Higher Engineering Mathematics
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Partially Ordered Set III

Hello friends, welcome to my lecture on Partial Ordered Sets, this third lecture on Partial Ordered Sets that is posets. Let us first define what do we mean a maximal element in a poset. An element a in a set S which is a poset is said to be maximal if no other element succeeds a. That is if a precedes x then it must imply that a is equal to x. A maximal element in a poset need not be unique. So this is a very important point. A maximal element in a poset need not be unique. There could more than one maximal element. And there may not be any maximal element.

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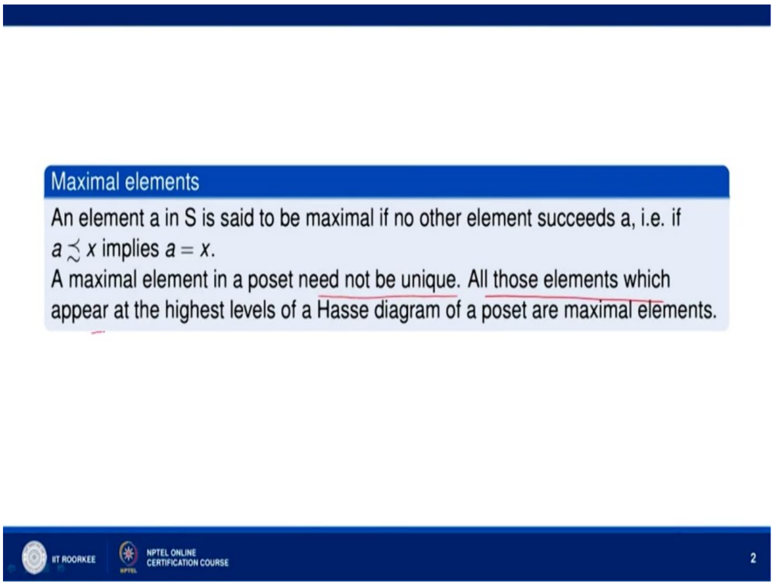

All those elements which appear at the highest levels of Hasse diagram of a poset are maximal elements. So from the Hasse diagram we can easily see what are the maximal elements.

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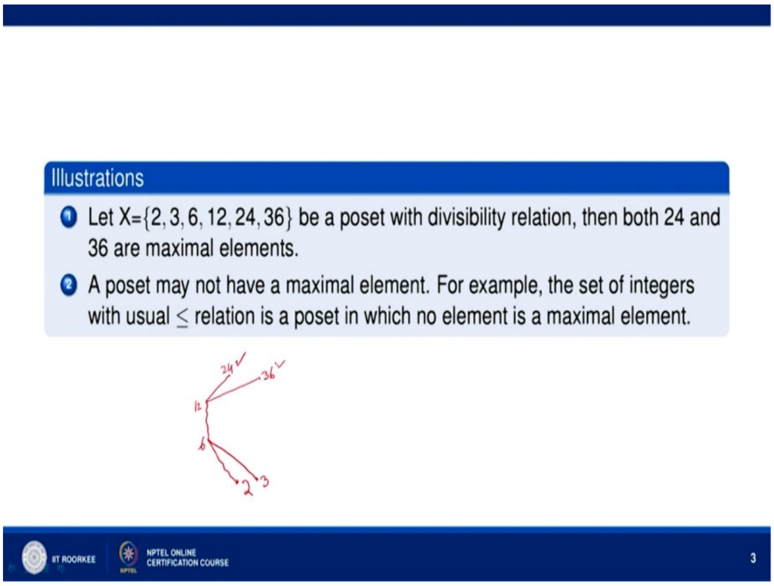

Now let us see for example we have this set. X ={ 2, 3, 6, 12, 24, 36} and let it be a poset with divisibility relation. Then we have to show that 24 and 36 are maximal element. So we will draw its Hasse diagram. And then from the Hasse diagram we shall be able to see the maximal elements. So let us draw the Hasse diagram. So this is 2. Then we have, now since it is a divisibility relation, 2 does not divide 3 or 3 does not divide 2, Ok.

2 divides 6, Ok so we have 2 divides 6, Ok. Then 6 divides 12, Ok. 6 divides 12. So we have 12 here. 12 divides 24, Ok so we have 24 here. And then we have 3, then we have 3 Ok, 3 divides 6 so we have 3 divides 6, Ok. 6 divides 12 and then 12 divides 24, 12 divides 36 Ok. 12 divides 36, Ok.So we can see this Hasse diagram, 2 divides 6, Ok. 6 divides 12. 12 divides 24, Ok. 3, 3 divides 6, Ok. 6 divides 12, 12 divides 36, Ok. Now we can see here 24 and 36, they are the maximal elements. Why they are the maximal elements? Let us go to the definition again of the maximal elements.

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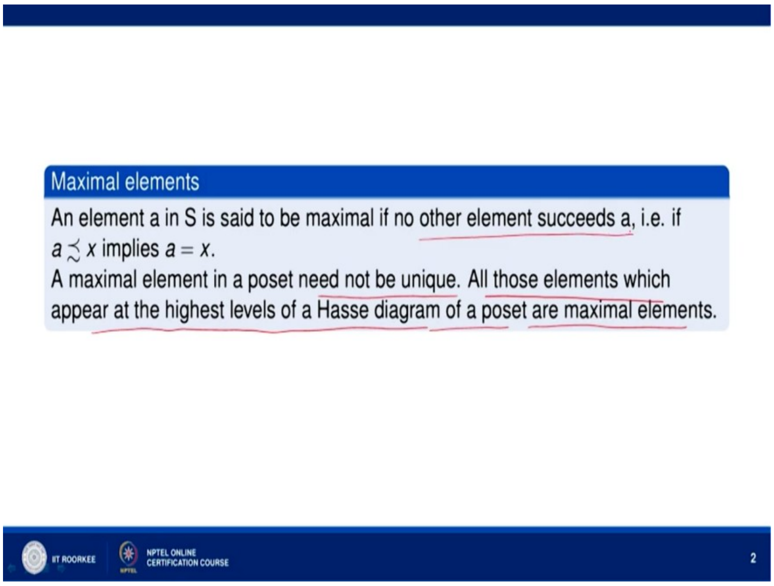

An element is called maximal if no other element succeeds a, that is a ≤ x implies a= x.

