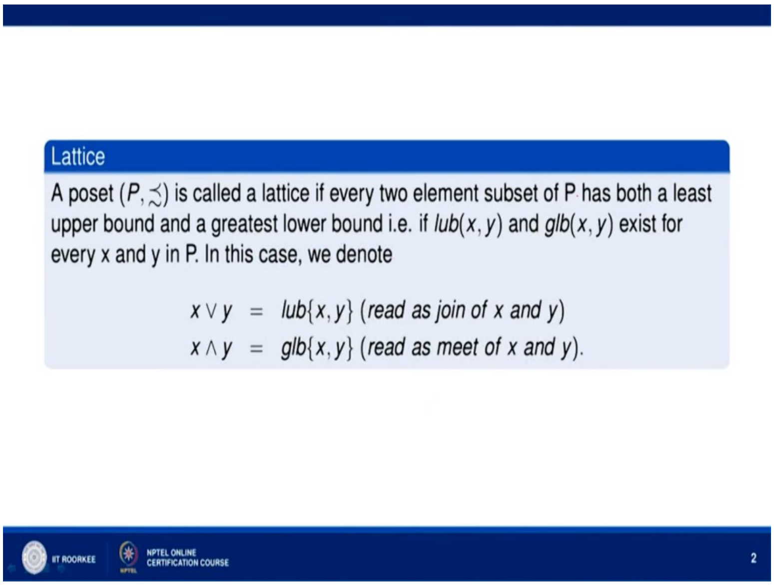
Higher Engineering Mathematics
Professor P. N. Agrawal
Department of Mathematics
Indian Institute of Technology Roorkee
Lattices I

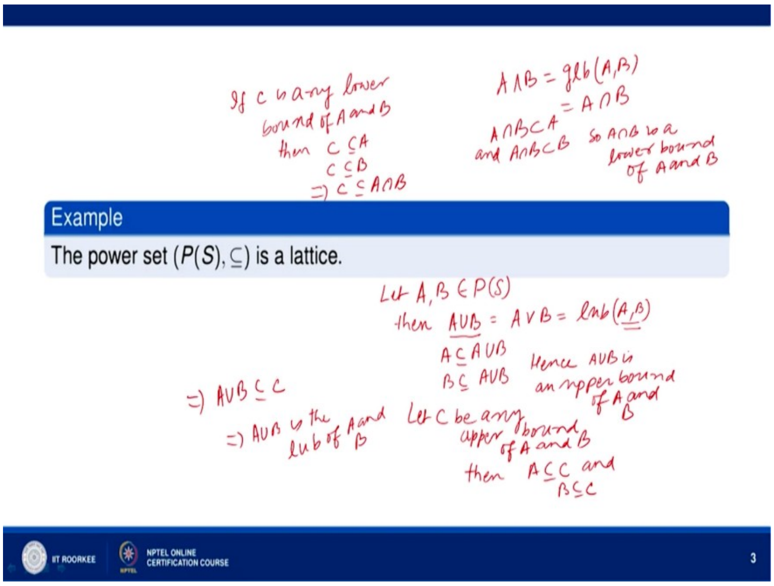
Hello friends, welcome to my lecture on Lattices. We will define a lattice and we shall see some of its properties. A poset P is called a lattice if every two elements, if every two element subset of P has least upper bound and a greatest lower bound

(Refer Slide Time: 00:43)



least upper bound we denote by l u b. So if x and y are any two elements of P then l u b{x, y} means least upper bound of x, y and g l b {x, y} means greatest lower bound of x, y.So P will be called a lattice if you take, pick up any two elements of the set P, then you have the greatest lower bound as well as the least upper bound of the, both the elements x and y of P. So in this case, this notation, this notation is called join of x and y. So join of x and y is least upper bound of x, y and meet of x and y, Ok ∨ notation means join of x and y, ∧ notation means meet of x and y, so meet of x and y is the greatest lower bound of x and y.

