Higher Engineering Mathematics

Professor P. N. Agrawal
Department of Mathematics
Indian Institute of Technology Roorkee

Lattices II

Hello friends, welcome to my lecture on Lattices. This is second lecture on lattices. Let us prove some properties of lattices. The theorem, this theorem we have proved in the last lecture.

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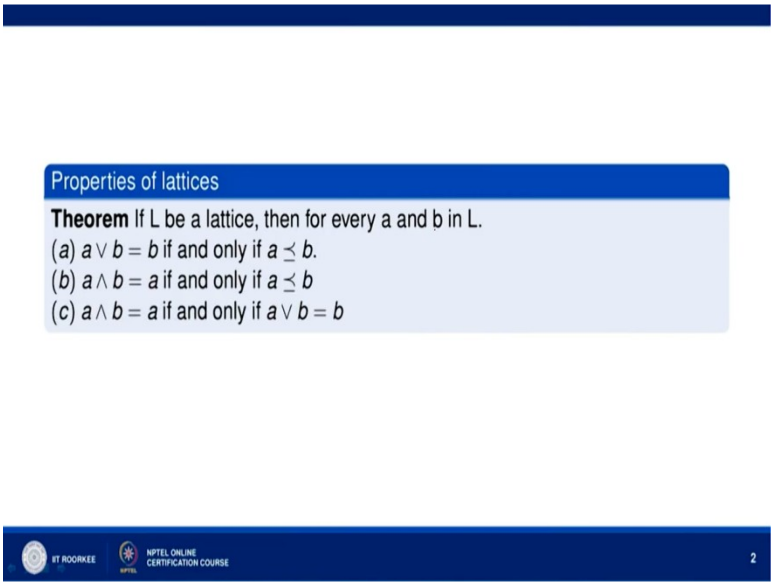

If L be a lattice, then for every a and b in L, a $\vee$ b = b if and only if a precedes b. a $\wedge$b = a if and only if a precedes b. a $\wedge$ b = a if and only if a $\vee$ b = b. This we had proved in the last lecture.

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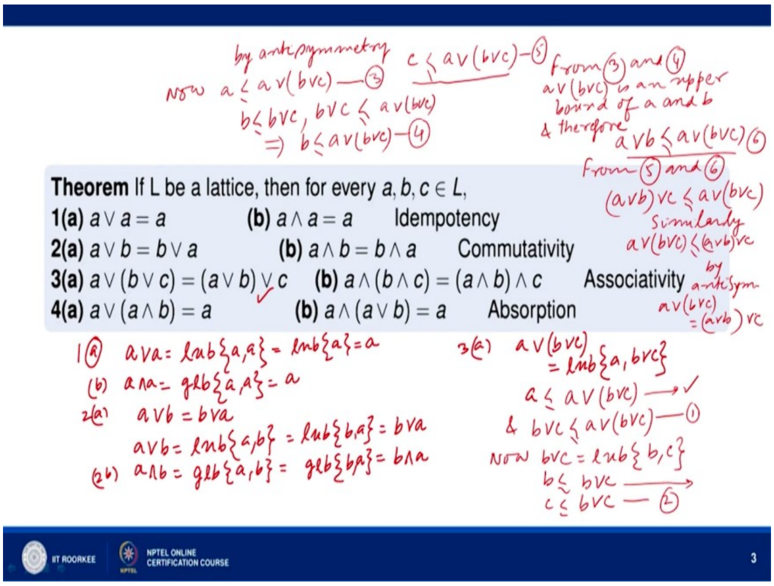

Now let us go to the second theorem. If L be a lattice, then for every a, b, c belonging to L a ∨ a = a, a ∧ a = a. These are idempotency laws. Then a ∨ b = b ∨ a, a ∧ b = b ∧ a, these are commutativity laws. Then a ∨ (b ∨ c) = (a ∨ b) ∨ c and a ∧ (b ∧ c) = (a ∧ b) ∧ c, these are associativity laws. And then we have the absorption laws. So a ∨ (a ∧ b) = a and a ∧ (a ∨ b) = a. These are absorption laws.

Let us prove these properties of a lattice. So let us say, first (a) part. a ∨ a. By definition a ∨ a = least upper bound of {a, a}, Ok. So this is least upper bound of {a} which is equal to a, Ok. So a ∨ a = a. Similarly a ∧ a = greatest lower bound of {a, a}. Greatest lower bound of {a}, is = a, Ok. Now let us prove, so this is part (b). Let us do 2(a) part. So a ∨ b = b ∨ a, Ok.

So a ∨ b = least upper bound of {a, b}. Least upper bound of a comma b is same as least upper bound of {b, a}. Ok so we have b ∨ a, Ok. Similarly a ∧ b = greatest lower bound of{ a, b} which is = greatest lower bound of {b, a}. So we have b ∧ a, Ok. So these two are commutativity laws, Ok and these two are, the first two are idempotency laws.

Now let us show 3(a) part, Ok so 3(a) part. We have to prove that a ∨ (b ∨ c) = (a ∨ b) ∨ c, Ok. So a ∨ (b ∨ c)= least upper bound of a and b ∨ c, Ok.

So since it is least upper bound of a and b ∨ c, so a precedes, a precedes, a ∨ (b ∨ c), Ok. And similarly

b ∨ c precedes a ∨ (b ∨ c)………………..(1)

Now $b \vee c$ is least upper bound of b and c, Ok. So b precedes $b \vee c$ and

c precedes $b \vee c$…………………………………(2)

Ok. So let us call this as equation number (1), this as equation number (2), Ok. Then c precedes $b \vee c$ and $b \vee c$ precedes $a \vee (b \vee c)$. So by antisymmetry we have c precedes $b \vee c$. $b \vee c$ precedes $a \vee (b \vee c)$. So c precedes $a \vee (b \vee c)$.

