**Ordinary and Partial Differential Equations and Applications**

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**Lecture – 08**
**Properties of Homogeneous Systems**

Hello friends, I welcome you to my lecture on properties of homogeneous system, we shall try to find the all solutions of the homogeneous system given by x dot = Ax, where as you know x represents the vector valued function with having an components x1t, x2t and so on xnt, na is the matrix aij, where i runs from 1 to n and j runs from 1 to n, so such a system we know is a homogeneous system of first order linear differential equations.

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Now we shall try to find all solutions of $\dot{x} = Ax$ (1)
First we show that constant times of a solution and the sum of two solutions are again solution of (1).

***Theorem 1:*** Let x(t) and y(t) be two solutions of (1). Then

a) c x(t) is a solution, for any constant c, and
b) x(t)+y(t) is again a solution.

***Lemma :*** Let A be an $n \times n$ matrix. For any vectors x and y and constant c,

a) A(cx)=c(Ax) and

b) A(x+y)=Ax+Ay.

Now, first we show that the constant times of a solution and the sum of 2 solutions are again solution of this homogeneous system 1, let us say xt and yt be any 2 solutions of the homogeneous system 1, then first we show that c xt is a solution for any constant c and second is xt + yt is again a solution. Now, we shall make use of the lemma here, let A be a n by n matrix for any vectors x and y and constant c, A * cx is = c times Ax and A * x + y = Ax + Ay.

These are the properties of matrices and they are very simple to prove, so we shall make use of this lemma, now you can see that dx/dt is = this x dot = Ax gives dx/dt = Ax, so when you take; when you want to prove that c xt is a solution, you need to show that d over dt of c xt is = A * cxt, okay, so d over dt of cxt will be c times dx over dt, which will be c times x and c times Ax is = A times cx, so we will get d over dt of cxt = A * acxt.

And therefore, cxt is a solution of the differential equation, if x dot = Ax, now, again if you want to show that xt + yt is a solution then if xt is a solution, then x dot = Ax and yt is a solution means dy over dt is = Ay, so if you want to prove that xt + yt is the solution, then d over dt of xt + yt will be dxt over dt + dyt over dt, which will be = Ax + Ay and Ax + Ay, you can see in lemma 2, lemma b part, Ax + Ay is = A * x + y.

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**Corollary of Theorem:** Any linear combination of solutions of (1) is again a solution of (1) i.e. if $x^1(t), \ldots\ldots, x^j(t)$ are solutions of (1), then $c_1 x^1(t) + \ldots\ldots + c_j x^j(t)$ is again a solution for any choice of constants $c_1, c_2, \ldots\ldots, c_j$ .

**Example 1:** Consider the system of equations

$$\frac{dx_1}{dt} = x_2, \; \frac{dx_2}{dt} = -4x_1$$

or

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} x, \; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad (2)$$

So, d over dt of x + y = A * x + y and therefore if xt and yt are the solutions of the homogeneous system x dot = Ax, then xt + yt is also a solution of the homogeneous system. Now, let us look at the corollary of this theorem 1, so any linear combination of solutions of 1 is again a solution of 1 that if x1t, x2t and so on xjt are j solutions of equation one, then c1 x1t + c2 x2t and so on cj xjt is again a solution of the equation 1 for any choice of constants; c1, c2, cj.

So, this follows from the theorem 1 because d over dt of c1 x1t + and so on cj xjt will be = c1 times dx1 over dt + c2 times dx2 over dt + and so on cj times dxj over dt, now which will be = c1 times Ax1t + c2 times Ax2t and so on cj times Axjt, which will be = A operating on A * c1x1t + c2x2t and so on cjxjt, so if x1t and x2t and so on xjt are the solutions of equation 1, then their linear combination is also solution of equation 1.

Now, let us consider, let us take an example here, let us consider the system of equations dx1/dt = x2, dx2/dt = -4x1, so this you can see is a homogeneous system of 2 linear; 2 differential equations of first order, so this can be written in the vector in the form of the matrix equation, dx/dt = now, here the coefficient of x1 is = 0, so we have 0 here, coefficient of x2 is 1 and in the second equation, coefficient of x1 is -4, coefficient of x2 is 0.

