Higher Engineering Mathematics

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Partially Ordered Set II

Hello friends, welcome to my second lecture on Partially Ordered Sets. So let us suppose we have two sets, S and T Ok. Let us take two elements, (s, s') in S,(t, t') in T.

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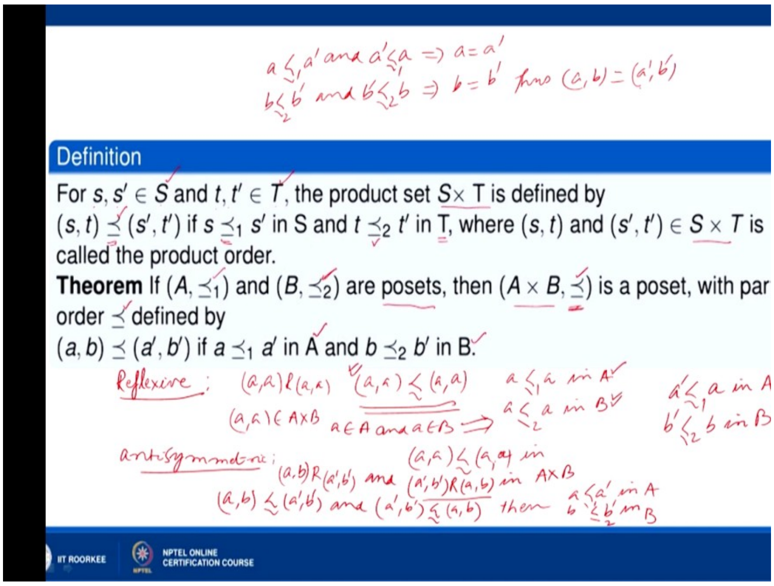

Then we define the product set S ✕ T as follows, Ok.(s t) precedes (s' t') if s precedes s' in the set S. So this is the precedes notation and suffix_1 here, Ok. That means for s, Ok. And then t precedes t' in T so that is why we have written 2 here. So this is, this notation means it is the partial order relation in T and this is partial order relation in S. And this is partial order relation in S ✕ T. We shall show that this is partial order relation in S ✕ T.

So (s, t) and (s,' t') belong to S ✕ T. So if you take two sets S and T then their product set S ✕ T is defined as (s, t) precedes (s,' t') if s precedes s' in S, t precedes t' in T where (s, t) and (s,' t') belong to S ✕ T. This is called as product order.Now let us show that if A is a partially ordered set with this notation, B is a partially ordered set with this notation they are posets, Ok. Then A ✕ B, Ok is a poset with partially order relation defined like this. (a, b) precedes (a', b') if a precedes a' in A, b precedes b' in B, Ok.

So let us prove that this relation is a partial order in the set A ⨯ B. So let us first show 0:02:13.2 reflexive property. Ok so we have to show that (a, a) is R related to a that is we have to show that (a, a), yes and this, for this, Ok a is less than, precedes a in A, and a precedes a in B. Ok we know that a is reflexive. We can...so...we have to prove this, Ok. We have to prove this, that a is related to a, Ok in A ⨯ B, Ok. And relation is this. That means we have to show this one.

Ok now let us notice that A is partially ordered set, B is partially ordered set. So (a, a) ∊ A ⨯ B then a will belong to A, (a, a) belong to A ⨯ B means (a,b) ∊ A and (a,b) ∊ B. And (a,b) ∊ A and (a,b) ∊ B means a precedes a in A and a precedes a in B, Ok. Now these two together then imply that a a precedes a a in S ⨯ T, Ok. Now let us say antisymmetric. So assume that (a, b) is related to (a', b'). And (a', b') is related to (a,b) in S ⨯ T, Ok. Then we have to show that (a,b) and a' b' are same, Ok.

So (a,b) belong, (a,b) is related to a' b' means what? (a,b) is related to a' b' means (a,b) precedes a' b'. And a' b' is related to (a,b) means a' b' precedes (a,b), Ok. So by definition, Ok a precedes a', b precedes b', Ok. Then a precedes a' in S, Ok. b precedes b' in T. So this is 1 here, this is 2 here because in T we are taking 2. So, Ok and a' b' precedes (a,b) means a' precedes a in S and b' precedes b in T, Ok

Now let us see. a precedes a' in S and a' precedes a in S. Together we consider a precedes a' and a' precedes a, Ok. And S is a partially ordered set. So this implies that a = a'. Next b precedes b' and b' precedes b, Ok and T is a partially ordered set with the, this partially ordered relation. So not T, it is A and B we are taking, Ok. So S is A and, so this is A and this is B sorry. This is A, this is B. Wherever S T we have written, there actually A and B. So this is in A and this is in B, Ok. Now what? So b precedes b' in B and b' precedes b in B implies that b = b', Ok. Thus (a,b) is = a' b'. So that is the proof for the antisymmetry property.