(Refer Slide Time: 01:40)



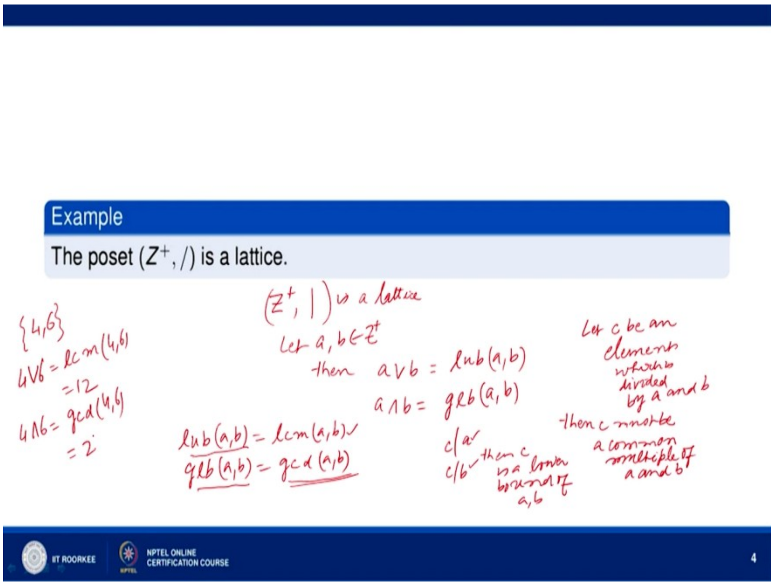
Now let us see an example of a lattice. The power set P(S) with inclusion relation is a lattice. Let A and B belong to P(S) which is the power set of a set S. Ok then $A \cup B$, $A \cup B$ is equal to the join of A and B. Why? Because A, join of A and B means least upper bound of, least upper bound of A and B. Now let us show that $A \cup B$ is the least upper bound of A, B. We can see it very easily. A is subset of $A \cup B$. B is subset of $A \cup B$, Ok. Hence $A \cup B$ is an upper bound of A and B.

Now let us show that it is the least upper bound of A and B. So let us say, C be any other upper bound of A and B. Let C be any upper bound of A and B. Then A is subset of C and B is subset of C, Ok which implies that $A \cup B$ is a subset of C, Ok. So we have seen that $A \cup B$ is an upper bound of A and B and if C is any other, any upper bound of A and B then $A \cup B$ is contained in C. So $A \cup B$ is the least upper bound of A and B, Ok.

Similarly we can show that $A \cap B$ is the greatest lower bound of A and B. So A, $a \wedge b$ which is greatest lower bound of A and B, let us show that it is $A \cap B$, Ok. So we have to show that $A \cap B$ is the greatest lower bound of A and B. We know that $A \cap B$ is contained in A. And $A \cap B$ is contained in B, Ok.

So $A \cap B$ is a lower bound of A and B, Ok. Now if C is any lower bound of A and B, if C is any lower bound of A and B then C is subset of A, C is subset of B, Ok. So this implies that C is subset of $A \cap B$, Ok and therefore $A \cap B$ is the greatest lower bound of A and B. So when A, B belong to P(S) then the greatest lower bound of A and B is $A \cap B$ and the least upper bound of A and B is the A ∪ B, Ok. And since it is P(S), Ok, so it contains all subsets of the set S.So A ∪ B and A ∩ B also belong to P (S) and therefore if you take any two sets A and B in P(S) then their join and their meet, they also belong to P(S). And therefore P (S) is a lattice.

(Refer Slide Time: 05:55)



Now let us show the poset $Z^{+¿¿}$, Ok set of positive integers. So let us now show that $Z^{+¿¿}$, with divisibility relation, Ok is a lattice, Ok. So let us take a and b belong to $Z^{+¿¿}$, Ok. Then a, join of a and b, it is l u b, least upper bound of a and b and a, a∧b is equal to your, greatest lower bound so g l b of a and b. Ok let us show that least upper bound of a, b, least upper bound of (a, b) is = l c m of (a, b), Ok. Least upper bound of (a, b) = l c m of (a, b).

