Higher Engineering Mathematics

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Partially Ordered Set 1

Hello friends, welcome to my lecture on Partially Ordered Sets. So let us begin with the first lecture on Partially Ordered Sets. A binary relation or simply relation from a set A to a set B is a subset of A × B. So if you denote the relation by R then R is a subset of A × B.

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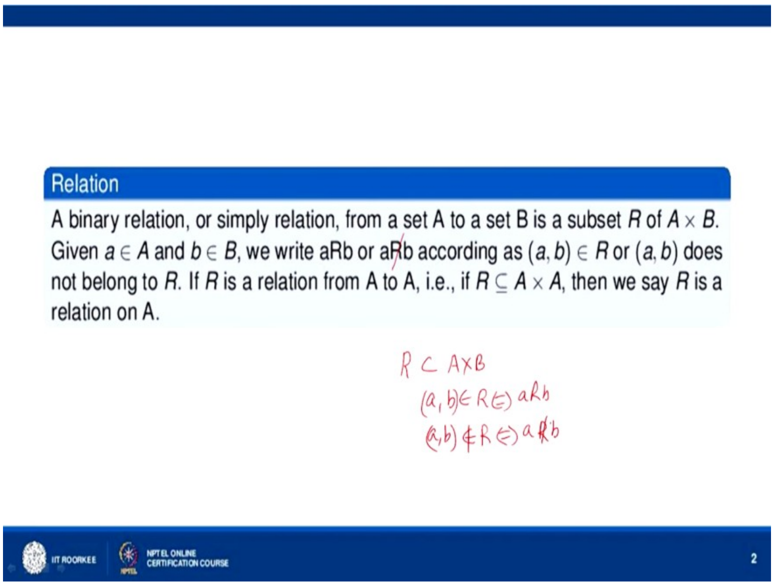

Let us say we have an element a belonging to A and b belonging to B then we write a R b or a not at R b, Ok according as (a, b) ∈ R. So if (a, b) ∈ R it means that a is related to b. Ok. And if (a, b) does not belong to R then we say that a is not R related to b, Ok.

So given two elements a in A and b in B, we say that a is related to b or a is not related to b according as (a, b) is in R or (a, b) is not there in R. If R is a relation from A to A, then R is a subset of A × A and we say that R is a relation on A.

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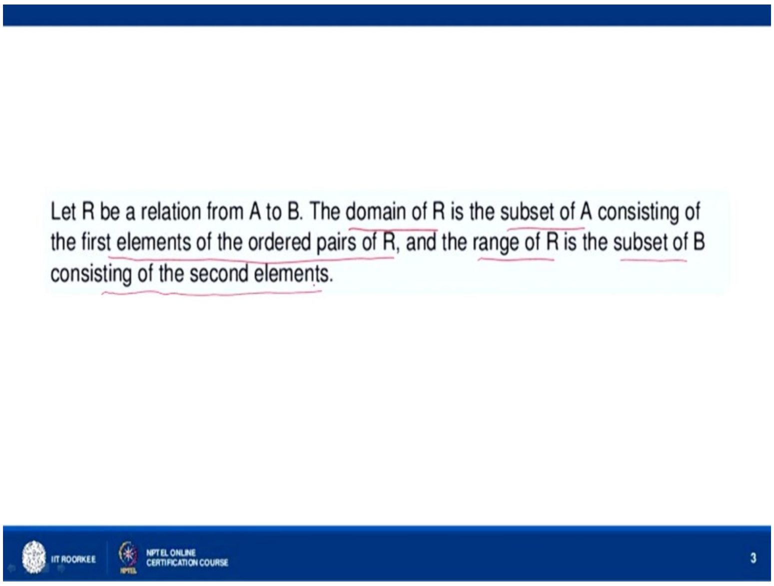

Now let us say R be a relation from A to B. That means R is a subset of A × B. Then what is the domain of R? The domain of R is the subset of A consisting of the first elements of the ordered pairs of R, Ok. So in the domain of R, consider all the ordered pairs of R. Then their first elements, their first elements constitute the domain of R while the range of R is the subset of B consisting of the second elements of all the ordered pairs of R.

So if we consider all the ordered pairs of R then you figure out all the second elements there, Ok. They constitute the range of R. And obviously it will be a subset of B. Now let us say the, let us define the inverse of the relation R.

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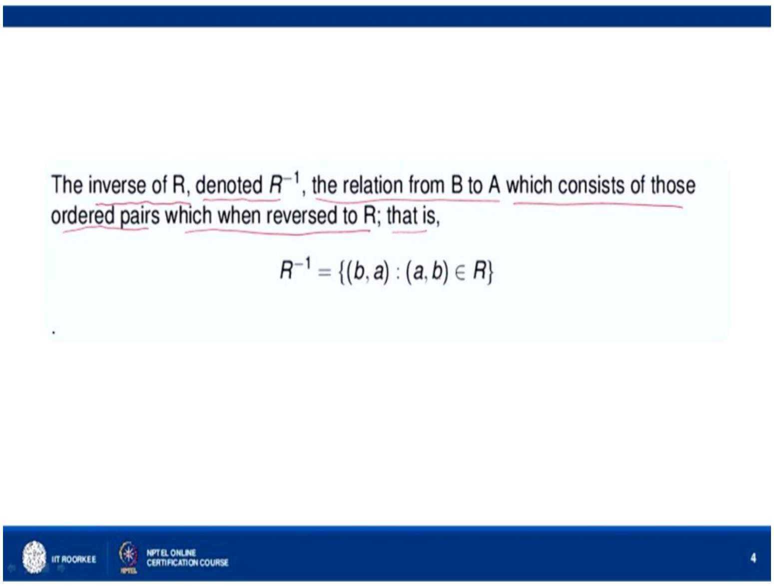

Then inverse of R denoted by R^{-1} inverse is defined as the relation from B to A which consists of those ordered pairs which when reversed to R, that is R^{-1} inverse is (b, a) when (a, b) $\in$ R. So whatever ordered pairs are there, say a, b there in R; R inverse will consist of the ordered pairs (b, a). So the second element b in R will become the first element in $R^{-1}$and the first element a in R will become the second element a in R^{-1}.