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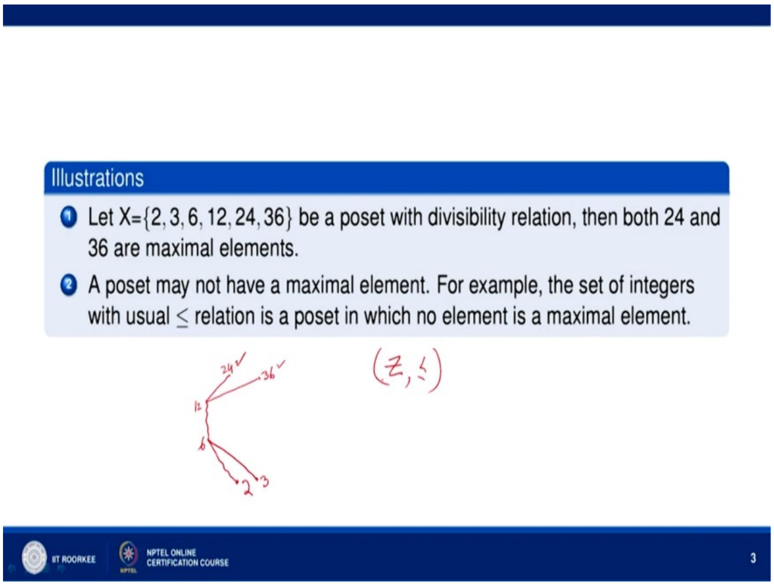

So you can see that there is no element that precedes 24 and 36 here. So 24 and 36 are maximal elements of the poset X. A poset may not have maximal element. For example let us consider the set of integers, Z. Ok let us consider the set of integers with the usual ≤ relation, Ok. Then no element is a maximal element, Ok. You take any element. We can always find element which is greater than that. So there is no element which is a maximal element.

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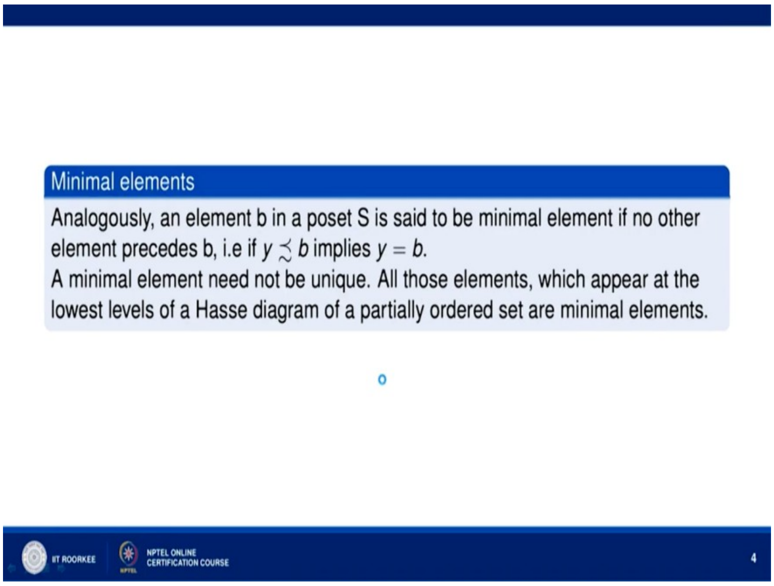

Now let us go to minimal elements. Analogously an element b in a poset S is called a minimal element if no other element precedes b. That is, if y precedes b then y must be equal to b. A minimal element again need not be unique. There could be more than one minimal element and there may not be any minimal element. All those elements which appear at the lowest levels, Ok at the lowest levels of the Hasse diagram of a particular partially ordered set are minimal elements. So it is easier to locate the minimal elements of a poset if we draw the Hasse diagram. The elements that occur at the lowest levels will be the minimal elements.

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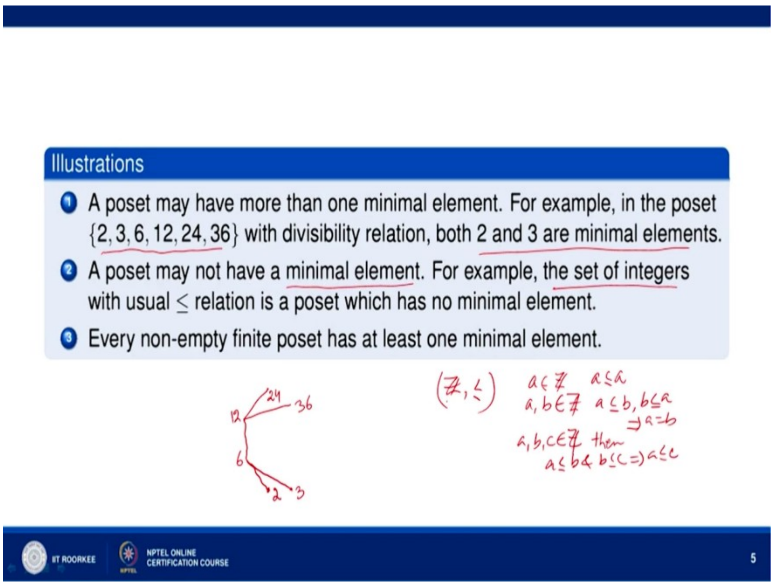

Now a poset may have more than one minimal element. Let us say for example, let us consider this poset. {2, 3, 6, 12, 24, 36} Ok with divisibility relation. Then 2 again we can see, 2 divides 6, Ok, 6 divides 12, 12 divides 24, Ok and then 3, 3 divides 6, 3 divides 6, 6 divides 12 and 12 divides 36, Ok So here you can see 2 and 3 occur at the lowest levels of this Hasse diagram. Therefore 2 and 3 are minimal elements here. A poset may not have a minimal element. You see here, a poset may not have a minimal element.