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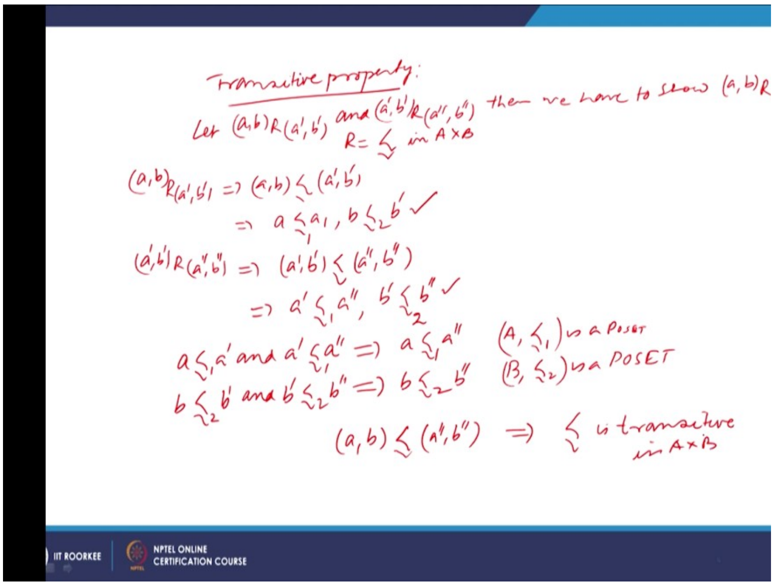

Now let us go to the transitive property. Ok so let us say (a,b) be related to a’ b’ and a’ b’ be related to (a’’, b’’), Ok. Then we have to show that (a,b) is related to (a’’, c’’) . And relation R is this precedes in $A \times B$, Ok. So (a,b) is related to (a’ b’) means (a,b) precedes (a’ b’), which is nothing but a precedes a’ in A, b precedes b’ in B, Ok.

And (a’ b’) precedes (a’’, b’’), no (a’ b’) R (a’’, b’’), means that (a’ b’) precedes (a’’, b’’). But this means that a’ precedes a’’ and b’ precedes b’’, Ok. Now let us consider this one and this one. So a precedes a’ and a’ precedes a’’ implies that a precedes a’’, Ok because a is partially ordered, this is a poset. This is a poset, Ok.

And b is partially, precedes b’ and b’ precedes b’’ implies that b precedes b’’, Ok because b, this is a poset. Ok so a precedes a’’ in A, b precedes b’’ in B. This means that (a,b) precedes, (a,b) precedes (a’’, b’’). And hence this, this relation is transitive in $A \times B$. Ok so $A \times B$ is a poset when A and B are posets.

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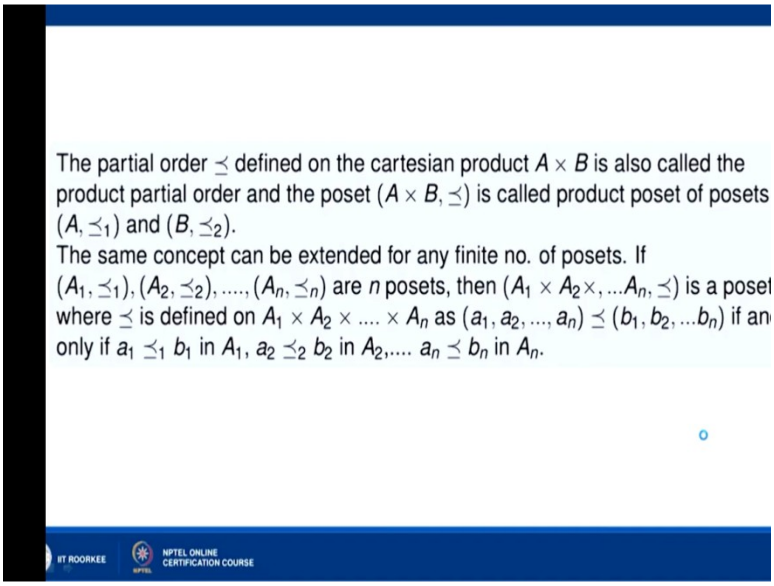

Now let us see, we go to the partial order relation precedes defined on the Cartesian product A ⨯ B is also called the product partial order. And the poset A ⨯ B with this partial order is called as product poset of posets. A precedes 1 and B precedes 2. The same concept can be extended for any finite number of posets. IfA_1 is partial ordered set with this notation, $A_2$is partial ordered with this, precedes 2 and then A n with this n posets then $A_1 \times A_2 \times A_n$with this notation for precedes is a poset where this notation is defined on $A_1 \times A_2 \times$ A n as a 1 a 2 a n precedes b 1 b 2 b n if and only if a 1 precedes b 1 in A_1, a 2 precedes b 2 in A_2, a n precedes b n in A_n . So we can easily extend that.

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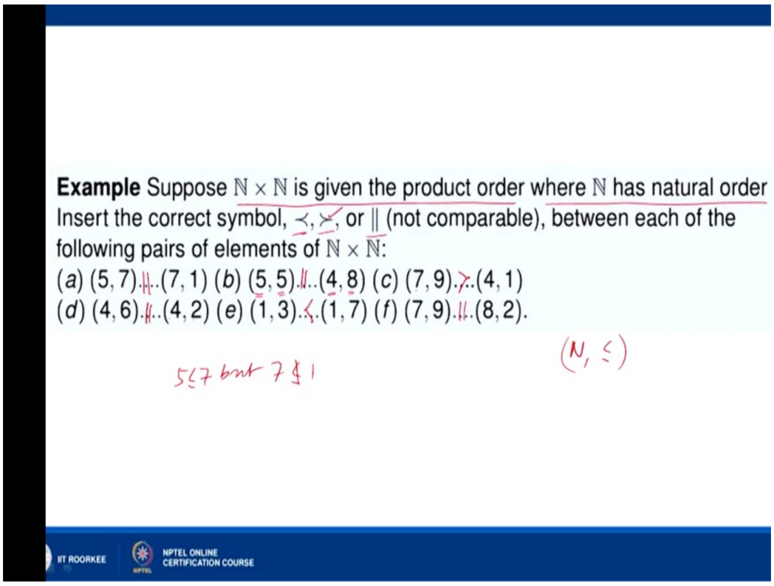