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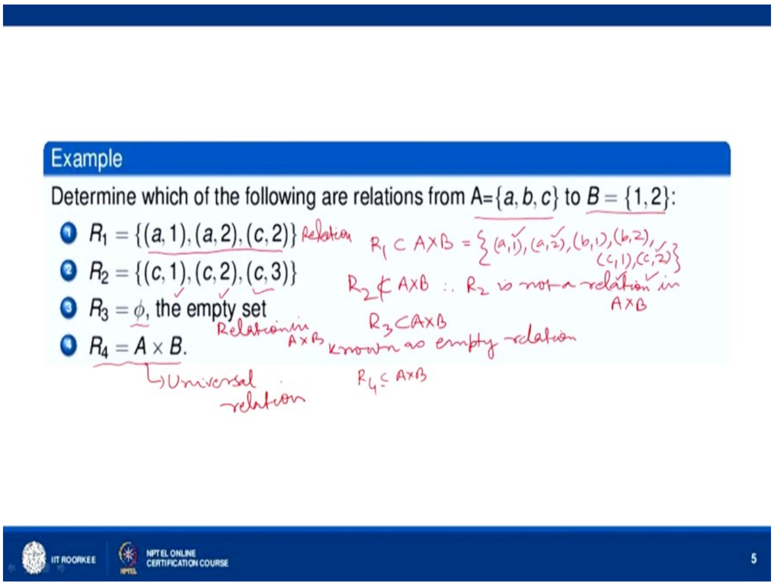

Now let us determine which of the following are relations from A={a, b, c} to B={ 1, 2}, Ok. So here R_1 is given as (a, 1),(a, 2),(c, 2) Ok. So you can see R_1 is clearly a subset of A × B, Ok. A× B is consisting of all the elements say, (a ,1), (a, 2),(b, 1), (b ,2), (c ,1), (c ,2), Ok. So $R_1$is a subset of A× B because (a 1) is there in A× B, (a 2) is there in A × B and (c, 2) is also there in A × B. So R_1 is a subset of A× B and therefore R_1 defines a relation in A × B, Ok. So this is a relation.

Now let us look atR_2. R_2 consists of (c, 1), (c, 2) which are there in A × B but (c, 3), (c, 3) is not there in A × B. So R_2 is not a subset of A× B. And therefore R_2 is not a relation in A × B. So R_2 Now R_3. R_3,equal to empty. The empty set is always a relation because phi, R_3, equal to phi, R_3, is always a subset of A × B, Ok.

Phi is a subset of A × B. This is empty set. This is called, there is always a relation, there always defines a relation in A × B. It is known as empty relation, Ok, known as...And when here R_4, is A× B, the entire set A × B so clearly $R_4$is a subset of A × B, Ok. So $R_4$is a relation in A × B and it is called as, because it is the complete set A × B so we call it as universal relation.So out of these four, the second one that is $R_2$is not a relation in A × B while the others are all relations in A×B.

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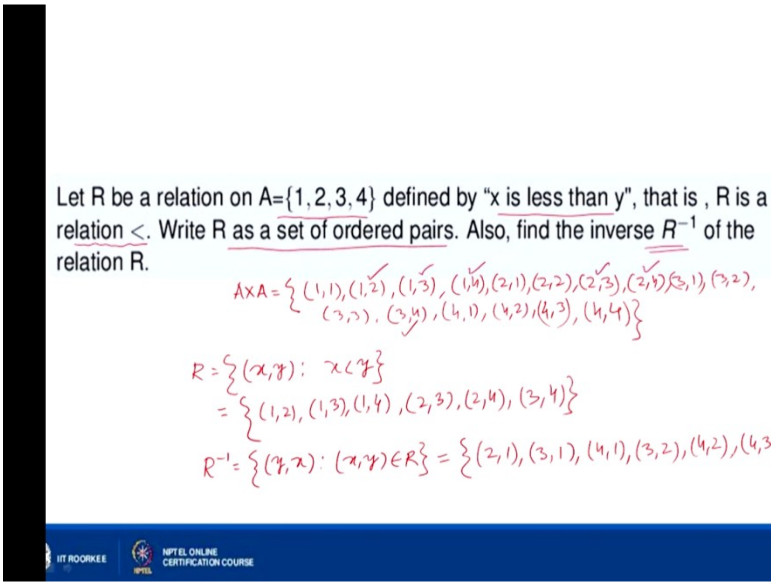

Now let us consider this example. Let R be a relation on A, Ok. A is {1, 2, 3}. Ok. R be a relation on A means R is a subset of A × A and it is defined by x is less than y, Ok. That is R is a relation less than, Ok. So write R as a set of ordered pairs. So let us see, A × A will consist of elements like (1, 1), (1 2),(1 3), (1 4), (2 1), (2 2), (2 3), (2 4), (3 1), (3 2), (3 3),(3 4), (4 1), (4 2), (4 3) and (4 4). Ok now R will consist of those ordered pairs where x is less than y. So if (x, y) is an element of R then x should be less than y.

So R is the set of those ordered pairs {(x, y) : x < y}. So let us see how many, which are the ordered pairs which satisfy that the first element x is less than the second element y, Ok. So here (1 1), (1 2). (1 2) we can take because 1 is less than 2, (1 3) we can take, (1 4) we can take, then we can take, (2 3) Ok, because 2 is less than 3, then we can take (2 4). We can take then (3 4), Ok, then yeah.

So R is equal to the set of ordered pairs (1 2), (1 3), (1 4), (2 3) and then (2 4) and then (3 4), Ok. Because in all these ordered pairs the x, that is first element is less than the y, second element. In all others this condition does not hold good, Ok. Now, so now write R is a set of ordered pairs, also find the inverse, R inverse of the relation R.

So R inverse is equal to, if (x, y) ∈ R then (y, x) ∈ R^{-1}. So we have (y, x), Ok where (x, y) ∈ R. So this will consist of (1, 2) will become (2, 1), (1, 3) will become (3 1), (1 4) becomes (4

1), (2 3) becomes (3 2), (2 4) becomes (4 2) and (3 4) becomes (4 3). So here the first element Ok is y and y is then greater than x, it is greater than the second element. So this is how we find the inverse of R.