Now, now further we have

a precedes $a \vee (b \vee c)$……………………..(3)

Ok and b precedes, so this is this equation and this equation, Ok. Let us consider now a precedes $a \vee b \vee c$, Ok and b precedes $b \vee c$ and $b \vee c$ precedes $a \vee (b \vee c)$, Ok. b precedes $b \vee c$. $b \vee c$ precedes $a \vee (b \vee c)$. So again by antisymmetry this implies

b precedes $a \vee (b \vee c)$………………………(4)

Now let us call them as equations (3) and (4), Ok. So from (3) and (4), we find that $a \vee (b \vee c)$ is an upper bound of, upper bound of a and b. And therefore $a \vee b$, $a \vee b$ is the least upper bound of a and b, so $a \vee b$ precedes $a \vee (b \vee c)$, Ok.

Now what do we notice? Let us consider this.

$a \vee b$ precedes $a \vee (b \vee c)$…………………….(5)

And

c precedes $a \vee (b \vee c)$……………………….(6)

So $a \vee (b \vee c)$ is an upper bound of $a \vee b$ and c. So this is, I can call this as 5 and this as 6. So from 5 and 6, $(a \vee)b \vee c$ precedes $a \vee(b \vee c)$, Ok. $(a \vee)b \vee c$ precedes $a \vee(b \vee c)$. And similarly we can show $a \vee (b \vee c)$ precedes $(a \vee b) \vee c$.

Then we can use antisymmetry, Ok. $(a \vee b) \vee c$ precedes $a \vee (b \vee c)$ and $(a \vee b) \vee c$ precedes $a \vee (b \vee c)$. So by antisymmetry then, antisymmetry $(a \vee b) \vee c = a \vee (b \vee c)$, Ok. This is how we establish the 3(a) part. 3(b) part can be similarly proved, Ok. Now let us go to the 4(a) part, Ok. In the 4(a) part we have to prove absorption law, absorption law. So $a \vee (a \wedge b) = a$. $a \vee (a \wedge b) = a$, this is what we have to prove,

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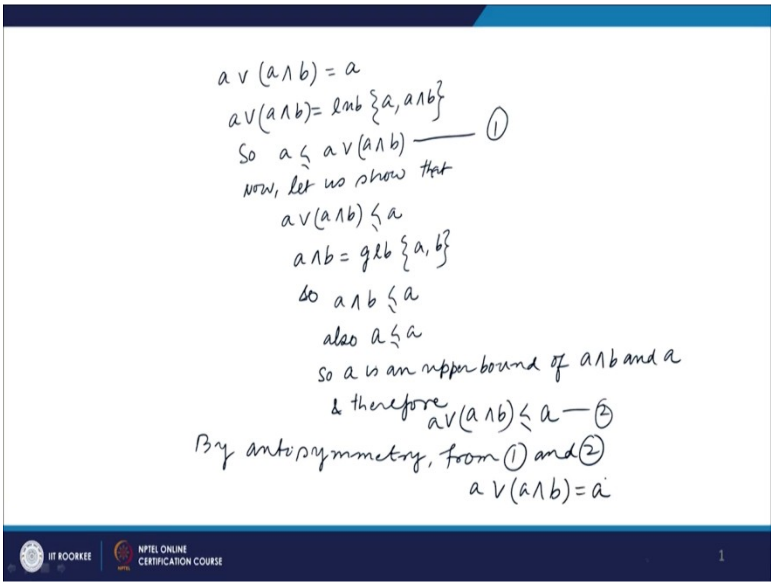

Ok a ∨ (a ∧ b) = a, we have to show this. So here we notice that, Ok, alright, Ok. So a ∨ (a ∧ b) = l u b of a and a ∧ b, so a is preceded by, a precedes, sorry

a precedes a ∨ (a ∧ b)……………………..(1)

Ok. a precedes a ∨ (a ∧ b). Now let us show that, Ok, now let us show that a ∨ (a ∧ b) precedes a, Ok. a ∧ b, a ∧ b = g l b of a and b, Ok. So a ∧ b precedes a, Ok. Also a precedes a, Ok. So a is an upper bound of, Ok so a is an upper bound of a ∧ b and a, Ok and therefore,

a ∨ (a ∧ b) precedes a……………………..(2)

Ok. So we have, this is equation 1, and this is equation 2, Ok. So from 1 and 2 by antisymmetry a ∨ (a ∧ b) = a. Similarly we can prove the other absorption law. a ∧ (a ∨ b) = a.