So, we have the coefficient matrix A as 0, 1, -4, 0 x; x is x1, x2, so we can write the given system of 2 differential equations in convenient form, which is dx/dt = Ax; A is the matrix 0, 1, -4, 0.

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This system of equations is obtained from

$$\frac{d^2y}{dt^2} + 4y = 0$$

Hence, $$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \cos 2t \\ -2\sin 2t \end{bmatrix} + c_2 \begin{bmatrix} \sin 2t \\ 2\cos 2t \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \cos 2t + c_2 \sin 2t \\ -2c_1 \sin 2t + 2c_2 \cos 2t \end{bmatrix}$$

is a solution of (2) for any choice of constants $c_1$ and $c_2$.

Now, this system of equations is obtained from the second order differential equation with constant coefficients, d square y over dt square + 4y = 0, to see this, what we do is; let us put x1 = y.

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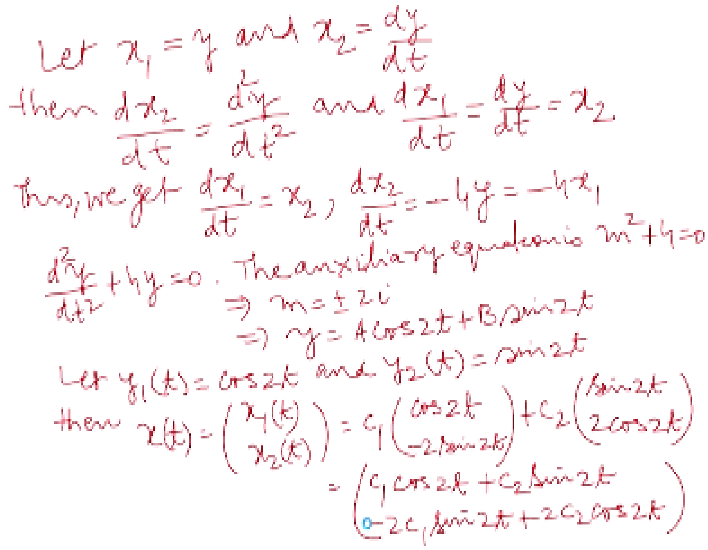


Let x1 = y and x2 = dy over dt, then dx2 over dt will be = d square y over dt square and dx1 over dt will be = dy/dt, which will be = x2, okay, so thus we have; thus we get, dx1/dt = x2, this is one equation, second equation; dx2/dt is = d square y/ dt square and we have the second order equation has d square y/ dt square + 4y = 0, so this = -4y; -4y means – 4x1. So, you can see we get 2 equations; dx1/dt = x2 and dx2/dt = -4x1.

So, now in order to find the general solution of this system of differential equations, what we notice is that d square y over dt square + 4y = 0, this is a second order linear differential equation with constant coefficients, so we know how to find a solution, the auxiliary equation is m square + 4= 0 which gives us 2 values of m + - 2i and therefore, y is = some constant a times cos 2 t + some constant B times sin 2t.

And therefore, cos 2t and sin 2t are 2 solutions of this differential equations, now let us say, y1t = cos 2t and y2t = sin 2t, okay, then now we are looking for the solution of the homogeneous system of equations; dx1/dt = x2, dx2/dt = -4x1, so xt is = x1t x2t will be equal to some constants c1 times, now, x1t, let us take the x1t to be cos 2t, so cos 2t and then it is derivative because x2 is = dx1/dt, so we get -2 sin 2t derivative of cos 2t is -2 times sin 2t + c2 times.