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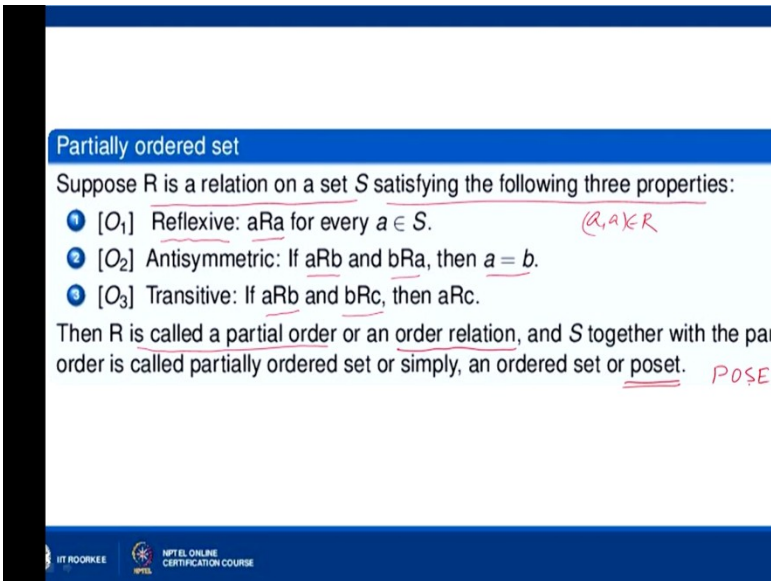

Now let us define partially ordered set. Suppose R is a relation on a set S which satisfies the following three properties. First property is reflexive. a is related to a, that is (a, a) $\in$ R, Ok. That is the first property for every a $\in$ S. Antisymmetric, if a related to b that is (a, b) $\in$ R and (b, a) $\in$ R that will happen only when a is equal to b. So if relation is symmetric then if a R b and b R a, Ok both hold, a is related to b and b is related to a, then a must be equal to b.

Now the third property transitive, if a is related to b and b is related to c, then a is related to c. That means if a, b $\in$ R and b, c $\in$ R then a, c also $\in$ R. We shall then call R to be a partial order or an order relation, an order relation on S together with the partial order relation will be then called partially ordered set. Or we call it as an ordered set. Or in brief we call it as a poset, POSET means partially ordered set.

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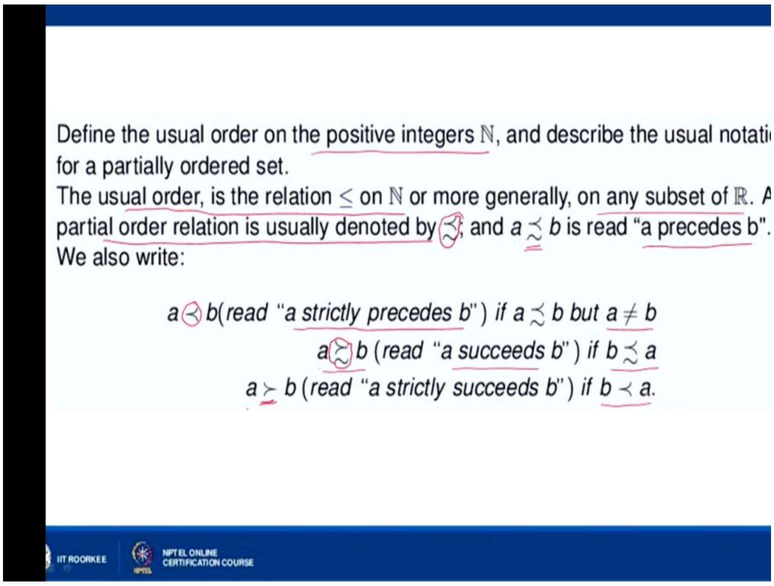

Now let us define the usual order, Ok on the set of positive integers, Ok. On the set of positive integers and the usual order is the relation $\leq$, Ok which is also there on any subset of the set of real numbers. A partial order relation is usually denoted by this symbol, this symbol Ok, this symbol and what is the meaning of this symbol? If this is the symbol is there between a and b we say that $a \leq b$, Ok. $a \leq b$ if we have this symbol Ok where this one is not there, Ok. Then we say that $a < b$. Now $a < b$ means $a \leq b$ but $a = b$, Ok.

Now if we have $\geq$ symbol, Ok this means that $a \geq b$, $a \geq b$ if $b \leq a$, Ok. You can see this symbol, $\leq$ symbol means precedes. So if $b \leq a$ then $a \geq b$. Now if $\geq$ symbol is there we say that a strictly suceedes b, Ok if $b < a$. So we shall make use of these symbols on a partially ordered set.

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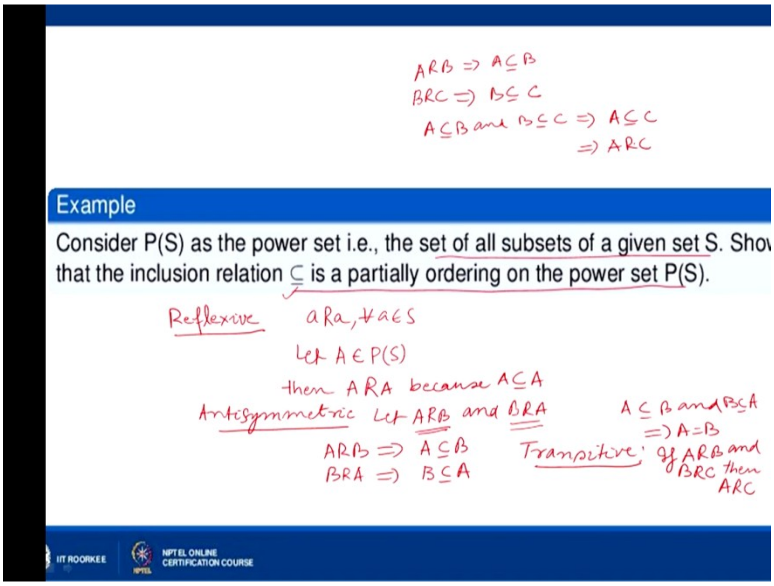

Now consider P(S) as the power set. Let us consider the power set of, the set of, that is the set of all subsets of a given set S so that the inclusion relation is a partial ordering on the power set S. Ok. So first thing that we have to see is reflexive. Reflexive means a is related to a for all a belonging to S. This is the first thing we have to see. So let us take a set in P S, Ok. Let a ∊ P S, the power set of S, Ok.

Then A is related to A because A is contained in A, Ok. This is always true. Relation here is this inclusion relation. So A is always contained in A. So we say that A is related to A. So this reflexive property holds. Then antisymmetric, so let A is B R related to B and B be R related to A, Ok. Then we have to show A is equal to B. Now A R B means A is contained in B, Ok. And B is related to R means, B is related to A means B is contained in A, Ok.

So A contained in B and B contained in A implies that A is equal to B. If A is subset of B and B is subset of A then they must be equal. So A is equal to B and so therefore if A is related to B by inclusion relation and B is related to A by the inclusion relation then A must be equal to B. So antisymmetric property holds. And the third one is transitive property. So if A is related to B and B is related to C, Ok then we have to show A is related to C. Ok.

So A is related to B implies that A is contained in B and B is related to C means B is contained in C. Now A is contained in B and B is contained in C implies that A is contained in C. Which means that A is related to C, Ok? So the transitive property also holds. And therefore the relation, inclusion relation is partial ordering on the power set S of the given set S.