For example if you consider the set of integers again, set of integers Z, Ok with the usual relation $\leq$. Then we know that it is a poset. Because you take any two integers, Ok, any integer then a belongs to Z, this $\leq$ is a reflexive property because $a \leq a$. And if you take (a, b) belong to Z then $a \leq b$, $b \leq a$ and $b \leq a$ implies $a = b$, Ok.

If you take a, b, c belonging to Z, Ok then $a \leq b$ and $b \leq c$ implies that a is $\leq$ c. So the set of integers with this relation $\leq$ is a poset, Ok. And clearly it has no minimal element. Because you take any element we can always have an element which is, which precedes that element, Ok. So this has no minimal element. So the set of integers with the usual relation does not have a minimal or a maximal element.

Now every non-empty finite poset has at least one minimal element and one maximal element. So let us show that, every non-empty finite set has at least one maximal element and then, similar proof we can give for minimal elements. So let us say, let S be a finite non-empty poset, Ok with the order relation, partial order relation, this precedes, Ok.

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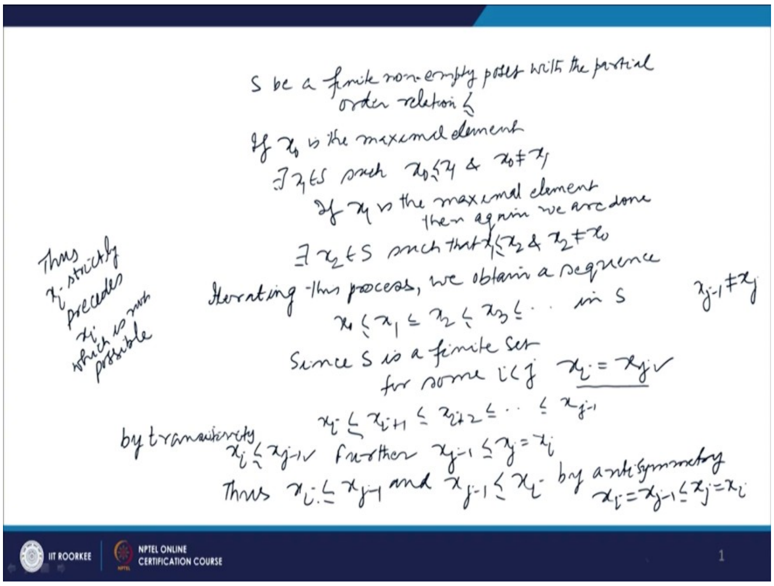

So since S is a non-empty set it must have an element, say x_0. So if x_0 is the maximal element, if x_0 is the maximal element then we are done.If x_0 is not the maximal element then there exists x_1 belonging to S such that x_0 precedes x_1 and $x_0 \neq x_1$, Ok. If x_0 is not the maximal element, then there will always exist an element x_1 belonging to S such that x_0 precedes x_1 and $x_0 \neq x_1$.

If x_1 is the maximal element then again we are done. If x_1 is not the maximal element then there will exist x_2 belonging to S such that x_2, x_1 precedes x_2, Okx_1 precedes $x_2$and $x_2\neq x_0$, Ok. So we can go on iterating this process, if x_2 is not a maximal element then we will get an element x_3 such that x_2 precedes x_3 and $x_2 \neq x_3$. So in this manner we get a sequence, Ok, iterating this process we obtain a sequence x_0 precedes x_1 , x_0 , x_1 , x_2, x_3 and so on.

We get a sequence in S, Ok which satisfies this order relation, x_0 precedes x_1, x_1 precedes x_2 , x_2 precedes x_3 and so on Ok. Since S is a finite set, Ok when you get the sequence x_0 , x_1, x_2 this infinite sequence then there are two possibilities. At some state we get the maximal element, Ok. We get the maximal element, we get a maximal element in S, say some, Ok. We stop.

If we do not have a maximal element then because that, because of the fact that S is a finite set what will happen that for i, for some i less than j, x_i will be equal to x_j . x_i will be equal to x_j. Now x_i precedes, x_{i+1} precedes x_{i+2} and so on, x_{j-1}, Ok so x_i precedes x_{j-1} Ok and what we have here, x_i equal to x_j, Ok x_{j-1} precedes x_j Ok. Further x_{j-1} precedes x_j and x_j is equal to x_i , Ok. x_j is equal to x_i , Ok. x_j is equal to x_i so by transitivity, see x_i precedes x_{i+1} . x_{i+1} precedes x_{i+2}, and so on. x_{j-1} , so by transitivity, by transitivity x_i precedes x_{j-1} .

Now x_i precedes x_{j-1} and x_i precedes x_{j-1} , further x_{j-1} precedes x_j and x_j is equal to x_i .So we have x_{j-1} precedes x_i . Thus, thus x_i precedes x_{j-1} and x_{j-1} precedes x_i , Ok. So by antisymmetry, x_i equal to x_{j-1}. But x_{j-1} precedes x_j, Ok. x_{j-1} precedes x_j , Ok x_{j-1} strictly precedes x_j . x_{j-1} is not equal to x_j , Ok, x_{j-1} is not equal to x_j and x_{j-1} precedes x_j and so x_{j-1} precedes x_j and x_j is equal to...so, so x i, x i is equal to x_{j-1}, x_{j-1} precedes x_j , Ok x_{j-1} strictly precedes x_j. So this means x_i strictly precedes x_i , Ok.