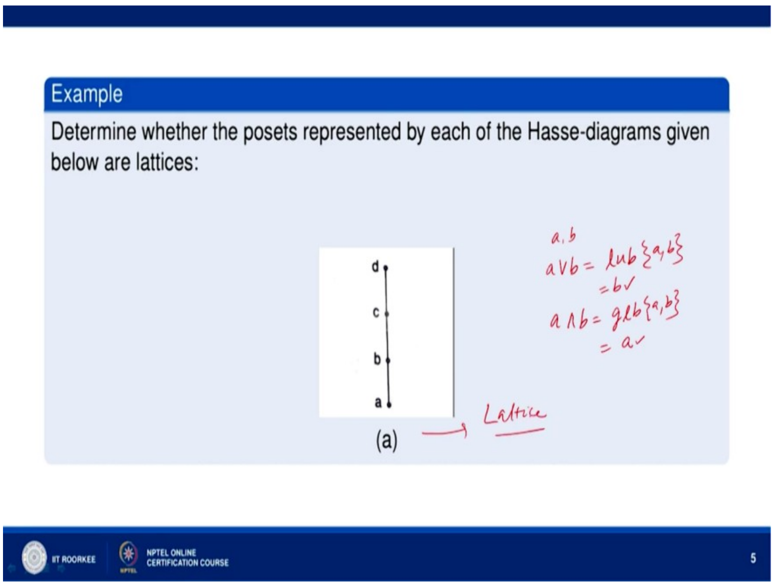
And greatest lower bound of (a, b) = g c d of (a, b). Ok. l c m means least common multiple of a and b, and g c d (a b) means greatest common divisor of a and b.Now how it follows? You see, because of this divisibility relation, Ok. Least upper bound of (a, b) means an upper bound which, first we should find an upper bound, Ok that is an element which is divided by a as well as b, Ok.

So let us say, let c be an element which is divided by a and b. Ok so c must be a common multiple of a and b, Ok. So this c, let us consider an upper bound, let us consider an upper bound of a and b.Upper bound of a and b means an element which is divided by both a and b. Now that element then must be a common multiple of a and b, Ok. So least upper bound means least such element, Ok that is least common multiple of a and b. So l u b of (a, b) becomes least common multiple of a and b.

Similarly lower bound of (a, b) means an element, a lower bound of a, b means an element which divides a as well as b, Ok. So if we find an element c which divides a and c also divides b Ok, then it will be a lower bound of a and b, Ok. If c divides a, c divides b then c is a lower bound of a and b, Ok. And we find greatest lower bound. This means that... So c is a common divisor of, c is a common divisor of a and b. We will find the greatest lower bound means greatest common divisor of a and b. So greatest lower bound means g c d of a and b.

For example let us say, we take elements 4 and 6, Ok. Let us take elements 4 and 6. Then the $4 \vee 6$, $4 \vee 6$, means l c m of 4 and 6. So this is equal to 12. And if you take $4 \wedge 6$, Ok, $4 \wedge 6$, then this is g c d of 4 and 6 and which is clearly equal to 2, Ok. So this is how we find the g c d and, I mean the least, g c d of 4 and 6 and l c m of 4 and 6 and we see that both g c d of 4 and 6 and l c m of 4 and 6, they are again positive integers. So when you take any two positive integers a, b belonging to, a and b belonging to $Z^{+¿¿}$ then their meet and their join, they are again positive integers and therefore they belong to $Z^{+¿¿}$. So $Z^{+¿¿}$ $\vee$is a lattice.