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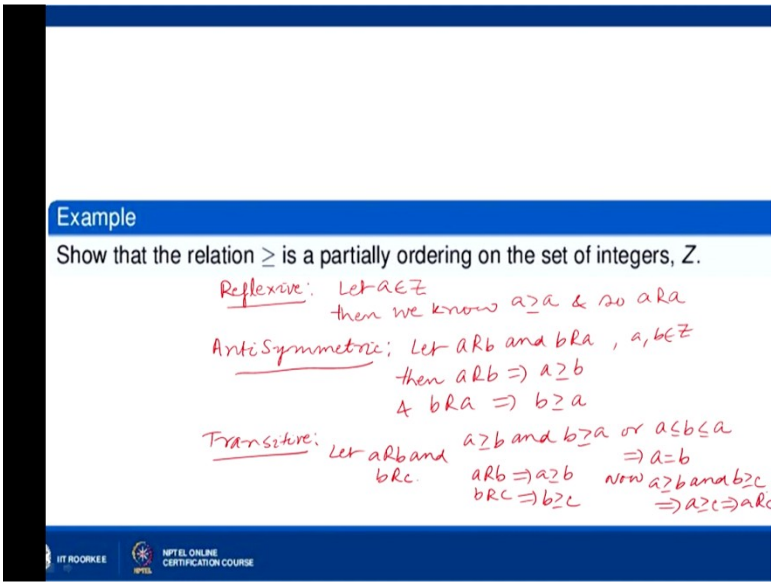

Now let us show the relation greater than or equal to is a partial ordering on the set of integers Z, Ok. So first we prove that reflexive property. Take any a belonging to Z. Let a be any integer, Ok. Then we know a is always greater than or equal to a, Ok. And so a is R related to a, Ok. Then reflexive, after that reflexive, we go to antisymmetric. So let us say a is R related to b and b is R related to a, Ok. a, b belong to Z,

Ok. Then a is R related to b implies $a \geq b$. And b R related to a implies $b \geq a$, Ok. So this means what? So $a \geq b$ and $b \geq a$ together can hold, Ok only when they are equal because this also I can write as b greater than or equal to a means a less than or equal to b and b less than or equal to a, Ok.

So a less than or equal to b less than or equal to a implies a is equal to b, Ok. Now then we go to transitive. So let us say, a is R related to b and b is R related to c, Ok. Then we have to prove that a is R related to c. So a is R related to b implies $a \geq b$ and b is R related to c implies $b \geq c$.

Now $a \geq b$, and $b \geq c$ implies that $a \geq c$. So, so this means that the relation $\geq$ is a partial ordering on the set of integers Z.

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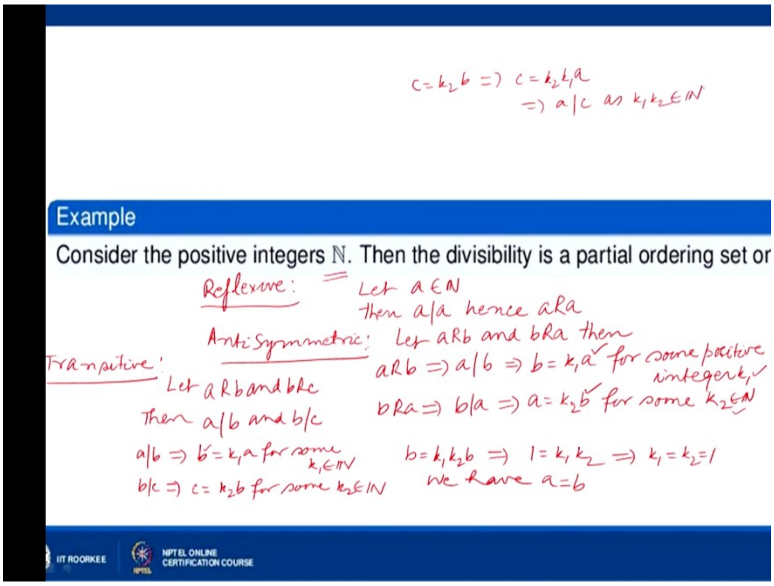

Now consider the set of positive integers N, the set of natural numbers N. Then the divisibility is a partial ordering on the set N. Let us prove that. So reflexive, Ok so let a belong to N, a belong to N, a be an integer, positive integer then a | a. We know a | a? Ok. Hence a is related to a. Ok. This is valid for any a belonging to N?

Then antisymmetric, let a be related to b and b be related to a. a R b, and b R a, Ok. Then a R b implies a | b which means that b= k_1 × a for some positive integer k, for some positive integer k_1, Ok. b is R related to a implies b | a, Ok which implies that a is k_2 times b for some k_2 belonging to N for some positive integer k_2 belonging to N. Now from here what we can see, from this equation and this equation we find that b = k_1. a. a we can put as k_2. b. So b= k_2 k_1b, Ok.

Now b is a positive integer. So we can divide by b so this implies 1= k_2 k_1. Now k_2, $k_1$are both positive integers and their product is 1. It means thatk_1= k_2 = 1, Ok. So what we will get, k_1 and k_2 both are equal to 1, so we will get b equal a, a equal to b. That means we have, so transitive, so, so antisymmetric property holds. Now next we go to transitive property, Ok. So let us say a be R related to b, b be R related to c, Ok, b be R related to c then a | b and b | c, Ok. a | b implies b equal to k_1 a for some k_1 belonging to N, Ok. And b | c implies c is equal to k_2 b for some k_2 belonging to N, Ok.

So what do we get? a | b gives b equal to k_1 a and b | c gives c is equal to k_2.b, Ok. So let us replace the value of b equal to k_1 a in the equation c equal to k_2 b. So c equal to k_2. b gives c equal to k_2.k_1 .a, Ok. k_1, k_2 are positive integers so k_1 into k_2 is also positive integer. So this means that a | c as k_1, k_2 $\in$ N, Ok, so the, when we consider the set of positive integers N that divisibility is a partial ordering on the set N.