Now, this time we take x1t to be sin 2t and then derivative of dx1, which give cx2, this cx2, so we get the derivative sin 2t as 2 cos 2t and this gives you c1 cos 2t + c2 sin 2t – 2c1 sin 2t + 2c2 cos 2t, so this gives us a xt = this, is the general solution of the given system of equations for any choice of constants, c1 and c2, you can see xt is = - c1 times cos 2t + c2 sin 2t, then -2c1 sin 2t + 2c2 cos 2t, this is a solution of the given system for any choice of constants c1 and c2.

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Our next step is to determine the number of solutions we must find before we can generate all solutions of (1). For this we will use linear algebra.

***Vector space:*** A vector space (also called linear space) is a set V on which two operations ' + ' and ' . ' are defined, called vector addition and scalar multiplication.

The operation + (vector addition) must satisfy the following conditions:

*Closure:* if u and v are any vectors in V, then sum u+v belongs to V.

1) *Commutative law:* u+v=v+u, for all u,v ЄV.
2) *Associative law:* u+(v+w)=(u+v)+w, for all u,v,w ЄV.
3) *Additive identity:* There exists an element denoted by 0 in V such that 0+v=v+0=v, for all v ЄV.

Now about the next step is to determine the number of solutions we must find before we can generate all solutions of the system of equations x dot = Ax, for this we will have to use a linear algebra, so let us begin with the definition of a vector space, a vector space is also called a linear space, so it is a set B on which we define 2 operations; one is denoted by +, the other is denoted by dot, the plus operation is known as the vector addition.

And the dot is known as scalar multiplication, so with respect to the vector addition operation, there must be the following conditions, this must be satisfied. First is closure; if u and v are any 2 vectors in V, then their sum u + v must be an element of V, then commutative law, u + v is = v + u for all uv belonging to V. Associative law; u + v + w is = u + v + w for all uvw belong to V. Additive identity; there exists an element denoted by 0 in V, such that 0 + v = v + 0 = v for all v belonging to V.

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4) *Additive inverse:* for each vector v in V, the equations $v+x=0$ and $x+v=0$ have a solution x in V, called additive inverse of v and denoted by $-v$.

The operation ' . ' (scalar multiplication) must satisfy the following conditions:

1) *Closure:* If $v \in V$ and c be a scalar then $c.v \in V$
2) *Distributive law:* *a)* $c.(u+v)=c.u+c.v$ for all scalar c and $u,v \in V$.
   *b)* $(c+d).v=c.v+d.v$, for all scalars c,d and $v \in V$.
3) *Associative law:* $c.(d.v)=(cd).v$, for all scalars c and d and vector $v \in V$.
4) *Unitary law:* $1.v=v$ for all $v \in V$.

Additive inverse for each vector v in V, the equations v + x = 0 and x + v =0 have a solution x in V, which is called the additive inverse of V and we denoted by –v, with respect to the scalar multiplication operation, there must hold the following conditions; closure, if v is an element of V and c be a scalar, then c * v belongs to V. Distributive law; c. u + v = c . u + c. v for all scalar c and any vectors u, v belong to V.

And c + d . v = c . v + d. v for all scalar cd and v belongs to V, the associative law; c.d.v= cd.v for all scalars c and d, and vectors v belong to V and then finally, Unitary law; 1* v = v for all v belonging to V, now you can see that if V denotes the set of all solutions of the homogeneous system, x dot = Ax, then v is a vector space because first of all we have already seen that if u and v are any 2 solutions of x dot = Ax.

Then, vector addition that is u + v also is an element of V that is u + v is also the solution of V, so V is close with respect to the vector addition operation. Then, if you take any 2 solutions, u and v belonging to V then the commutative law, u + v = v + u is clear, associative law; u + v = w for all u, v, w belong to V, if u, v, w are solutions of the equation x dot = Ax then this associative law is also clear.

Now, additive identity, there exists an element denoted by 0's in V such that v0 + v = v + 0 = V for all v belong to V, so here the additive identity in the case of V which is the space of all solutions of the system x dot = Ax is the 0 solution that is xt = 0000, all the components of the solution xtr 0's, so such a solution obviously satisfies the differential equation dx/dt = Ax, if you put x = 0 then x dot is 0.