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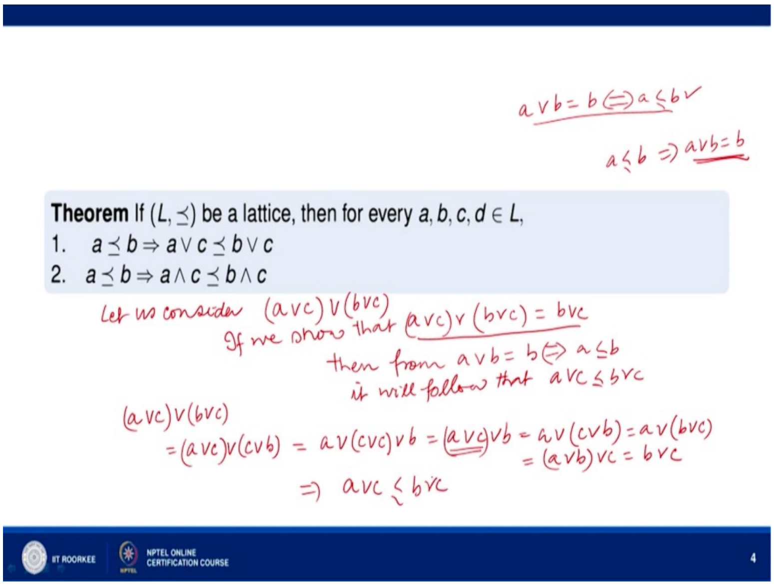

Now let us go to next theorem. Let L precedes be a lattice, then for every a, b, c, d belonging to L a precedes b implies a ∧ c precedes b ∧ c, Ok. So let us consider (a ∧ c) ∧(b ∧ c) Ok. What we are going to do is this. In order to prove that a ∧ c precedes b ∧ c, we will use the first theorem here, this theorem, Ok. a ∧ b = b if and only if a precedes b, Ok. So a ∧ b = b if and only if a precedes b, Ok. So we are going to use this theorem. In order to prove that a, a ∨ c precedes b ∨ c, we simply have to show that (a ∨ c) ∨ (b ∨ c) = b ∨ c.

Let us consider this. If we show, if we show that (a ∨ c) ∨ (b ∨ c) = b ∨ c, then using this result, Ok, then from a ∨ b = b ⇔ a precedes b. It will follow that a ∨ c precedes b ∨ c, Ok. So let us prove this, Ok. (a ∨ c) ∨ (b ∨ c), we can consider, we can write it as (a ∨ c) ∨ (c ∨ b), because of the commutative law, b ∨ c = c ∨ b and by the associative law I can write it as a ∨ (c ∨ c)∨ b, Ok. a ∨ (c ∨ c) ∨ b.

Now c ∨ c = c, Ok c ∨ c equal c. So a ∨ c ∨ b, Ok. Right, now a ∨ c, we have to consider a ∨ c, Ok. c ∨ b, we have a, we have a, we have not made use of this, a this, Ok this implies a ∨ b = b, Ok So I can write it as...again use associative law, Ok. I can write it as a, Ok I can write it as a ∨ c ∨ b, c ∨ b I can use commutative law so a ∨(b ∨ c), Ok. Now I can write it as, using associative law, (a ∨ b) ∨ c, Ok. And a ∨ b = b because a precedes b. So b ∨ c, Ok. So a ∨ c, (a ∨ c) ∨ (b ∨ c) = b ∨ c so this implies that a ∨ c precedes b ∨ c. Similarly we can prove a precedes b implies a ∧ c precedes b ∧ c.

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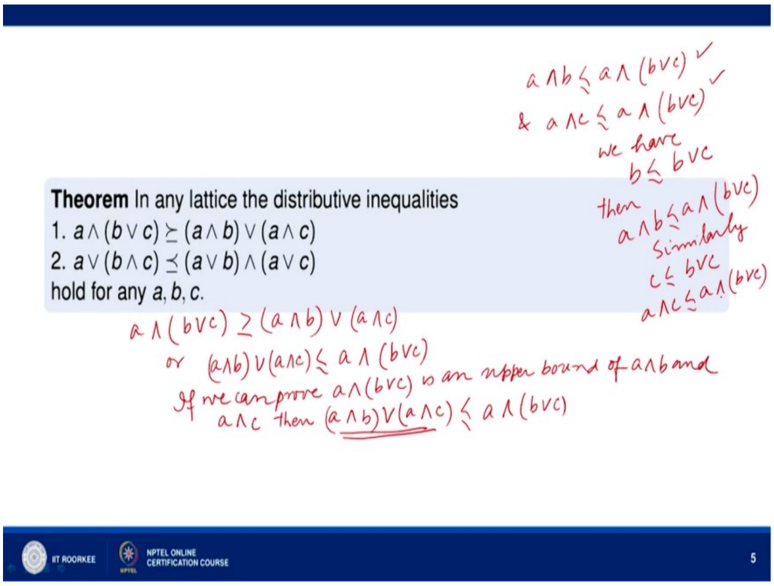

Now let us prove the distributive inequality. a ∧ b, let us prove a ∧ (b ∨ c) succeeds (a ∧ b) ∨ (a ∧ c), Ok. Or we can say; we have to show (a ∧ b) ∨ (a ∧ c) precedes a ∧(b ∨ c), Ok. So if we can show that, if we can prove that a ∧(b ∨ c), is an upper bound of, of a ∨ b and a ∨ c. If we can prove that a ∧(b ∨ c), is an upper bound of a ∧ b and a ∧ c, then a ∨, (a ∧ b) ∨ (a ∧ c), Ok will be, will precede a ∧(b ∨ c), because this is nothing but the least upper bound of a ∧ b and a ∧ c, Ok

Now to show, that means we have to show that a ∧ b precedes a ∧(b ∨ c),, Ok and we have to show that a ∧ c precedes a ∨, sorry a ∧(b ∨ c), Ok. So we have to establish this. Now a ∧ b, we have to show that a ∧ b is, precedes a ∧ b, Ok so we have b precedes b ∨ c, Ok. b precedes b ∨ c. Ok so we can use now this property, this one, this one we have to get, we have to get ∧, Ok.