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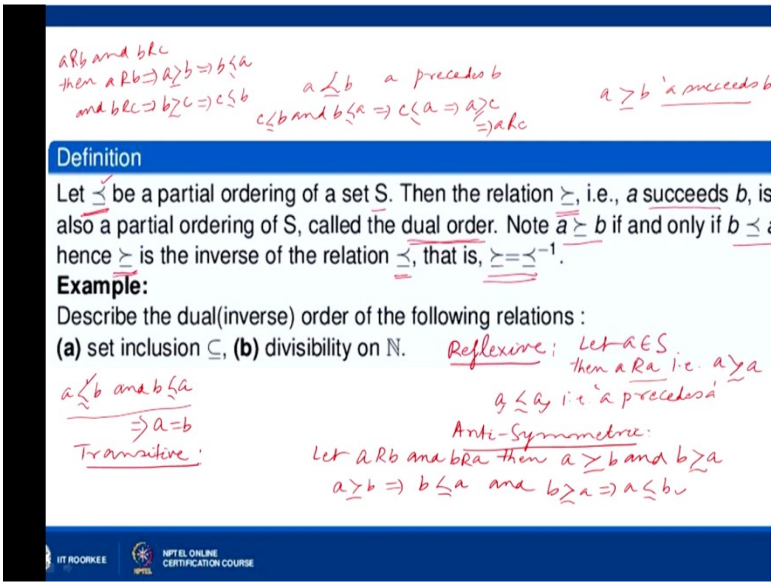

Now let us go to this symbol which we talked about earlier. This means when ≤ symbol is there between a and b, Ok we say that a ≤ b, Ok. So let this be a partial ordering of S at S. Then the relation this, Ok, this relation means ≥, Ok. When it is there between a and b it will mean that a ≥ b, Ok.So to show that this relation is also a partial ordering of S; it is called the dual order, Ok. So let us see. We have to show that, this is a, first we show the reflexive property. Reflexive property means we have to show that let a $\in$ S. Let a $\in$ S. Then a is R related to a. a is related to a.

This means we have to prove that, that is this, Ok. Now there is already a partial ordering of the set S. That means, sorry a ≤a. This is given, this is already there. That is a ≤ a, Ok. This is partial ordering on the set S. So with respect to this partial ordering S is reflexive. This is a reflexive relation on S so this means a ≤a is always there. When a ≤ a then this a a ≥ a this a, Ok. So a ≥ a, Ok. So a is related to a.

And then let us prove the antisymmetric. Antisymmetric, let us say, let a be related to b and b be related to a, Ok. Then we have to show that a and b are equal. So a is related to b means a ≥ b and b ≥ a, Ok. a ≥ b and b ≥ a. Now a ≥ b means b ≤ a, Ok and b ≥ a implies a ≤ b, Ok.Now so that what we have? a ≤ b and b ≤ a, Ok. Then by the fact that this partial ordering is transitive, Ok, so a ≤ b means, a ≤ b and b ≤ a, Ok.

Now this is a partial ordering on the set. So with respect to this partial ordering we have antisymmetric property and that implies that $a = b$, Ok. Because this is nothing but, this is nothing but the antisymmetric case for this partial ordering, Ok. Now the third one, transitive. So let us say a be related to b and b be related to c, Ok. Then a R b implies $a \geq b$, Ok and b related to c implies $b \geq c$.

Now $a \geq b$ means $b \leq a$, Ok. And $b \geq c$ means $c \leq b$, Ok. So now $c \leq b$, and $c \leq b$ and $b \leq a$, Ok and this is a partial ordering of the set S. Therefore with this partial ordering, this partial ordering is transitive. So $c \leq b$ and $b \leq a$ implies $c \leq a$, Ok which is same as $a \geq c$, Ok. $a \geq c$ that means a is related to c, Ok. So that is how we show that this notation, Ok, which is the notation that $a \geq b$ is also a partial ordering of S.

And it is then therefore called; it is called dual order, Ok. Now $a \geq b$, if and only if $b \geq a$. $a \geq b$ if and only if $b \leq a$, Ok. Hence this notation Ok which is the notation for $\geq$, $a \geq b$. So this notation is the inverse of the notation $a \leq b$, Ok.This is the notation for $a \leq b$ and this is the notation for $a \geq b$. So this is inverse of this relation and we show it by writing this notation for $a \leq b$ is equal to $a \geq b$ inverse, this notation, inverse of this notation.

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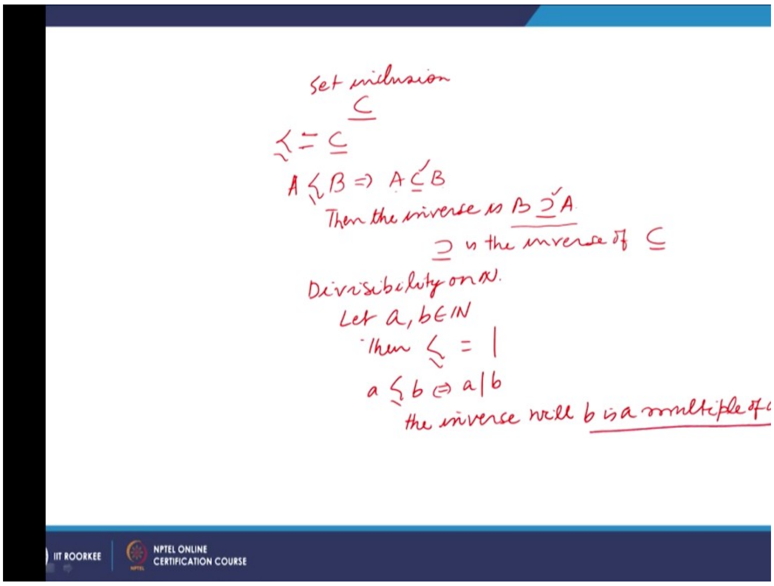

Now describe the dual order of the following relations.Set inclusion, Ok. So let us go to set inclusion first. Ok so this. We want to know what is the inverse of this, Ok. So inverse of this means let us say this is the notation this. This is same as this, Ok. So $A \leq B$ Ok means that A is subset of B, Ok. So the inverse of this will be then the inverse of, then the inverse of...Then the inverse is B contains A.

So when you write the inverse of A contained in B what you get is B contains A. Ok so this is the notation for set inclusion. This is the notation for set containment. So B contains A.So this notation is the inverse of this. Now the next case was divisibility on N, Ok. So let us see what is the inverse of divisibility on N? So divisibility, so let us say, let a, b belong to N. Then this notation Ok is division, Ok.

So a means that a | b, Ok. a | means, a | b Ok that means we can say, inverse will be, the inverse will be b is a multiple of a, Ok. When a | b its inverse is b is a multiple of a, Ok.