Now suppose N ⨉ N is given the product order, Ok, N ⨉ N is given the product order where N has the natural order, Ok less than or =. We have to insert the correct symbol. This symbol, this symbol or this symbol, Ok between each of the following pairs of elements of N ⨉ N, Ok. So we have to see whether we have to put this symbol or this symbol or this symbol or this symbol, whether they ... where N has the natural order, Ok. So N with less than or =, Ok is a well-ordered set.

Ok it is, you take any two elements, one is always less than or = the other, so it is always well-ordered, or totally ordered with this.Now we have to insert the correct symbol. So (5, 7), (7, 1). You can see (5, 7),(7, 1), so we see that 5 is less than or = 7, Ok but 7 is not less than or = 1, Ok. 7 is not less than or = 1. And therefore we can say that they are not comparable. Ok, they are not comparable. Now let us look at (5, 4); (4, 8). So 5 is greater than or = 4, Ok and 5 here is less than or = 8. So again they are not comparable.

Now (7, 9) if you consider, 7 is greater than 4, 9 is greater than 2. So we can, thus we can put this strictly, it strictly succeeds (4 1), Ok (7 9) strictly succeeds (4 1). So we have to put this notation for strictly succeeds. And then (4, 6), (4, 2), Ok? 4 less than or = 4, 6 less than or = 2, so that is not true. So we can say, not comparable, Ok. And then (1 3), 1 is less than or = 3 less than or = 7. So we put, because 3 is less than 7. So we can put this one, this one here Ok.

And then (7, 9). 7 is less than 8, 7 is less than 8. 9 is not less than 2. So this is not comparable. Ok so we can see a part, b part, d and f they are not comparable while c and e are comparable. In c we have to put this strictly succeeding and here we, strictly preceeding;

because although 1 is = 1, 3 is strictly less than 7. And here 7 is strictly less than, 7 is strictly greater than 4, 9 is strictly greater than 1, Ok.

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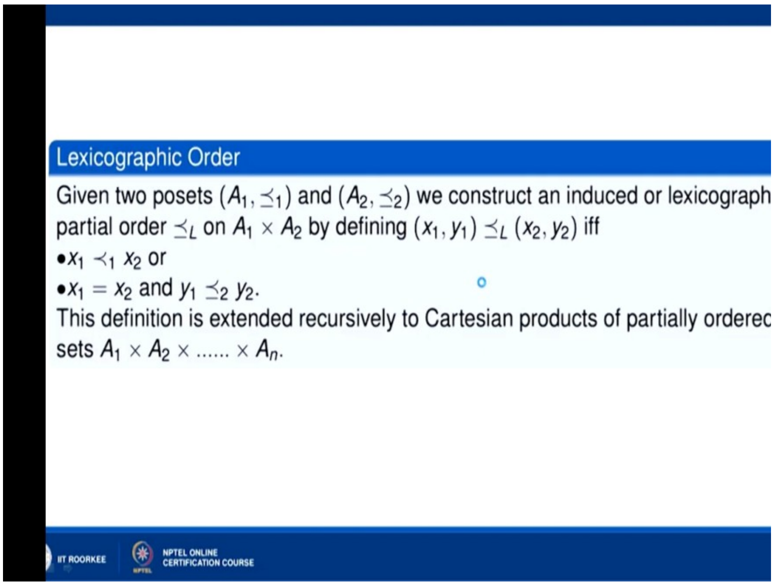

So now let us go to lexicographic order. Given two posets a 1 and a 2 be constructed, induced or lexicographic partial order preceding L, L represents lexicographic on $A_1 \times A_2$ by defining (x_1, y_1) precedes in the lexicographic order (x_2, y_2) if and only if x_1 is strictly precedes, x_1 is strictly precedes x_2, either this should hold or if $x_1$is = x_2, then y_1 precedes y_2 , that should hold, Ok. So this definition is defined recursively to Cartesian products of partial ordered sets $A_1 \times A_2 \times \ldots A_n$. Let us see how we do this. So we can take an example on this.

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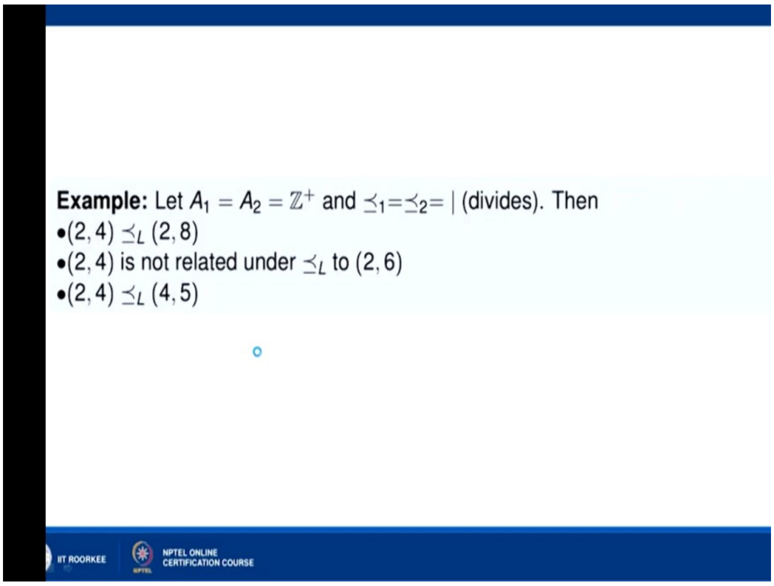