So thus x_i strictly precedes x_i, Ok which is impossible. Because x_i strictly precedes x_i means x_i is not equal to x_i and x_i precedes x_i . So here what we do? x_i precedes x_{j-1}, x_{j-1} precedes x_i Ok, therefore by antisymmetry x_i is equal to x_{j-1} but x_{j-1} strictly precedes x_j. And x_j is equal to x_i . So x_i precedes, strictly precedes x_i . So that is impossible So the finite set S must have a maximal element. And similarly we can show for minimal element, the existence of a minimal, minimal element.

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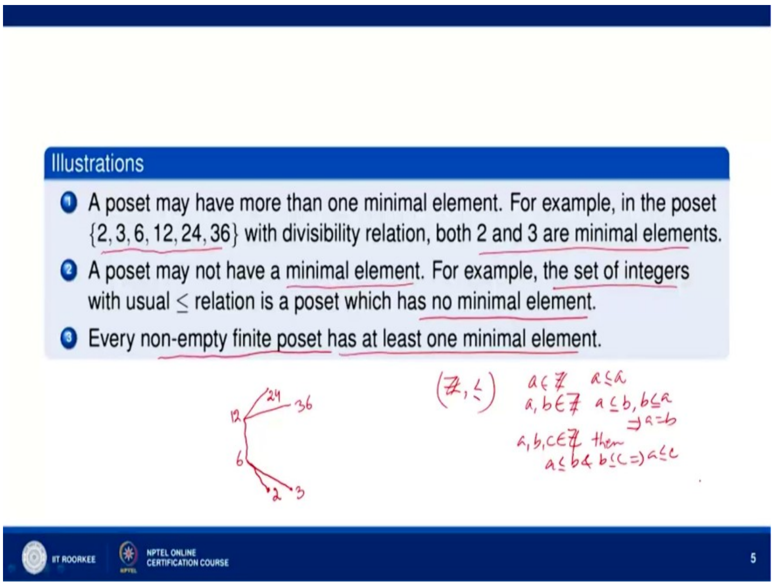

Now let us go to the, this example.

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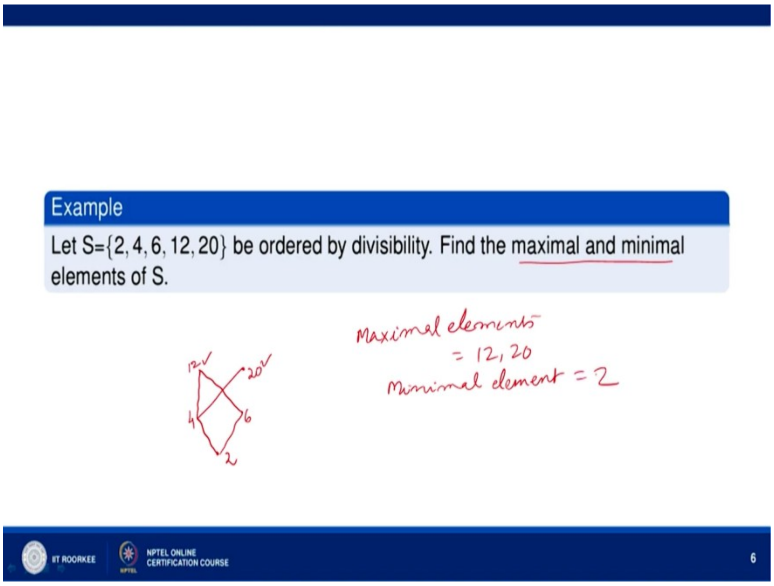

Let S be equal to {2, 4, 6, 12, 20} and S be a poset by the order relation divisibility then let us find the maximal and minimal elements of S. So let us draw the Hasse diagram. 2, 4, Ok then we have 6, 2, 4 then 4 divides 12 so we have 12 here, Ok and then we, 12 does not divide 20, Ok. Now what have, 2 divides 6, Ok. So we have 6 here, 2 divides 6, 6 divides 12, Ok, 6 divides 12 and 4 divides 20, Ok, 4 divides 20. So now in this Hasse diagram what do we notice? There are two maximal elements. Maximal elements are 12 and 20. They are at the highest levels of the Hasse diagram. Maximal elements are 12, 20. And minimal element, minimal element is the one which occurs at the lowest level of the Hasse diagram. So minimal element is 2, Ok.

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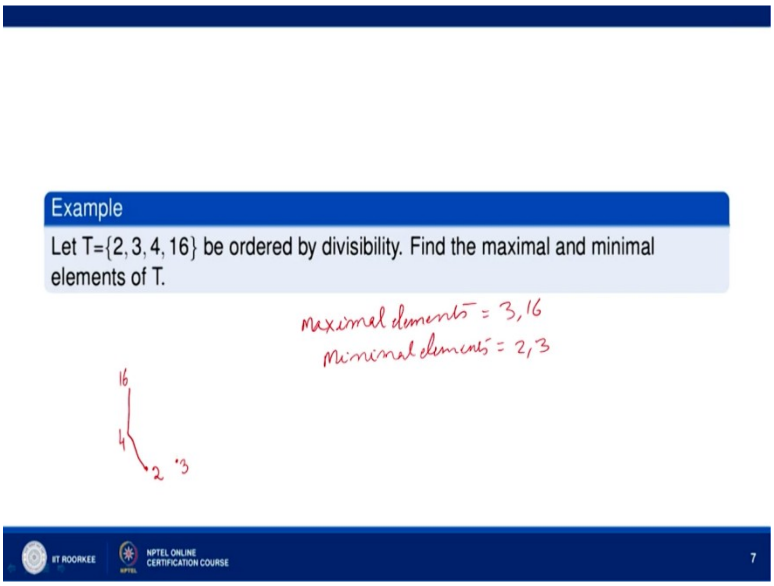

Now let us go to this example, let T be equal to {2, 3, 4, 16}. Let it be ordered with the divisibility. Then again let us find the maximal, minimal elements of T. Then let us draw the Hasse diagram for this. So 2 divides 4, Ok, 4 divides 16, Ok and we have 3. Ok, 3 and 2 are not comparable because 2 does not precede 3, 3 does not precede 2 Ok and 3 does not precede 4, 3 and 4 are also non-comparable. 3 and 16 are also non-comparable. So here the maximal elements are 3 and 16, Ok. The minimal elements are 2 and 3.