So if a precedes b then a ∧ c precedes b ∧ c, Ok. Let us use this property of the last theorem, Ok. So then a ∧ b precedes a ∧(b ∨ c), Ok by the preceding theorem. Similarly we can say that, similarly c precedes b ∨ c, Ok. So by the preceding theorem a ∧ c precedes a ∧(b ∨ c),, Ok. Thus we see that a ∧ b precedes a ∧(b ∨ c), and a ∧ c precedes a ∧(b ∨ c),, Ok. So a ∧(b ∨ c), is an upper bound for a ∧ b and a ∧ c, Ok. And therefore the (a ∧ b) ∨ (a ∧ c) which is the least upper bound of a ∧ b and a ∧ c is ≤ a ∧ (b ∨ c), Ok. So this is what, this is how we prove this result. A similar argument we can give for the proof of the other one.

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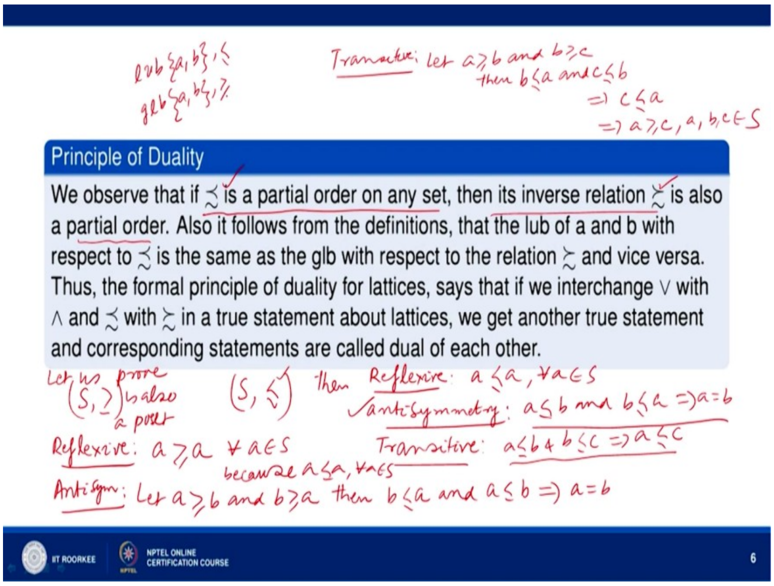

Now let us go to principle of duality. We observe that precedes is a partial order on any set,

Ok then its inverse relation succeeds is also a partial order. This can be easily shown. Suppose S is a set, Ok and it is a poset with this partial order, Ok, then first property is reflexive. a precedes a for any a belonging to S, Ok. The second one is antisymmetry. a precedes b and b precedes a implies a = b, Ok and third one is transitivity, Ok. So a precedes b and b precedes c implies a precedes c, Ok.

Now let us show that S succeeds is also a poset. Let us prove, Ok. So a precedes a means a succeeds a, Ok. So first thing is reflexive. a succeeds a for every a belonging to S because a succeeds a for every a belonging to S, Ok. Second thing antisymmetry, let a succeeds b and b succeeds a, Ok then we have to prove that a = b. Now a succeeds b means b precedes a and, b succeeds a means a precedes b, Ok.

So a precedes b and b precedes a, Ok using the antisymmetry, Ok. This implies that a = b. Ok we can use this result now. This, that S is a poset with this notation, Ok and similarly transitive. Let us say a precedes b and b precedes c, Ok. Then, no sorry, let a succeed b and b succeed c. Then b precedes a and c precedes b, Ok. Now we apply the transitive property here, Ok. So c precedes b and b precedes a implies that c precedes a. Or we can say a succeeds c, Ok.

So let a succeed b and b succeed c, let a succeed b and b succeed c then a succeed c. So this is true for all a, b, c belonging to S, Ok and therefore if this is a partial order, this one is also a partial order on S. Now it follows from the definitions that the l u b of {a, b} with respect to this, l u b of, it is clear that l u b of {a, b} with respect to this order, precedes is g l b of {a, b} with respect to this order, Ok. So, and vice versa. So thus the former principle of duality for lattices says that if we interchange, ∨ and ∧ and precedes and succeeds in a true statement about lattices, we get another true statement. And corresponding statements are called dual of each other.

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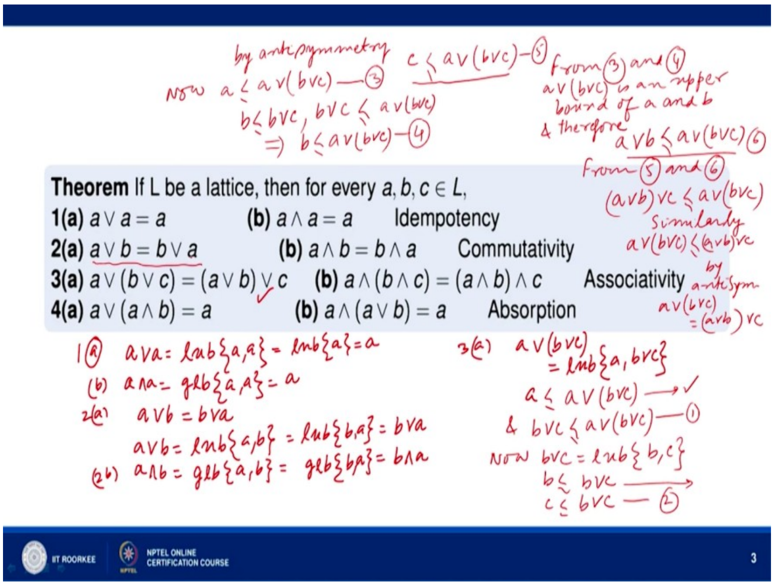

So you can see here, where we have written these, yeah, you can see we have this result, Ok. We can interchange, Ok ∨ and, in order to write this, we can interchange ∨ and, by ∧, Ok. And then we get the corresponding commutative law, commutative law for this one. And similarly here you can interchange ∨ and ∧, Ok, here ∨ and ∧ and we get the other absorption law. So, so we can say that this is dual of this one, Ok this is dual of this one. And this is dual of this one. So they are dual of each other.