Suppose we have two sets A_1, A_2 Ok$Z^{+¿¿}$, means set of positive integers Ok and the A_1 is poset with this divisibility, A_2 is also a poset with the divisibility, then (2 4) is lexicographic to (2 8). How (2 4) is lexicographic order to (2 8), we can see. See this is (x_1 y_1).

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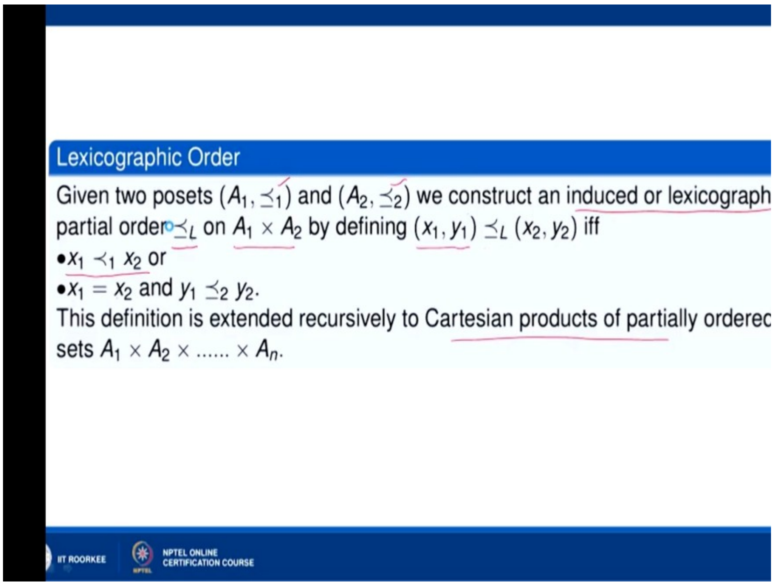

This is, if we go by this notation, so this is (x_1 y_1), Ok and this is (x_2, y_2) .

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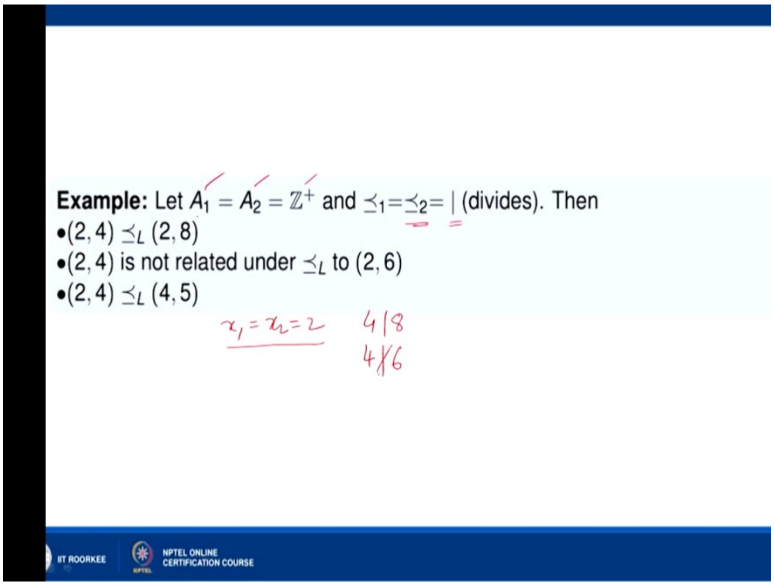

Then we have $x_1 = x_2$, Ok equal to 2. $x_1 = x_2$, = 2. And 4, 4 divides 8. Ok this 4 divides 8. So 4, so we have this condition holds true here. You see here. x_1 equals to x_2, and $y_1$is strictly, y_1 precedes y 2; precedes here is divisibility. So y_1 divides y_2. That means 4 divides 8. So we have 2 4 is lexicographic to, I mean 2 8. And then 2 4 is not related to, lexicographic to 2 6 because 2 is = 2, that is x_1= x_2, = 2 holds but 4 does not divide 6. Ok 4 does not divide 6. And here in (2 4), what we see? 2 is, 2 divides 4, Ok. We see that 2 divides 4, Ok. So that means that 2 4 is related to 4 in the lexicographic order. Now let us see, because either this should hold or if this is = this, then this should hold. So here we notice that this x 1 divides x 2 holds.