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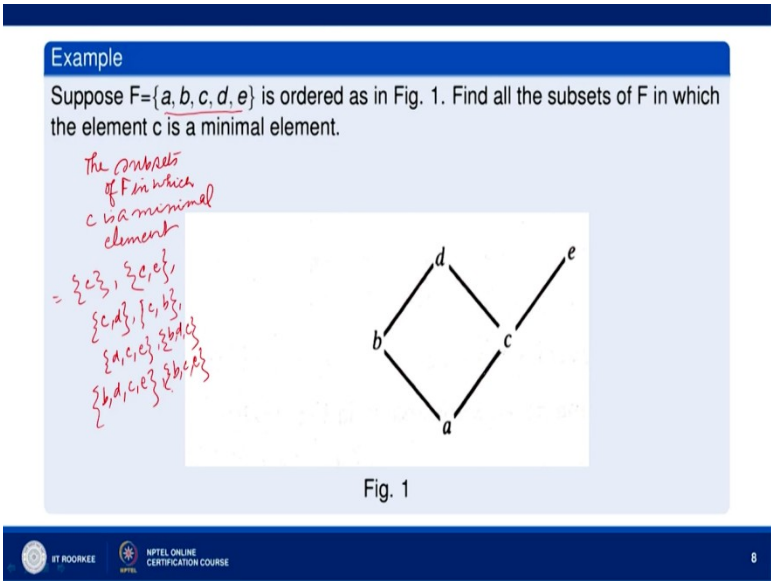

Now suppose F ={a, b, c, d, e}. Ok F={ a, b, c, d, e} be ordered as in this figure, Ok. Find all the subsets of F in which the element c is a minimal element.So subsets of F in which the element c will be the minimal value element means we should consider all those sets, Ok subsets of F which contain c but do not contain a, Ok. So the minimal, so the subsets of F are, the subsets of F in which c is minimal element, they are singleton set c, Ok, singleton set c, then we can write c, Ok we should see that the set contains c but it does not contain a, because a occurs at the lowest level of the Hasse diagram.

So {c}, {c e}, we can write {c d}, {c b}, then we have {d c e}, then {b d c} and then {b d c e], Ok. So all those subsets contain c, but do not contain a. Now singleton set {c} we have taken, {c e} we have taken, {c d}, {c d} we have taken, {b c} we have taken and then we have {d c e}, {d c e} and then we have {b d c}, and then we have {b d c e}, we also have {b c e}. Ok so there are 8 sets, subsets of F in which c is the minimal element, Ok.

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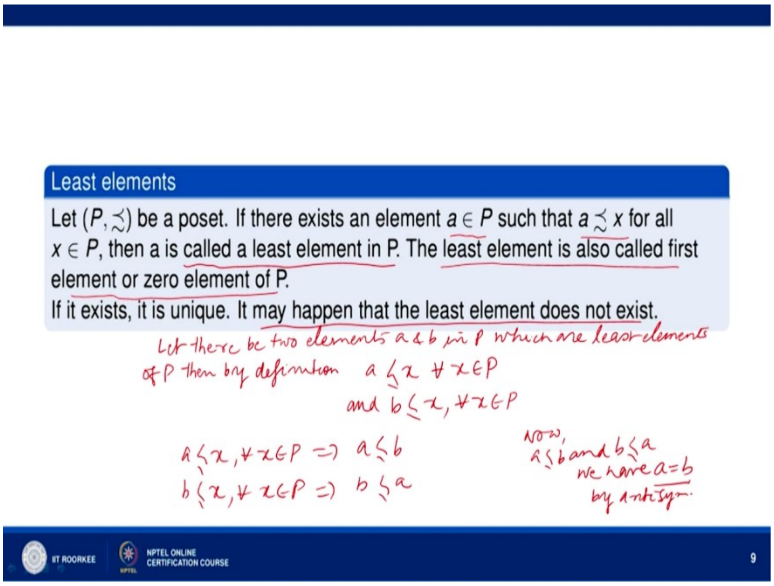

Now lets us go the case, least elements. Let P be a poset with this order relation precedes. If there exists an element a in P such that a precedes x for all x belonging to P then a is called a least element in P. The least element is also called first element or zero element of P. Now if the least element exists then we can easily prove that it is the unique element. It may happen that the least element does not exist, Ok.

So if the least element exists it will always be unique. So let us prove this fact. So let us say, let there be two elements, a and b in P which are least elements of P. We shall show that they have to be equal. Then by definition a strictly, a precedes x, Ok for all x belonging to P. And b precedes, say x for all x belonging to P, Ok. Now since a precedes x for all x belonging to P and b is an element of P. So a precedes x for all x belonging to P implies that a precedes b, Ok. Further b precedes x for all x belonging to P implies that b precedes a, because a is also an element of P.Now a precedes b and b precedes a, Ok. Let us use antisymmetry. So then if a precedes b and b precedes a we have a equal to b, Ok, by antisymmetry. So if at all the least element exists it will be unique.