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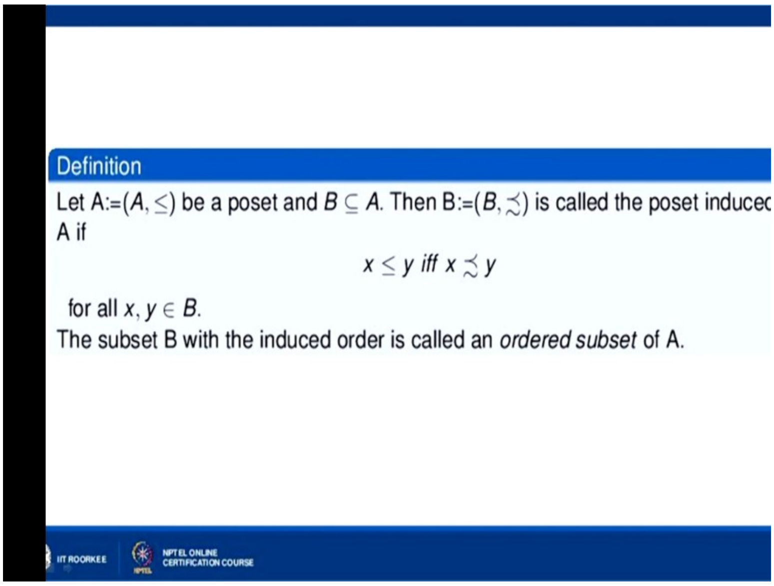

Now we go to this definition. So let (A, ≤) be a poset, Ok. This is the order relation on A and B ⊂ A. It is not necessary that it is less than or equal to, it is some relation on A.

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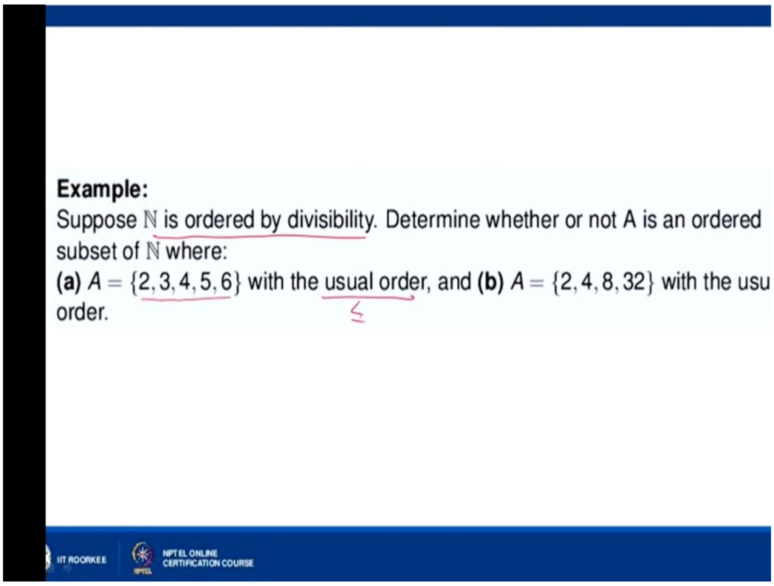

So let this be a poset. And B ⊂ A, then B:=(B,≤) is called the poset induced by A if x less than or equal to y if and only if x ≤ y, Ok. So suppose A is a partially ordered set. It is a poset with the partial ordered relation given by this notation, Ok. And B ⊂ A. Then B will be called a poset, Ok induced by A if x less than or equal to y, you take any two elements x, y belonging to B Ok then x less than or equal to y, Ok as, because this is relation in A.

So x less than or equal to y in A if and only if x ≤ y, Ok in B. Ok so that should hold. So the subset B with the induced order, with the induced order is called an ordered subset of A. Now let us look at an example on this. Suppose N is ordered by divisibility, Ok. N is ordered by divisibility. Determine whether or not A is an ordered subset of N where A is consisting of {2,3,4,5, 6} with the usual order. Usual order means less than or equal to, Ok. Usual order as we have seen earlier, it means less than or equal to.

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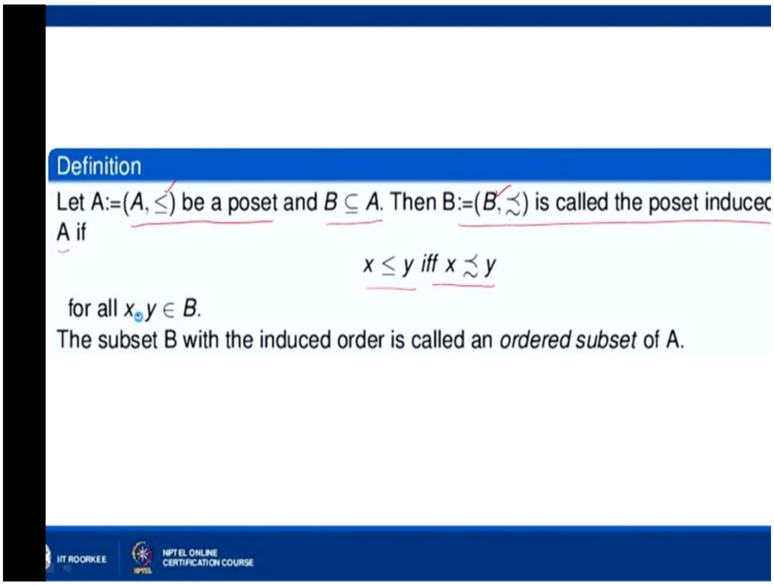

Now you can see A will be the set induced by S, induced by N provided A will be called the poset induced by N whenever you take any two elements in A, Ok we have A less than or equal to B if and only if A ≤ y. N less than or equal to is the notation for the partial order in the set A.

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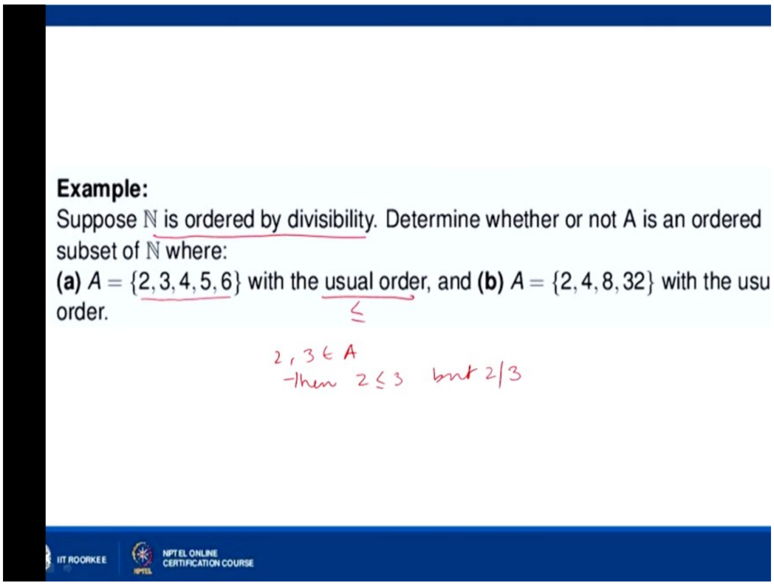

So what we have here? We can see that 2, 3 Ok, 2, let us take 2, 3 belonging to A, Ok. Then 2 ≤ 3, Ok. 2 ≤ 3 but, but 2 does not divide 3, Ok. In N, N is partially, partial order relation in N is divisibility, Ok. So we have taken x to be 2, y to be 3, Ok and then we see that with respect to the usual order in A, Ok, 2 and 3 are related. 2 ≤ 3. But 2 is not, 2 does not divide 3. So this condition does not hold, this one.