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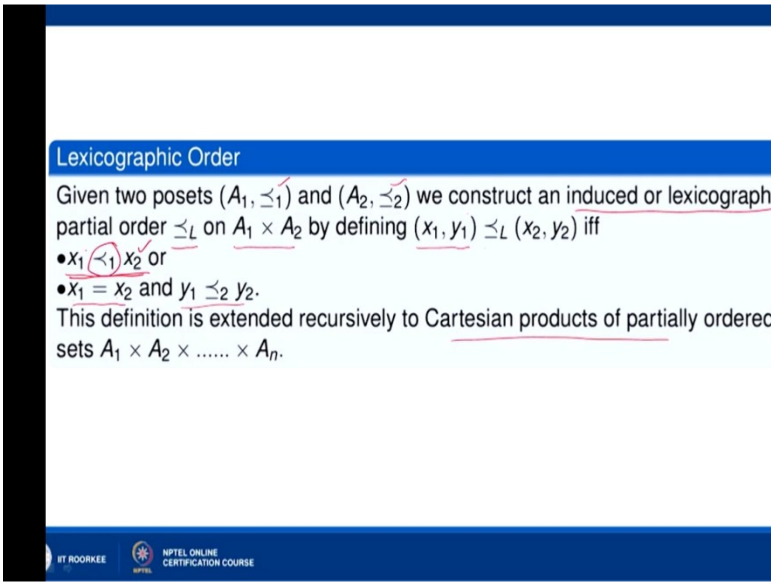

This notation we have defined here as the divisibility. You can see. So here 2 divides 4 and therefore we say that this lexicographic partial order holds between (2 4) and (4 5), Ok.

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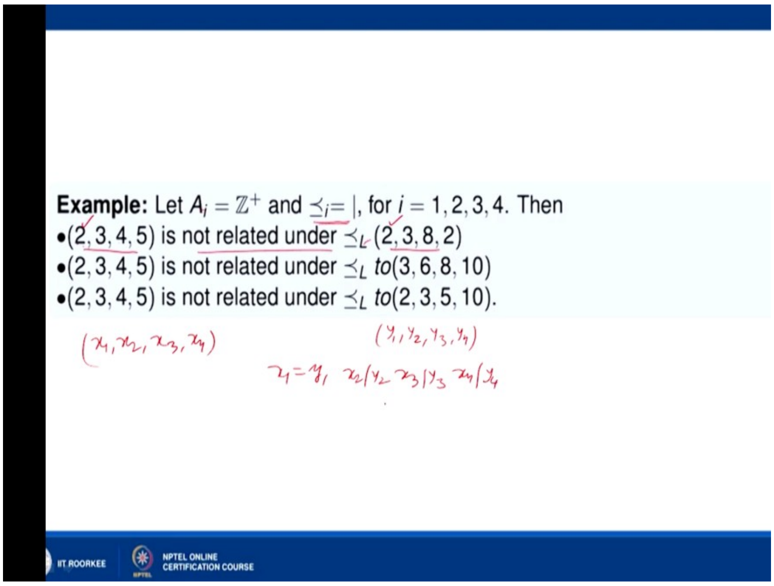

Now let us take this one, $A_i = Z^{+¿¿}$, set of positive integers Ok and all these A'^{s}_{i} are posets with this divisibility order, Ok, divisibility, they are posets with divisibility. This is valid for i = 1, 2, 3, 4 all of them. Then let us look at this, whether this is related to this under this lexicographic order. So you see 2 is = 2, Ok. 2 is = 2. Then 3 divides 3. 4 divides 8 but 5 does not divide 2, Ok.

And therefore this is not related under this order. Because we notice that when x 1 is = x 2 then y 1 should divide y 2, y 3, I mean this is suppose x 1, y 1 z 1, let me call it as x 1, x 2, x 3, x 4 Ok. And this one we call as y 1, y 2, y 3, y 4, Ok. So we notice that x_1 is = y 1, Ok. x_1 is = y 1 then x 2 should divide y 2, x 3 should divide y 3, x 4 should divide y 4 which is not true because 5 does not, 5 does not divide 2. Now here 2, 3, 4, 5 let us take and 3, 6, 8, 10. Then 2 is, 2 does not divide 3, Ok, 2 does not divide 3 and 2 is also not = 3. So this is not related under this relation. Now {2, 3, 4, 5};{ 2 3, 5, 10} Ok. So here 2 is = 2. Ok. Now 3 divides 3. 4 does not divide 5, Ok; 4 does not divide 5. So this is not related under this relation to this, 2, 3, 5, 10

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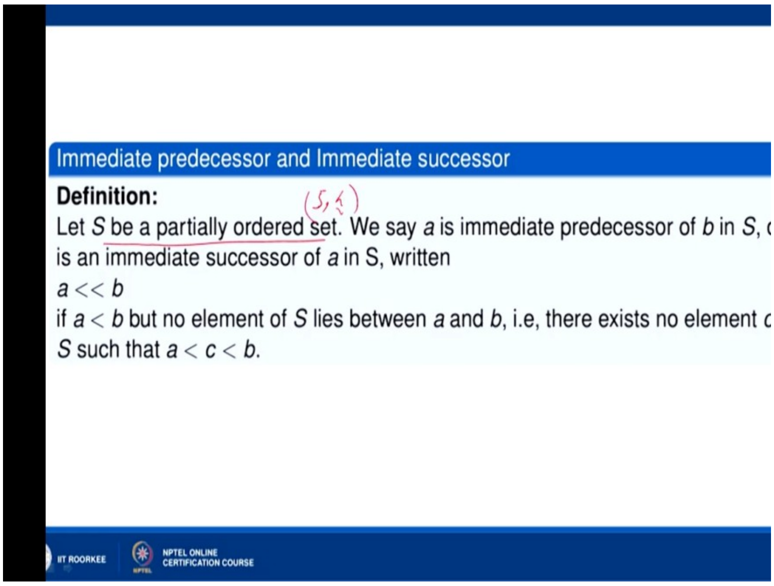

Immediate predecessor and immediate successor, let S be a partial ordered set. Ok S be partially ordered set. Say I can take the partial ordering to be this. Let S be a partially ordered set, Ok. On S this is the partial order, Ok, precedes. We say that a is immediate predecessor of b in S or b is an immediate successor of a in S written like this, Ok. a is immediate predecessor of b. If we want to write that then we put this symbol, Ok. So a is immediate predecessor of b or b is immediate successor of a, Ok if a is less than b but no element of S lies between a and b. That is there exists no element c in S such that a is less than c less than b and c is related to a and b by this partial order. Ok that we have to see.