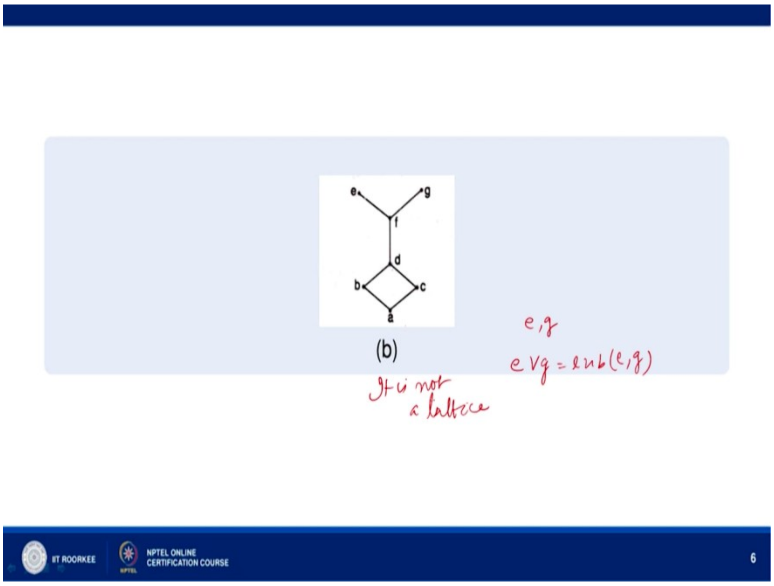
(Refer Slide Time: 10:33)



Now determine whether the posets represented by each of the Hasse diagrams shown below are lattices. Let us see this is a chain, Ok. So a precedes b, b precedes c, c precedes d, Ok. And therefore if we take a and b, if we take the pair a and b here then $a \vee b$, Ok. $a \vee b$ means, $a \vee b$ means least upper bound, least upper bound of a and b, Ok. $a \vee b$ is least upper bound of a and b.

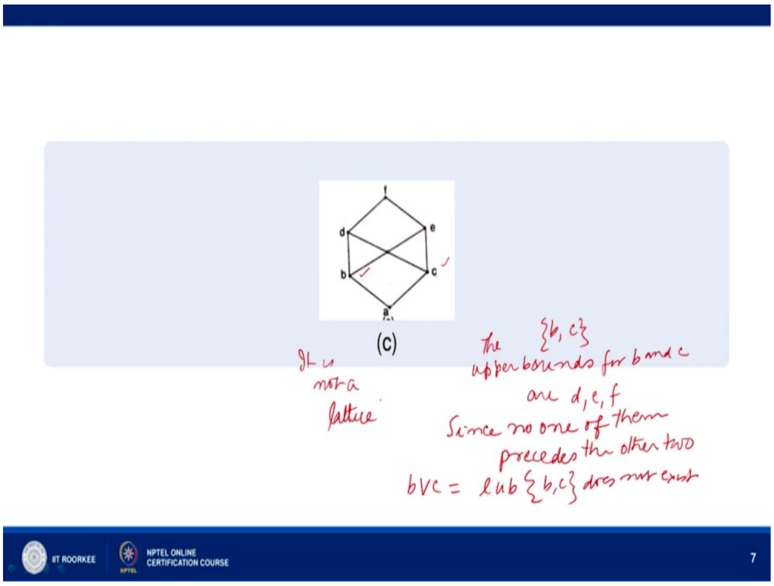
That is an upper bound, Ok, that is an upper bound and which is the least. So for {a b}, b is the least upper bound here, Ok. And then if you see a∧b then it is g l b of a, b Ok, so g l b is clearly a here. So if you take a, b then a∨b equal to b, a∧b equal to a which are again elements belonging to {a, b, c, d} set Ok So similarly if you take {a c} or you see {b, c} or you take {c, d} you again can find that the meet of the two or the join of the two are again a, b, c or d. So they belong to this given set and therefore it is a lattice.