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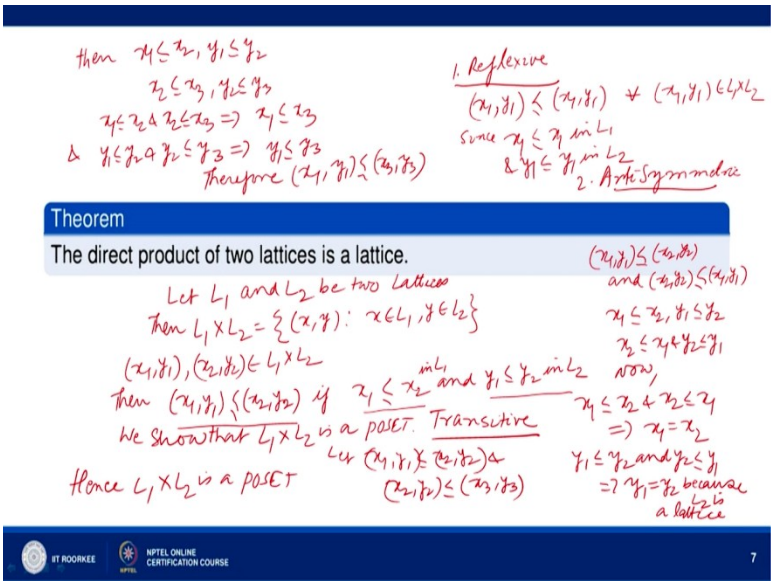

Now the direct product of two lattices is a lattice. Let us see, let L_1 and L_2. be two lattices, Ok. Then $L_1 \times L_2$, we define as (x, y) such that x belongs to L_1 , y belongs to L_2.

Ok. We will first prove that, let us say (x_1, y_1) and (x_2, y_2) belong to $L_1 \times L_2$. Then let us define (x_1, y_1) precedes (x_2, y_2) if x_1 precedes x_2 and y_1 precedes y_2 x_1 precedes x_2 in L_1 and y_1 precedes y_2 in L_2.

Now although we are using the same notation, but they are not same. In different, in L_1 and L_2 , they mean different, they have different meanings, Ok. So convenience we are using the same notation. So (x_1, y_1) precedes (x_2, y_2) if and only if x_1 precedes x_2 in L_1 and y_1 precedes y_2 in L_2. Let us define order like this, and then we will prove that $L_1 \times L_2$ is a poset with this definition. So we show that $L_1 \times L_2$ is a poset, Ok.

So first thing is reflexive, Ok. So we show that, let (x_1, y_1), (x_1, y_1) precedes (x_1, y_1) . First we have to show this for every (x_1, y_1) belonging to $L_1 \times L_2$, Ok.So (x_1, y_1) precedes (x_1, y_1) will be true because x_1 precedes x_1 in L_1 and y_1 precedes y_1 in L_2 . So since $x_1$precedes x_1 in L_1, and y_1 precedes y_1 in L_2, Ok because L_1 and L_2 are lattices, it follows that (x_1, y_1) precedes (x_1, y_1) whenever (x_1, y_1) belongs to $L_1 \times L_2$. So this is reflexive property.

Then we have antisymmetric. So let us assume that (x_1, y_1) precedes (x_2, y_2) and (x_2, y_2) precedes (x_1, y_1). Then we have to show that $(x_1, y_1) = (x_2, y_2)$. So (x_1, y_1) precedes (x_2, y_2)

means x_1 precedes x_2, y_1 precedes y_2 . And (x_2, y_2) precedes (x_1, y_1) means x_2 precedes x_1 and y_2 precedes y_1,Ok.Now x_1 precedes x_2 and x_2 precedes x_1, Ok implies that $x_1 = x_2$ because L_1 is a lattice. And similarly y_1 precedes y_2 and y_2 precedes y_1 implies that $y_1 = y_2$ because L_2 is a lattice.

Now let us prove transitive, Ok. So let (x_1, y_1) precedes (x_2, y_2) and (x_2, y_2) precedes x_3, y_3, Ok. Then x_1 precedes x_2, y_1 precedes y_2 , Ok. x_2 precedes x_3, y_2 precedes y_3, Ok. Now L_1 is a lattice. So x_1 precedes x_2 and x_2 precedes $x_3$implies that x_1 precedes x_3,

Ok. And similarly y_1 precedes y_2 and y_2 precedes y_3, implies that y_1 precedes y_3. Now x_1 precedes $x_3$and y_1 precedes $y_3$and therefore (x_1, y_1) precedes (x_3, y_3). Ok and so we have shown that in $L_1 \times L_2$, if we define order by this, (x_1, y_1) precedes (x_2, y_2) if and only if x_1 precedes x_2 in L_1 and $y_1$precedes y_2 in L_2. Then $L_1 \times L_2$ is a poset, Ok. So hence $L_1 \times L_2$ is a poset. Now we shall show that $L_1 \times L_2$is a lattice. So now let us prove that $L_1 \times L_2$is a lattice , Ok.