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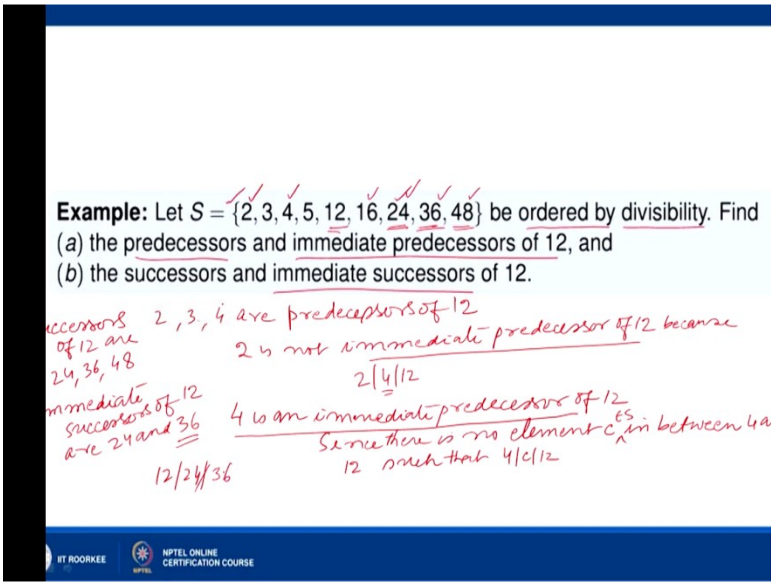

Now let us see this example, S = {2 3 4 5; 12 16 24 36 48}. Let us say it is this set is partial, poset with respect to the divisibility. Then we have to find predecessors and immediate predecessors of 12, Ok. Now so predecessors of, when we want to find predecessors of 12, we should find the elements which divide 12, Ok and which are less than 12. So 2, 4 divide 12, Ok, are predecessors of; 2 and 4 are predecessors of 12.

Now 2 is not immediate predecessor of 12 because 2 divides 4, 4 divides 12. So there is 4 in between 2 and 12, Ok. Because of the occurrence of 4, and 4 is an element of S, so because of this 4 which occurs between 2 and 12, Ok, this 2 is not immediate predecessor of 12. Now let us see whether 4 is immediate predecessor of 12 or not. 4 is of course, 4 is an immediate predecessor of 12 because there is no element in between 4 and 12, Ok which is, say you can say that, that element is c such that, that divides 12, Ok.

So since there does not occur, since there is no element c between 4 and 12 such that 4 divides c and c divides 12, Ok. c belonging to S, Ok and therefore 4 is an immediate predecessor of 12. Now let us look at the successors and immediate successors of 12. So 24, 36, 48 Ok because 12, 12 does not divide 16, Ok. So successors of 12 are 24, 36, 48. Ok so successors of 12 are 24, 36, 48.

Now we have to see, we have to find immediate successor of 12, Ok. So immediate successor of 12, immediate successors of 12 are 24 and 36 because, between 12 and 24, 16 is there Ok and 16, 12 does not divide 16, Ok so 16 does not divide 24. So there is no element between 12 and 24, Ok. Therefore, to which, I mean which is divisible by 12 and which divides 24, so we say that 12 is an, this 24 is an immediate successor of 12. Also 36 is an immediate successor of 12 because what do we notice? That 12, 24 although 24 lies between 12 and 36 but 24 does not divide 36, Ok. So 36 is also an immediate successor of, successor of, your 12.

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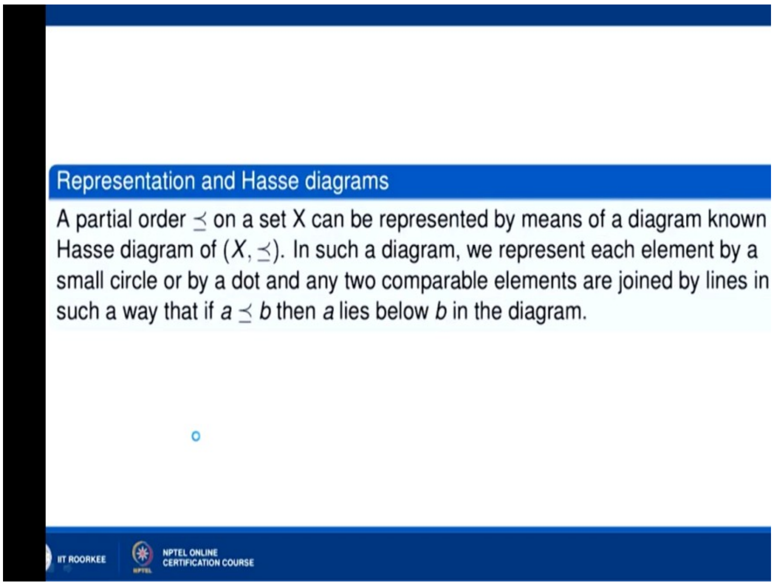

Now we go to representation in Hasse diagrams. A partial order precedes on a set X can be represented by means of a diagram known as Hasse diagram of X, poset X. Now in such a diagram we represent each element by a small circle or by a dot and any two comparable elements. Comparable elements let us see, two elements of a poset are called comparable if a precedes b or b precedes a is true, Ok. So any two comparable elements are joined by lines in such a way that if a precedes b then a lies below in the, below b in the diagram.

