these are elements of L, so L has a, L has the least element, Ok. Similarly we can show that L has the greatest element. And the greatest element is $a_1 \vee a_2 \vee a_3 \ldots.. \vee a_n$. So when L is a finite lattice, it is always bounded because it contains the least element and the greatest element. So that is all in this lecture. Thank you very much for your attention.