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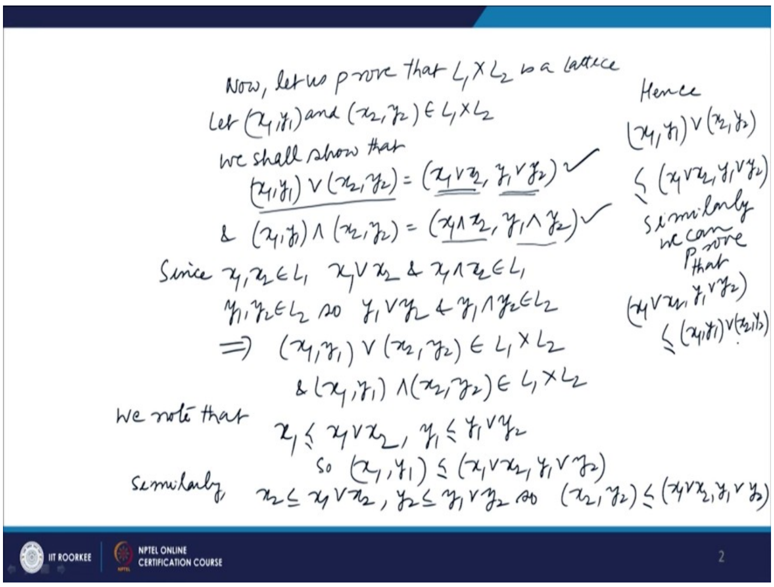

So let (x_1, y_1) and (x_2, y_2) belong to $L_1 \times L_2$. We shall show that $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \wedge y_2)$, Ok. And $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$. Ok. So since L_1 and $L_2$are lattices, Ok, $x_1 \vee x_2$, belongs to L_1 , $x_1 \wedge x_2$, belongs to L_1 ., $y_1 \vee y_2$belongs to $L_2$and $y_1 \wedge y_2$ belongs to L_2. And therefore (x_1, y_1) $(x_1, y_1) \vee (x_2, y_2)$ belongs to $L_1 \times L_2$and $(x_1, y_1) \wedge (x_2, y_2)$ belongs to $L_1 \times L_2$.

So since x_1, x_2, belongs L_1, $x_1 \vee x_2$ and $x_1 \wedge x_2$, , they belong to L_1, Ok.

And similarly y_1, y_2 belongs to L_2 . So $y_1 \wedge y_2$, and $y_1 \wedge y_2$belongs to L_2 , Ok. And this will then imply that $(x_1, y_1) \vee (x_2, y_2)$ belongs to $L_1 \times L_2$. And $(x_1, y_1) \wedge (x_2, y_2)$ belongs to $L_1 \times L_2$. So we just have to prove this, Ok, and this. Ok if we can prove that they are equal, then $L_1 \times L_2$ will be a lattice. Ok so in order to prove this, we know that, first we shall show that $(x_1, y_1) \vee (x_2, y_2)$; $(x_1, y_1) \vee (x_2, y_2)$ this is least upper bound of (x_1, y_1) and (x_2, y_2) .

So first we shall show that $x_1 \vee x_2$, $y_1 \vee y_2$is an upper bound of $(x_1, y_1) \vee (x_2, y_2)$, Ok, (x_1, y_1) and (x_2, y_2) , Ok So x_1 precedes $x_1 \vee x_2$, Ok. And y_1 precedes $y_1 \vee y_2$, Ok. So (x_1, y_1) precedes $x_1 \vee x_2$ and $y_1 \vee y_2$, by your definition, Ok, alright. Similarly x_2 precedes $x_1 \vee$ x 2, y 2 precedes $y_1 \vee y_2$, Ok. So (x_2, y_2) precedes $(x_1 \vee x_2; y_1 \vee y_2)$.

Thus what do we notice? $(x_1 \vee x_2; y_1 \vee y_2)$ is an upper bound for (x_1, y_1) and (x_2, y_2) . And hence $(x_1, y_1) \vee (x_2, y_2)$ precedes $(x_1 \vee x_2; y_1 \vee y_2)$. Ok it is an upper bound for $(x_1 \vee x_2; y_1 \vee y_2)$, it is an upper bound, $(x_1 \vee x_2; y_1 \vee y_2)$ is an upper bound of (x_1, y_1) and (x_2, y_2) . So $(x_1, y_1) \vee (x_2, y_2)$, because $(x_1, y_1) \vee (x_2, y_2)$ is the least upper bound of (x_1, y_1) and (x_2, y_2) . So it has to be less than or equal to this, Ok.

Similarly we can prove that $(x_1 \vee x_2, y_1 \vee y_2)$ precedes $(x_1, y_1) \vee (x_2, y_2)$, Ok. And therefore they are equal. So x_1, so hence by antisymmetry$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$, Ok so this is proved, Ok.Similarly we can show then $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$. And that proves that the least upper bound of (x_1, y_1) and (x_2, y_2) is there in $L_1 \times L_2$. And greatest lower bound (x_1, y_1) and (x_2, y_2) is also there in $L_1 \times L_2$. And therefore $L_1 \times L_2$is a lattice.