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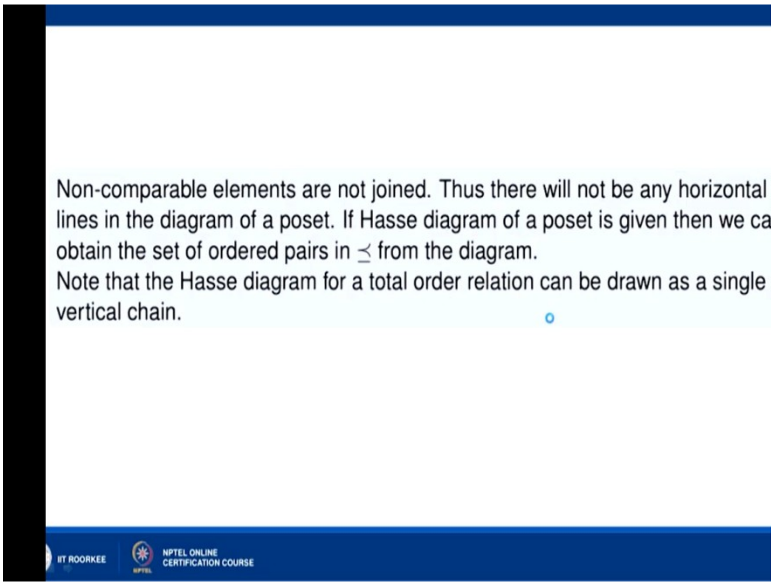

Ok now non-comparable elements, if there are two elements which are not comparable, that is a does not precede b and b does not precede a then such elements are not joined. Ok so we have to remember this. Now thus there will not be any horizontal lines in the diagram of a poset. If Hasse diagram of a poset is given then we can obtain the set of ordered pairs in this notation from the diagram. We can write the ordered, set of ordered pairs from the Hasse diagram. Now Hasse diagram for a total order relation, Ok if you have relation on the set X is a total order relation, total order relation means if you take any two elements of the set X then they are comparable. Either x is, precedes y or y precedes x so then in the Hasse diagram what we will get is a single vertical chain.

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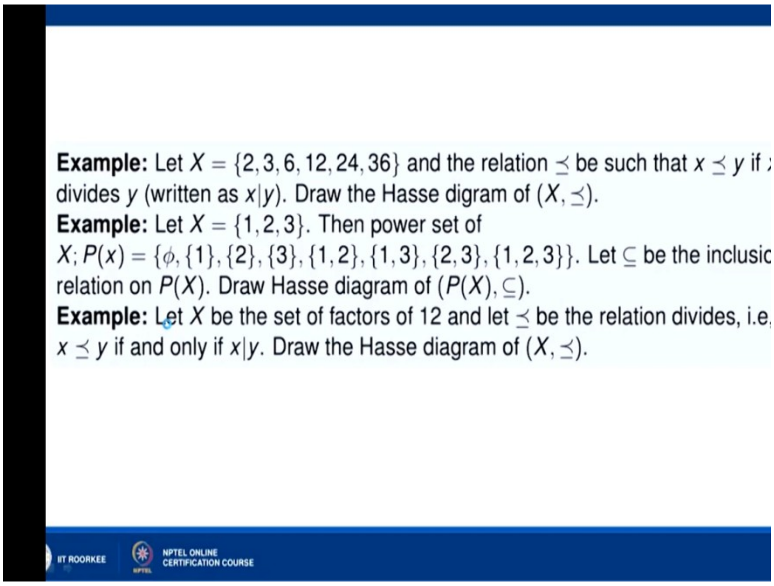

Now let us see how we consider, how we draw the Hasse diagram and some examples. Let X ={ 2, 3, 6, 12, 24, 36} and the relation in X is such that x is, x precedes y if x divides y. So in the set X then partial order is the divisibility, Ok. So x precedes y if and only if x divides y. That is the definition. Now let us draw the Hasse diagram of X. Ok so let us see, we have the Hasse diagram of X here.

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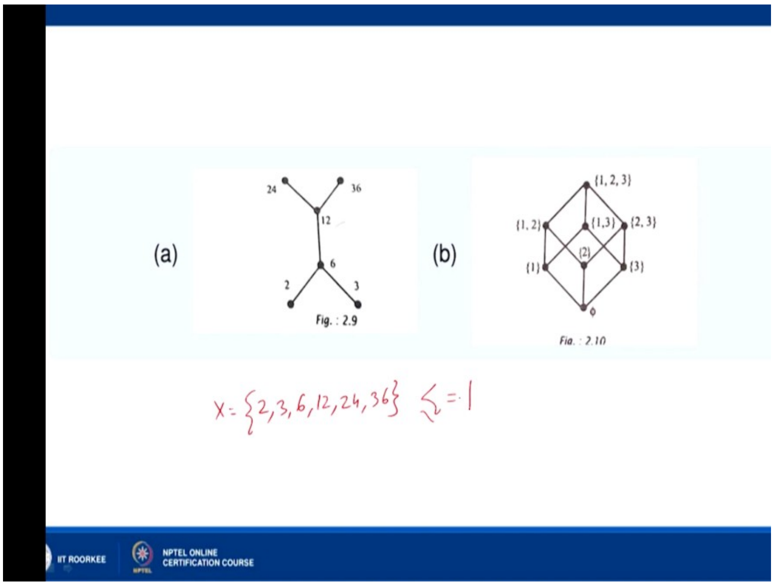