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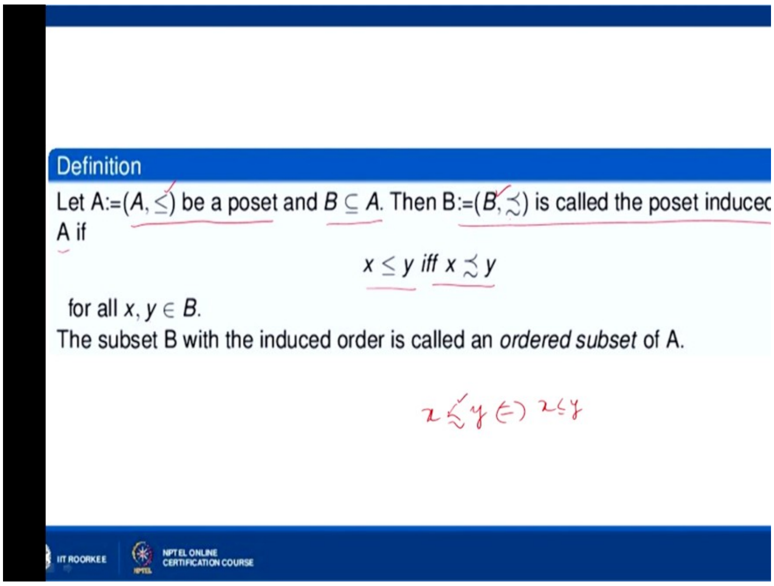

$x \leq y$, we should have $x \leq y$ if and only if $x \leq y$. This is the order relation in this subset. And this is the order relation in the given set. So here the order relation in N is divisibility, Ok. So this holds but this does not hold.

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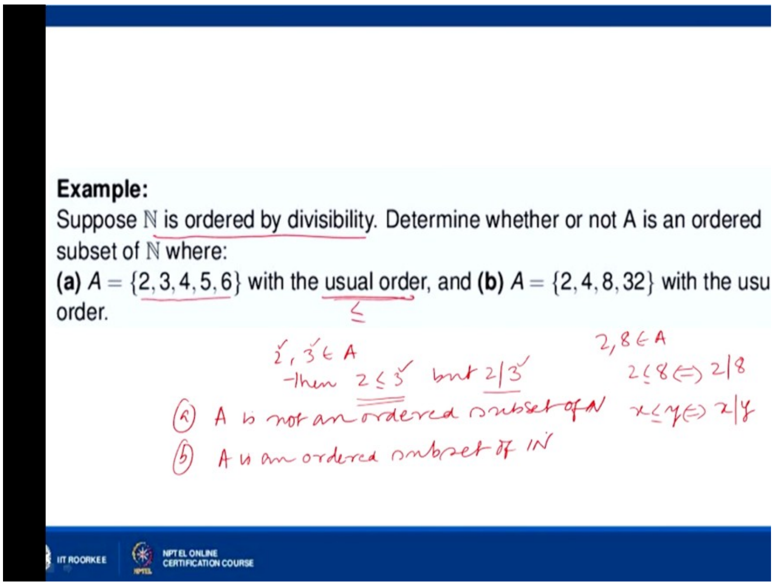

And therefore A is not an ordered subset of N, Ok So this should happen for any two elements in A, Ok but we find two elements in A, 2, 3 for which $2 \leq 3$ but 2 does not divide 3. So this is not, not an ordered set, not an ordered subset of N. Ok, now let us look at the second part A={2, 4, 8, 32}. So what do you do? Suppose I take any two elements here, say

2, 4 or you can take 4, 8 or you can take 2, 8 Ok or you can take 8, 32 or you can take 4, 32 or 2, 32.

Then x ≤ y, Ok. So let us take 2 and 8, Ok in A. Then 2 ≤ 8. And also 2 | 8, Ok. So this is valid, Ok. 2 divides this if and only if this, this condition is valid there. Take any two elements, Ok. If x is, in A we are taking the usual notation, yeah usual order. So x ≤ y if and only if x | y. So this holds true in the case (b). So A is an ordered subset of y, ordered subset of N sorry, ordered subset of N.

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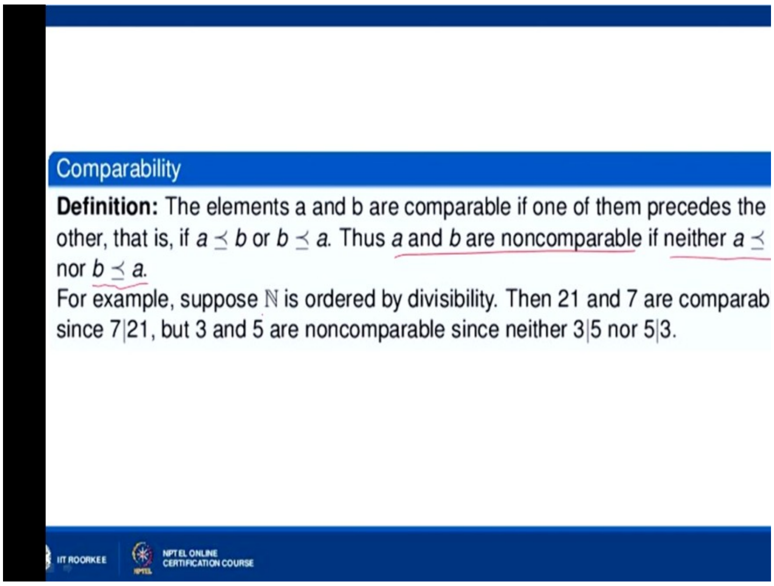

Now we go to comparability. The elements a and b are called comparable if one of them precedes the other. That is a is ≤ b or b ≤ a, Ok. Thus a and b are non-comparable if neither a ≤ b nor b ≤ a. For example, suppose N is ordered by divisibility, Ok. Then 21 and 7 are comparable because 7 | 21. But 3 and 5 are non-comparable because neither 3 | 5 nor 5 | 3.