(Refer Slide Time: 12:10)



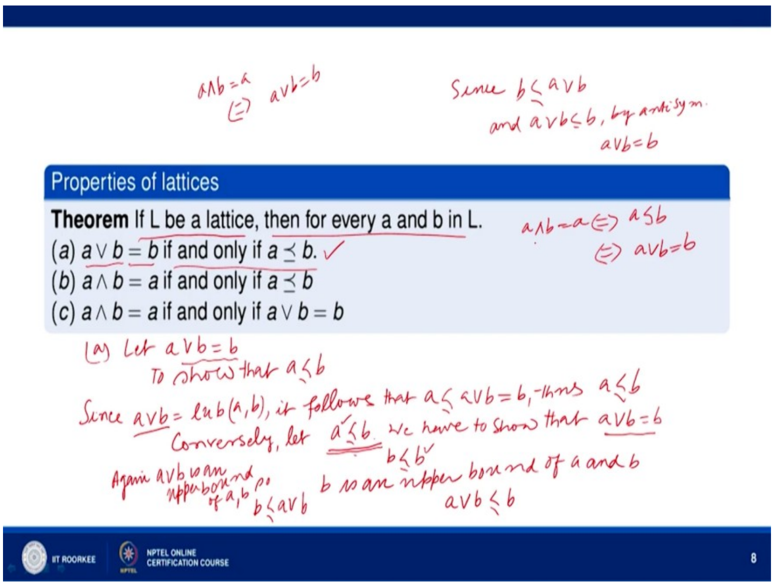
Now let us go to this figure. Let us look at this Hasse diagram. Here what do we notice? That e and g, Ok, e and g do not have, if you take e and g, then e∨ g, Ok is least upper bound of e and g, Ok. So e and g do not have least upper bound. They do not have any upper bound. Ok so they do not have any least upper bound and therefore this is not a lattice.

(Refer Slide Time: 12:50)



Let us go to this figure, Ok. Here we see that {b c,} if you consider {b c}; Ok then b c has three upper bounds d, e and f. The upper bounds for b and c are {d, e, f}. Now there is no, no one of them precedes the other two. Since no one of them precedes the other two, Ok we do not have least upper bound of {b, c}. So least upper bound of {b, c} which is join of b and c, this does not exist, Ok and so this is again not a lattice.

(Refer Slide Time: 14:13)



Ok now let us go to properties of lattices, Ok. So let us say that L be a lattice. L be a lattice then for every a and b in L, let us show a∨b = b if and only if a precedes b, Ok. So first let us consider, let join of a b be equal to b, Ok. Let us say we are given that join of a and b is equal to b. And we have to show that, to show that a precedes b, Ok. Now let us recall the definition of join of a and b. a, $a \vee b$ =least upper bound of $\{a, b\}$.

So what we can say, by the definition it follows that since $a \vee b$ is least upper bound of a and b it follows that a $a \vee b$. It is an, it is least upper bound. So it is an upper bound. Upper bound means a precedes a, $a \vee b$, but $a \vee b$ is given equal to b. So we have a precedes b, Ok. Now let us prove the converse. Ok conversely, conversely let a precedes b, Ok. We have to show that $a \vee b = b$, Ok.

Now we notice that b precedes b, Ok. b precedes b, so we are given that a precedes b and b precedes b therefore b is an upper bound of a and b. b is an upper bound of a and b. Now this is least upper bound of a and b, therefore $a \vee b$ must precede b, Ok. $a \vee b$ must precede b. Again $a \vee b$ is an upper bound of a and b, so b precedes $a \vee b$, Ok. Now what do we have?

b precedes $a \vee b$, and a and b precedes b therefore by antisymmetry, Ok so since b precedes $a \vee b$ and a and b precedes b, Ok, by antisymmetry, by antisymmetry $a \vee b = b$, Ok. So this is the proof of the part (a). Part (b) can be similarly shown, Ok and once you have the parts (a) and (b), part (c) follows by transitivity because we have to show that $a \wedge b = a$ if and only $a \vee b = b$, Ok.

Now meet of, now let us use the part (b), Ok. So a, $a \wedge b = a$ if and only if a precedes b. Ok now a, now let us use part (a). a precedes b if and only if join of a and b equal to b, Ok. So $a \wedge b = a$ if and only if $a \vee b = b$. So the third follows, (c) part follows by using the transitivity and the parts (a) and (b). So with that I would like to end my lecture. Thank you very much for your attention.