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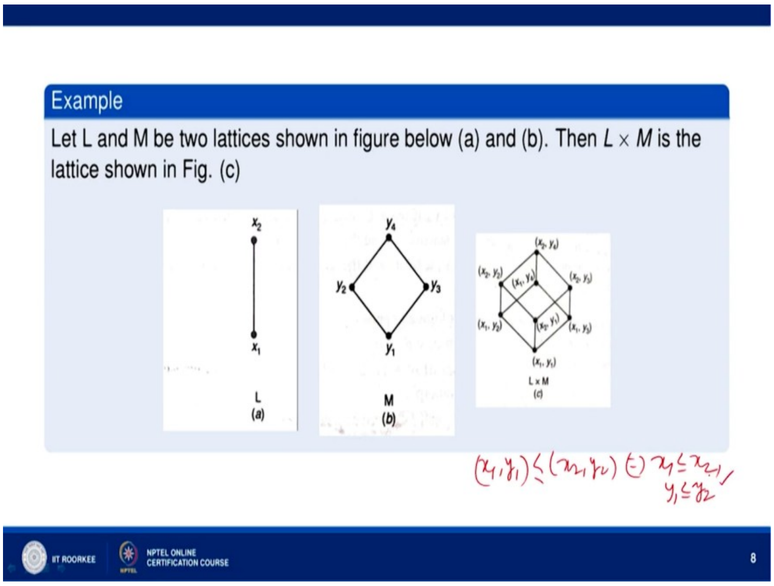

Now let L and M be two lattices shown in the figure below, (a) and (b), Ok. Then L × M is a lattice shown in figure (c). You see x_1, $x_2$this is the lattice L. And this is the lattice M, y_1, y_2, y_3, y_4. Then, then as we have seen (x_1, y_1¿precedes (x_2, y_2¿if and only if x_1 precedes x_2, y_1 precedes y_2. So using this definition, Ok, L × M, we can write L × M. See (x_1, y_1¿precedes (x_2, y_2). (x_1, y_1¿ precedes (x_1, $y_{3¿}$¿, Ok and ¿¿, $y_{1¿}$¿also precedes (x_1, y_4¿. Ok.(x_1, y_1) precedes x_2, $y_1$because x_1 precedes $x_2$and $y_1$precedes y_1, reflexive property, Ok. So (x_1, y_1)lies below (x_2, y_1¿, Ok. It precedes (x_2, y_1). And then ¿, y_1) precedes (x_2, y_2). ¿, y_1) precedes (x_2, y_3). And further (x_2, y_2) precedes (x_2, y_4¿. (x_2, y_3) precedes ¿¿, y_4). So this is the figure for L × M.

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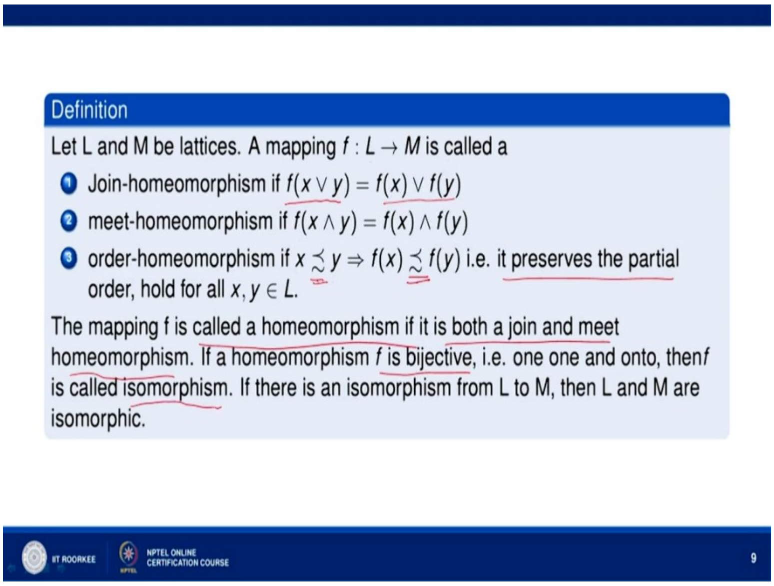

Now let us say, let L and M be two lattices. A mapping f from L into M is called a join homeomorphism if f (x ∨ y) = f (x) ∨ f (y) meet- homeomorphism if f(x ∧ y) = f (x) ∧ f (y), order homeomorphism if x precedes y implies f (x) precedes f(y). Now this precedes is not the same as the precede here, because this precede is for L and this precede is for M which can be different.

That is, it preserves the partial order, that is this, order homeomorphism means f will be said to have order homeomorphism if it preserves the partial order. x precedes y implies f (x) precedes f(y). And this should hold for all x, y belonging to L. The mapping f is called homeomorphism if it is both join homeomorphism and meet homeomorphism. If homeomorphism is bijective, that is one-one, onto then f will be called an isomorphism. And if there is an isomorphism from L to M then L and M are said to be isomorphic.