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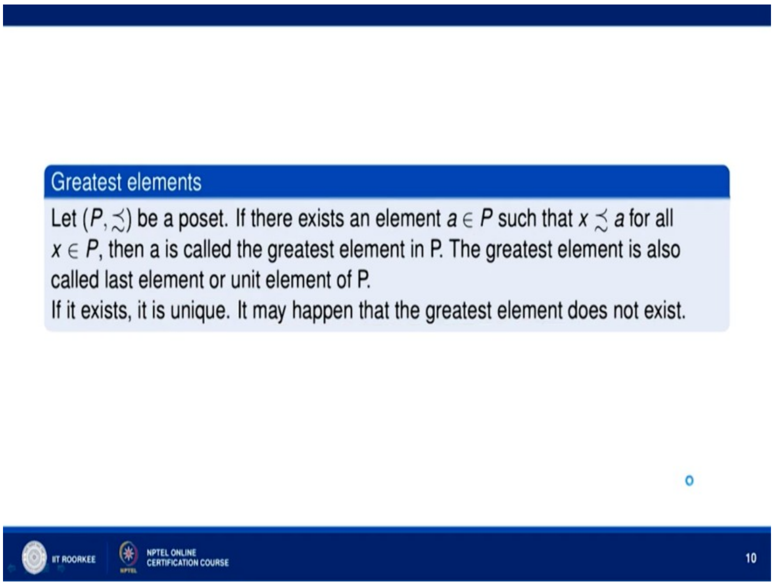

Now greatest element, let P be a poset. If there exists an element a belonging to P such that x precedes a for all x belonging to P then a is called the greatest element in P. The greatest element is also called the last element or unit element of P. If it exists, it is unique. Like, it can be easily shown that the greatest element has to be unique if it exists. We can give this argument similar to the case of the argument given in the case of least element. So it may happen that the greatest element does not exist.

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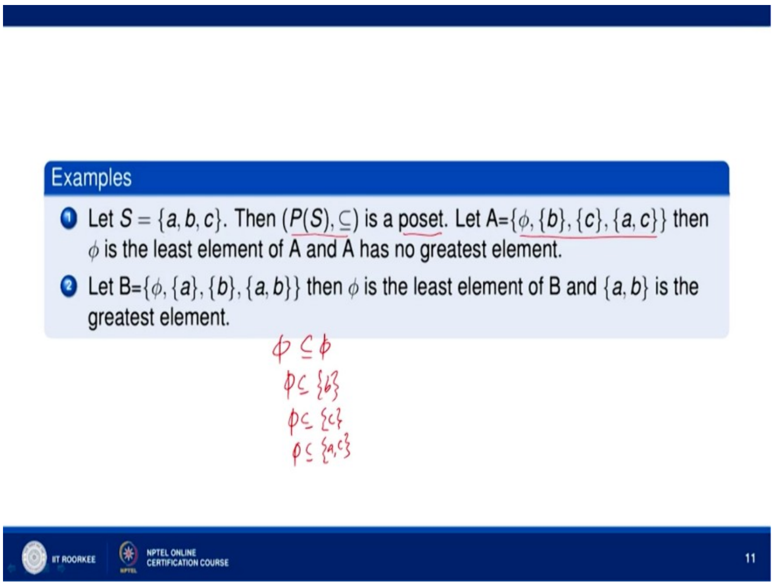

Now let us look at some examples. Suppose S ={a, b, c}, Ok. We consider the power set of S with the ⊂ relation, Ok. Power set of S with the ⊂ relation is a partially ordered set. That is, it is a poset. We know this. Ok now consider this subset of P (S)={ Φ, {b},{c},{ a c}}. The set A consisting of Φ, singleton set b, singleton set b, singleton set c and then the set a c, Ok is clearly a subset of P S.

Then, then Φ, Ok Φ, Φ is contained in every set. Φ is contained in every set belonging to the set A. Φ is contained in Φ, Φ is contained in b, Φ is contained in c, Φ is contained in a c, Ok.So Φ is contained in every set in the set A. So Φ is the least element by the definition of the least element. And it has no greatest element. Why? Because no, there is no element in A which includes every other element of A, Ok. That means all the elements, all the elements of A are subset of that element. No such element is there.

If you take this {a c}, then this{ a c} element contains c. It contains Φ but it does not contain b, Ok. So this has no greatest element. Now if you take the set B to be this, { Φ,{a}, {b}, {a b}}, Ok. Then again Φ is the least element of b, Ok and {a, b}, you can see {a, b} is the greatest element because {a, b}, I mean every subset here, every element of B is a subset of {a, b}, Ok. Φ is a subset of {a, b}, singleton set {a} is a subset of {a, b}, singleton set b is a subset of {a, b}, singleton set, this set {a b} is itself a subset of {a b}. So this {a, b} is the greatest element.

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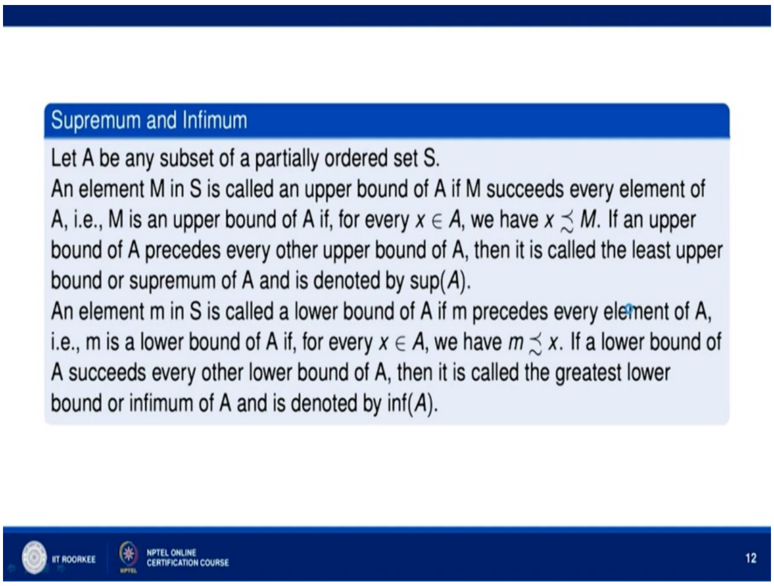

Now let us define supremum and infimum. Let A be any subset of a poset S. An element M in S is called an upper bound of A if M succeeds every element of A, Ok. That is M is an upper bound of A if for every x belonging to A we have x precedes M. If an upper bound of

A precedes every other upper bound of A then it is called the least upper bound, least upper bound or supremum of A and we denote by sup A.