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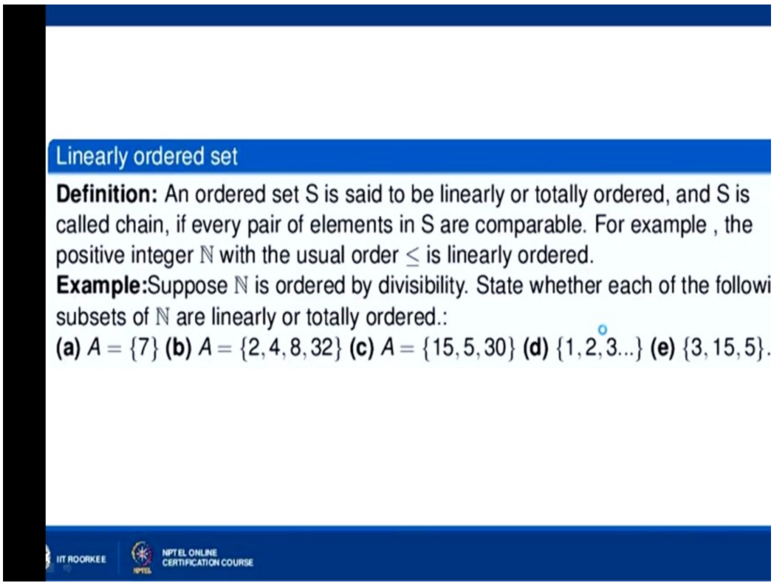

Linearly ordered set, an ordered set S is called linearly or totally ordered Ok and S is called a chain if every pair of elements in S are comparable

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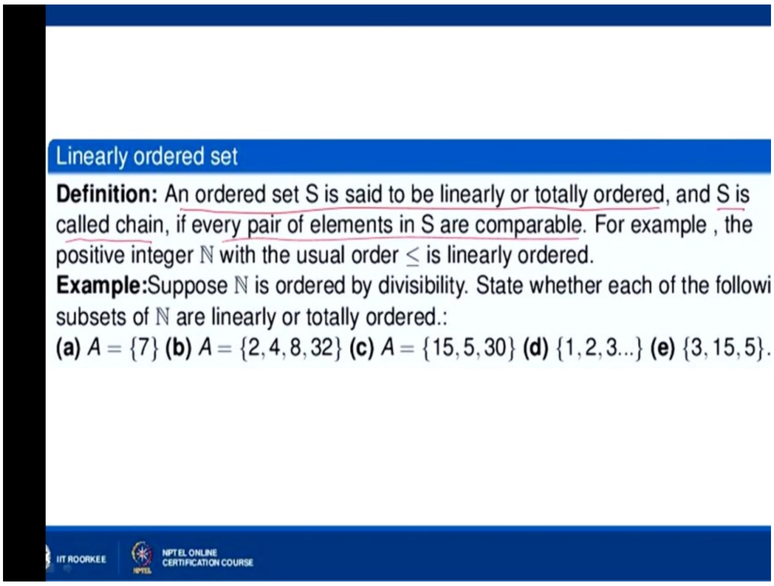

Ok so if you take a set S, Ok then you take any pair of elements in S, if they are comparable that is a ≤ b or b ≤ a happens then we say that S is totally ordered or linearly ordered. And the set S will be called as a chain. For example the positive integer, set of positive integers Ok, the set of positive integers with the usual order less than or equal to, linearly ordered. If you take any two elements in x y in N then either x ≤ y or y ≤ x. So it is the usual order, the set of positive integers is a linearly ordered or totally ordered set. Now suppose N is ordered by divisibility, Ok. Then we have to see whether the following subsets of N are linearly or totally ordered, Ok.

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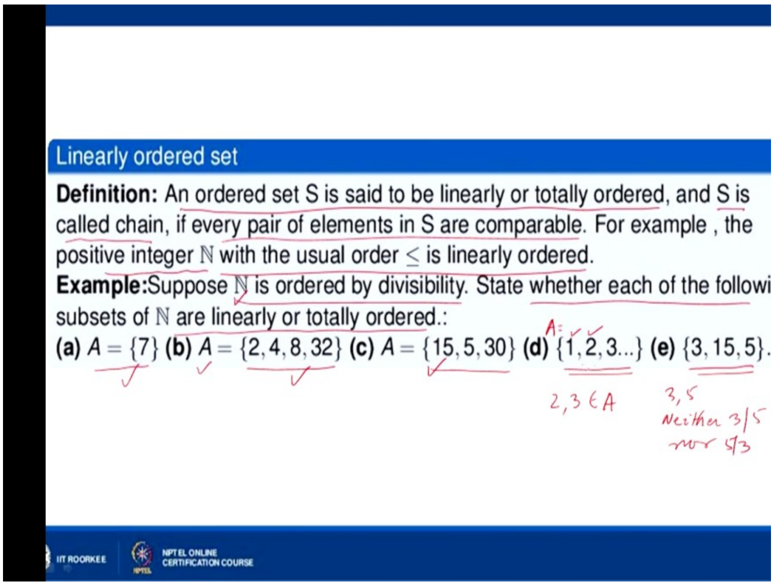

So let us look at the set a consisting of a single element {7}. It consists of a single element 7 so it is totally ordered because there is only one element. So when you take a pair of elements it will be {7, 7}. So, 7 | 7. So it is linearly or totally ordered set. Now here let us see {2, 4, 8, 32}, Ok. You take any two elements in this, Ok. {2, 4} or {4, 2} Ok. Then you take {4,8} or {8, 4}, {8, 32} then either 4 | 8 or 8 will divide, if you take 4, 8 then 4 | 8. If you take, {8, 32} then, 8 | 32. If you take, {2, 32}, 2 | 32. So if you take any pair of elements in a here in part b then one | the other, Ok.

Now here in the case of {15, 5, 30} again if we take a pair, Ok then we find either the pair will be {15, 5} or {5, 30} or {15 30}, there is always the situation that one element divides the other element. So this set and this set and this set are totally ordered or linearly ordered. If you take the entire set N, Ok entire set N, Ok then this N is not, this set is not linearly or totally ordered. Because if you take {1,2} Ok, if you take {1,2} then, if you take say {2,3}, if you take {2, 3} Ok in this set, let me call this set A, yes of course this a is equal to N.

So {2, 3} you take in N, neither 2 | 3 nor 3 | 2. So this A is not linearly or totally ordered. If you take {3, 15, 5}, Ok, {3, 15, 5} this is also not linearly or totally ordered because when we take {3, 5} then neither 3 | 5 nor 5 | 3.So this set and this set are not totally or linearly ordered, Ok, So that is all in this lecture. Thank you very much for your attention.