You can see. We have the elements as 2, 3, 6, 12, 24, 36.2, 3, 6, 12, 24, 36 Ok so this is my set X and we are taking the relation to be divisibility, Ok. Now we can see. X, if the element, let us say, take 2. 2, 2 does not divide 3, Ok. 2 does not divide 3 and 3 does not divide 2. So between 2 and 3, Ok, as we have seen earlier we have seen that non-comparable elements are not joined, Ok.

So the elements 2 and 3 are not joined, Ok. Now 2 divides 6, so we join 2 to 6 by this horizontal line, Ok. We join them, 2 to 6 and then 3. 3 divides 6 so we join 3 to 6. And then 6 divides 12, Ok. So we draw this horizontal line. Now one more thing you would remember, we have said that two comparable elements... Ok let us see this one, two comparable elements are joined by lines in such a way that if a, now this notation we are taking as divides, if a divides b then a lies below b in the diagram, Ok. So let us keep that into mind, Ok.

So this 3 divides 6 means 3 should lie below 6, Ok. So see 6 is up here. Then 6 divides 12, Ok. So 6 lies below 12 and then 12 divides 24 and 12 divides 36. So this is Hasse diagram of the set, poset X. Now let us take example 2. X ={ 1, 2, 3}. Then power set of X is consisting of sets ϕ, singleton set {1}, {2}, {3], {1 2}, {1 3}, {2 3}, {1 2 3}. And the relation is inclusion relation, Ok on the set P X. Let us draw the Hasse diagram for this, Ok

So we begin with phi, Ok. Now phi is contained in singleton set 1, Ok. Phi is contained in singleton set 1, phi is contained in singleton set 2, phi is contained in singleton set 3, Ok. So we join them by lines. And then 1 is contained in 1, 2, Ok. 1 is contained in 1, 2 so we join it by this line. And 1 is contained in 1 3. So we join by this line, this one Ok. And then 3 is contained in 1 3 so we join it by this line, Ok.

Now 2 is contained in {2, 3}. So we join by this line. And 2 is contained in {1 2}, so we join by this line. Now 2, 2 is a subset of {1 2} so 2 lies below {1 2}, Ok. 1 is a subset of {1 3}, so 1 lies below {1 3}. 3 is a subset of {1 3} so 3 lies below {1 3}. And then {1 2}, {1 3} and [2 3}, each one of them is a subset of [1 2 3}. So they join, we join them by these lines. And then this is what is the Hasse diagram for the example 2. Ok now let us go to the third one. Let X be the set of factors of 12 and this be the relation divides. That is x divides, x precedes y if and only if x divides y. So we take the division relation. Let us draw the Hasse diagram of this, Ok

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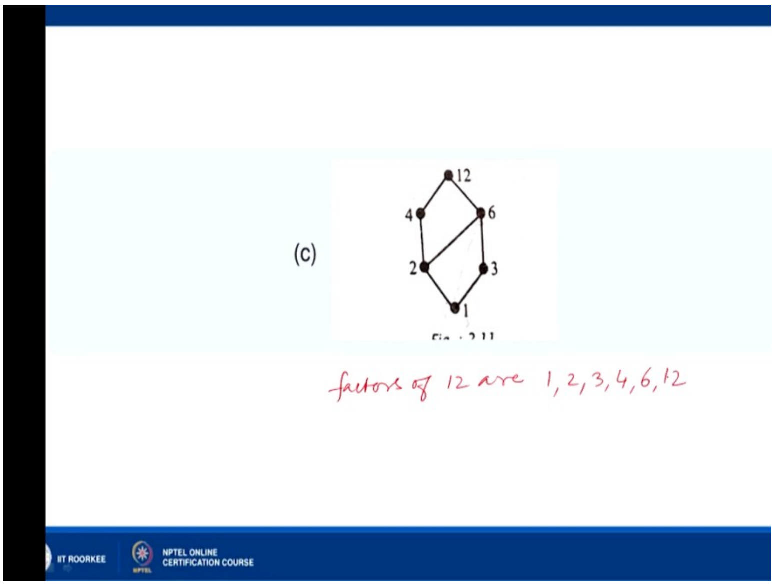

So we have 12. The factors of 12 are 1, 2, 3, 4, 6, 12, Ok. 1, 2, 3, 4, 6, 12. Now we begin with 1, Ok. So 1 divides 2, Ok. So we join 1 to 2. 1 divides 3 so we join 1 to 3. Then 2 divides 4 so we join 2 to 4, Ok and 3 divides 6. So we join 3 to 6 and 2 also divides 6. So we join 2 to 6. And then 4 divides 12, 6 divides 12. So we join 4 to 12 and 6 to 12. So this is the Hasse diagram for the third example. So with that I would like to end my lecture. Thank you very much for your attention.