An element m in S is called a lower bound of A if m precedes every element of A, that is m is a lower bound of A if, for every x belonging to A we have m precedes x. If a lower bound of A succeeds every other lower bound of A, Ok then we call it the greatest lower bound or infimum of A and we denote it by inf A.

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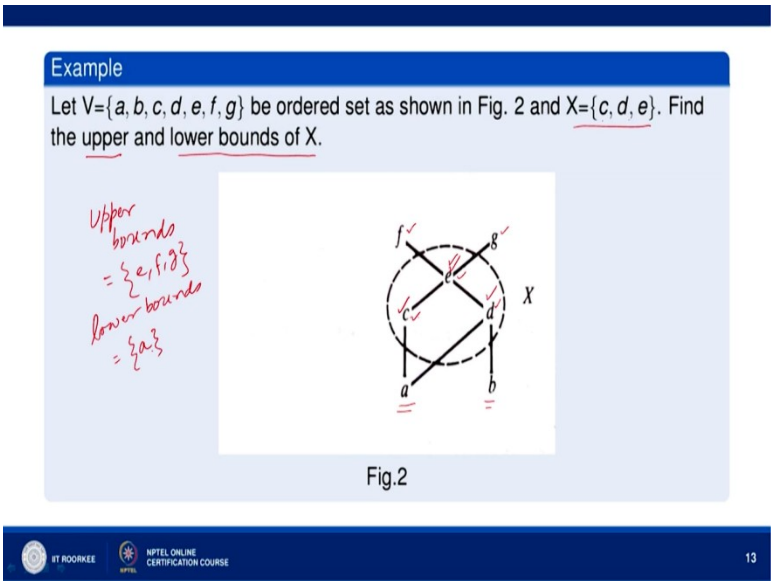

Now look at this figure, Ok this Hasse diagram. So let V={ a, b, c, d, e, f, g}. Ok. Let it be ordered as shown in this figure, Ok. X = {c d e}. We have to find upper and lower bounds of X, Ok So upper bound, let us again go to the definition of upper bound. Upper bound, an element M in S, Ok an element M in S will be called an upper bound of A if M succeeds every element of A. So let us see which element of V succeeds every other element of X, Ok.

So we have here, c you can see, {c d e}, Ok this is {c, d, e} Ok. So now you can see f, e and g. Ok. f, e and g are the elements in V which succeed every other element of X, Ok. e succeeds c, e succeeds e, e succeeds c, e succeeds d, Ok and f, f succeeds e. f succeeds c, f succeeds d. Similarly g succeeds e and g succeeds c and g succeeds d. Ok. So upper bounds here are e, f, g. Now let us look at lower bounds. So lower bounds are those elements of V, Ok which, which precede every other element of X. Ok so let us see. We have c, d, e here. This is c Ok, this is d, this is e, Ok. Now which elements of V precede c, d, e? There could be two elements, a and b Ok. Now a precedes c, a precedes e, a precedes d, Ok so a is clearly there. a is a lower bound, Ok Now b is not a lower bound of this set X because b does not precede c, Ok. b and c are not comparable, Ok. So since b does not precede c, Ok we have a lower bound as only a.

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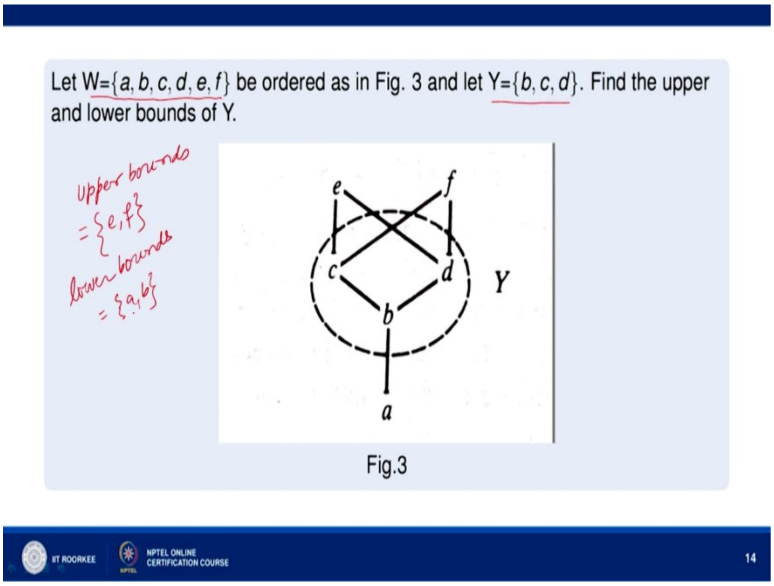

Let us take another problem. Let W = {a, b, c, d, e, f}. Let it be ordered as shown in this figure and Y be equal to {b, c, d}. We have to find the upper and lower bounds of Y. Ok so now upper bound of Y means every, I mean an element of W which succeeds every element of Y={b, c, d} Ok. So the elements e and f, the elements e and f succeed the elements c, d and b. Ok so upper bounds are e and f. And lower bounds are clearly a and b. Because a and b precede every element of the set Y. So lower bounds are a and b