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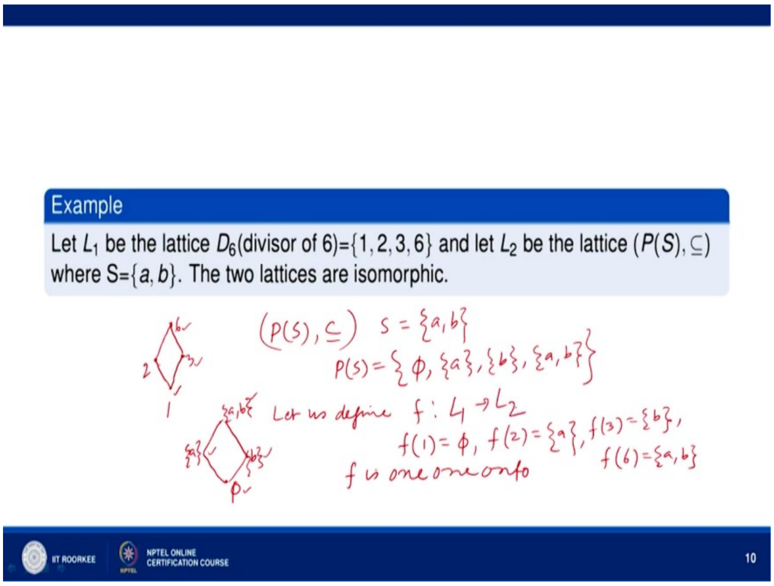

Now let us look at this. Let L be, L_1 be the lattice D_6. D_6. is divisor of 6, so we have {1, 2, 3, 6}. Let us draw the Hasse diagram, 1, 2, 6 Ok so 1, 2, 6 and we have 3 here, Ok, so 3 divides 6. So we have 1, 2, 6, 3, Ok. And L_2 be the lattice P (S). P (S), power set of S and we have the order relation inclusion, Ok. S ={a, b} so S ={ a, b} here. So P (S) = { Φ, {a}, {b}, {a b}} Ok. We have to show that the two lattices are isomorphic.

So what we do is let us define a function f from D_6. to P(S), Ok as f (1) = Φ, f(2) = a, f (3) = b, Ok f(6)={ a, b}. Ok. Then we can draw a Hasse diagram for this P (S). This is Φ here, and this is {a} this is [b}, and this is {a, b}. Ok so this 1 goes to Φ here, Ok, 2 goes to [a}, 3 goes to [b}, and 6 goes to {a, b}, Ok.And clearly f is one-one, onto, Ok, f is one-one, onto so there exists a bijection f from D_6. to P (S) this L_1 to L_2, Ok.So f is a map bijection from $L_1$to L_2. Instead of D_6. we should write L_1, L_2 Ok and therefore since there exists a bijection from L_1 to L_2, the two lattices are isomorphic.

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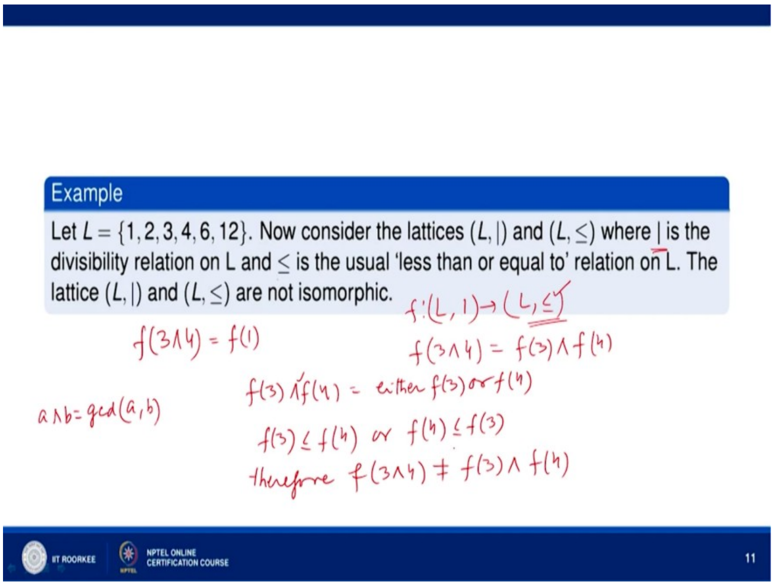

So now let us say, let L be this set 1, 2, 3, 4, 6, 12. We consider the two lattices, L with divisibility relation, L with less than or = where (L, |) notation is for divisibility on L and (L ,≤) notation is the usual, less than or = relation on L. We have to show that the two are not isomorphic. So we have to show that L, in order to show that it is isomorphic we have to show that it is, the two things we have to show, join homeomorphism, meet homeomorphism and there is a bijection. So let us show that, meet homeomorphism is not true here, meet homeomorphism does not exist, Ok.

So let us see. Suppose f is a mapping from (L, |) to (L ,≤) then what we will see? Let us consider f (3 ∧ 4), f (3 ∧ 4) let us consider, Ok. Then f (3 ∧ 4), f is a mapping from L to L. L with divisibility and the other one is L with less than or equal to, Ok. So f (3 ∧ 4), a ∧ b in the case of divisibility is greatest common divisor of a and b, Ok. So greatest common divisor of 3 and 4 is 1. So we have f(1), Ok.

And right side we have f (3 ∧ 4) = f(3)∧ f(4). Now what is f (3) ∧ f (4) ? f (3) ∧ f (4) = either f (3) or f (4), Ok because this ∧ here and in this set is less than or equal to. The relation is less than or =. So f (3) will be less than or equal to f(4), Ok or f(4) will be less than or equal to f (3), Ok in the case of, less than, so less than or equal to, so either f (3) will be less than or equal to f (4), or f (4) will be less than or equal to f (3). And therefore f (3 ∧ 4) is not equal to f (3) ∧ f(4), Ok. Either it is f (4) or it is f(3).

So f (1) is not equal to f (3) or f (1) is not equal to f (4) and therefore $\wedge$, it is not meet homeomorphism. So (L, |) with divisibility relation and (L ,$\leq$) with less than or equal to, they are not isomorphic to each other. With that I would like to end this lecture. Thank you very much for your attention.