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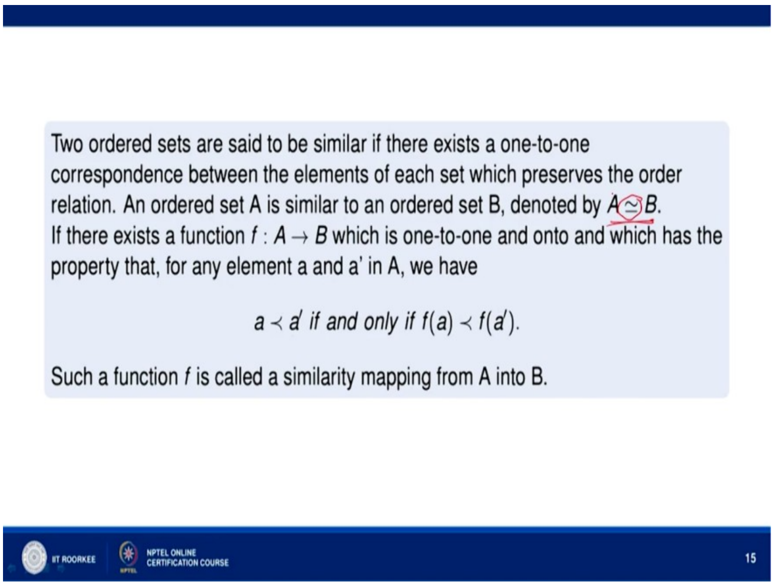

Now let us define similarity mapping. Let two ordered sets are called similar if there exists a one-to-one correspondence between the elements of each set which preserves the order relation, that is a precedes a’, a a’ belongs to A, a precedes a’ if and only if f a precedes f a’. So an ordered set A is similar to an ordered set B, we denote it by this notation, this one, Ok, so denoted by A, this is actually called isomorphism, isomorphic relation A and B are then isomorphic.

So two ordered sets A and B are called isomorphic if we can find an isomorphism, Ok this is isomorhpism, one-to-one, onto mapping from A to B which has the property that for any two elements a and a’ in A, a precedes a’ if and only if f a precedes f (a’). So the order relation is preserved here. Such a function is called similarity mapping from A to B.

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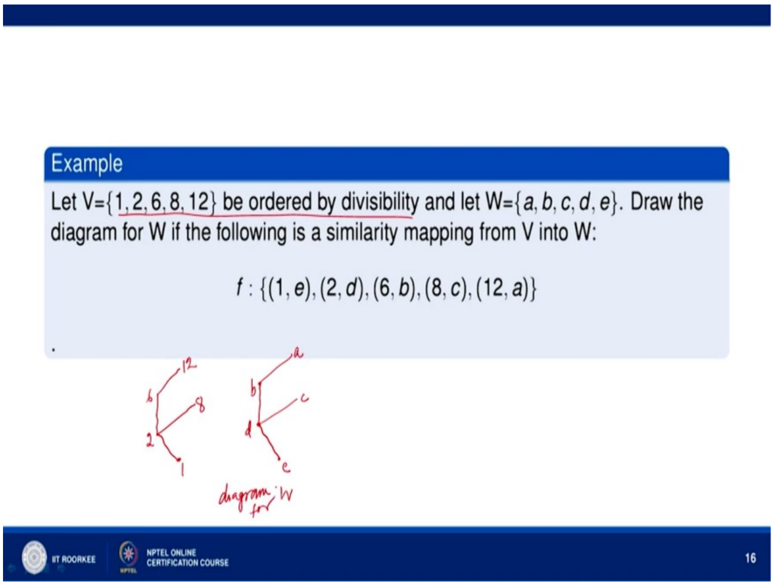

Now let V={1, 2, 6, 8, 12}. This be ordered by divisibility and W={ a, b, c, d, e}. Draw the diagram for W if the following is the similarity mapping from V into W, Ok. So let us see. This is ordered by divisibility, Ok. So 1, 1 divides 2, 2 divides 6, Ok, 2 divides 6, 2 divides 8 Ok and 6 divides 12, 6 divides 12 Ok. So 1 divides 2, 2 divides 6, 6 divides 12 and 2 divides 8 Ok. Now what we have here 1, under the mapping f, 1 goes to e, so we have e here, Ok and then 2 goes to d, so we have d Ok and then 6 goes to b, so we have b here. And then 8 goes to c, so we have c here, and then 12 goes to a, so we have a here, Ok. So draw the diagram for W, Ok. So this is the diagram for W, Hasse diagram for W, Ok using the similarity mapping.

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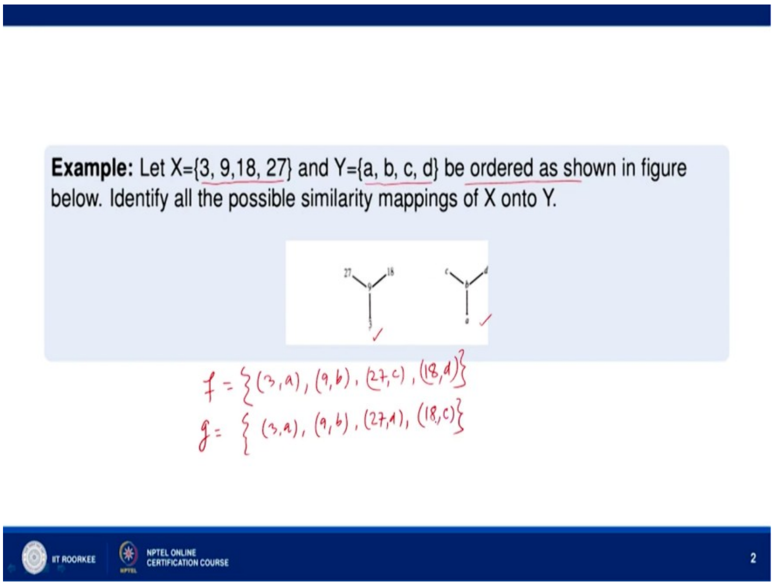

Now, let X be {3, 9, 18, 27}; X ={3, 9, 18, 27}. Y = {a, b, c, d}. Let they be ordered as shown in this figure, Ok, as shown in this figure. Then identify all the possible similarity mappings of X onto Y. So one similarity mapping could be f, (3 a), (9 b), then (27 c) and (18 d), Ok. Other similarity mapping could be written as say, g, (3 a), (9 b) and then we can say (27 d), and (18 c). So there are two possible similarity mappings in this case. That is the end of my lecture. Thank you very much for your attention.