Because all derivatives; derivatives of all components are 0's and = A times x, so A time 0, 0, so 0 solution obviously satisfies the homogeneous system, so that 0 solution belongs to V and plays the role of the additive identity here, you yield 0 solution to any solution v belonging to V, then you get v + 0 = v, so additive identity adjust in the space V and moreover if v is any solution then –v is also solution.

Because we have already have seen that if v is a solution then c times v is a solution of the homogeneous system x dot = Ax, takes c = -1, so then if v is the solution then –v is also a solution and that -v plays a role of the additive inverse because v + - v = 0 solution and –v + v is also = 0 which is the additive identity. So, additive inverse adjusts in the space v and we have already seen that x is a; if v is the solution of the homogeneous system, then c times v is the solution of the homogeneous system.

So, closure property holds, then distribute law; c times, c.u + v = c.u + c.v clearly holds if u and v are any solutions of the system x dot = Ax, and c is scalar and if you take any 2 scalars; c and d and v to V any solution of the x dot = Ax, then c + d. v = c.v + d.v is also true and similarly, associative law is true. Unitary law; 1.v = 1 also clearly holds, so if v denotes the space of all solutions of the homogeneous system, x dot = Ax.

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Let V be the set of all vector valued solutions

$$x(t) = \begin{bmatrix} x_1(t) & x_2(t) & . & . & . & x_n(t) \end{bmatrix}^t$$

of the vector differential equation

$$\dot{x} = Ax, \quad A = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{bmatrix}$$

Then, V is a vector space, now, let V be the set of all vector value solutions, xt = x1t, x2t, xnt, this is transpose, so that means this is a row vector, when we take transpose, it becomes a column vectors, so xt is a solution of the space, the equation x dot = Ax and V is the set of all such vector valued solutions of the equations x dot = Ax, where A is this n by n matrix vector differential equation.

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then V is a vector space under the usual operations of vector addition and scalar multiplication.

In order to solve the homogeneous linear system of equations (1) we shall use the following theorem :

***Theorem*** *(Existence-uniqueness theorem):* There exists one and only solution of the initial value problem

$$\dot{x} = Ax, \qquad x(t_0) = x^0 = \left[x_1^0, x_2^0, \ldots, x_n^0\right]^t.$$

Moreover, this solution exists for $-\infty < t < \infty$.

Now, then V is a vector space as I said already under the usual operations of a vector addition and scalar multiplication, now, in order to solve the homogeneous system of; homogeneous linear system of equations x dot = Ax, we shall need the following existence uniqueness theorem, there exist one and only solution of the initial value problem x dot = Ax, xt0 = x0, where xt0 is = x10, x20, the column vector; x10, x20, xn0 t; t is the transpose.

So, when we take the transpose of the row vector, we get the column vector and moreover this solution exist for - infinity < t < infinity, the overall values; real values of t, so this t do not confuse this t with this t, okay this t is the independent variable, okay here, now, so we shall know this theorem and in existence ad uniqueness theorem, we have already done for the equation, dx/dt = ft x, so the poof of this theorem follows along similar lines.

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***Remark:*** If $x(t)$ is a non-trivial solution of the IVP, then $x(t) \neq 0$ for any t because if $x(t^*)=0$ for some $t^*$, then $x(t)$ must be identically zero, since it, and the trivial solution satisfy the same differential equation and have same value at $t=t^*$.

We know that the space V of all solutions of (1) is a vector space.

In our next result we determine the dimension of V.

***Theorem:*** The dimension of the space V of all solutions of the homogeneous linear system of differential equations (1) is n.

So, right now we will make use of this theorem, if xt is the non-trivial solution, now let us look at this remark if xt is the non-trivial solution of the initial value problem, this is the initial value problem x dot = Ax, where x at t0 is given by v vector x0, so if xt is non-trivial solution of this initial value problem, then xt cannot be 0 for any value of t because if xt = 0, for some value of t say, t = t star, then xt must be identically 0.

Because the trivial solution also satisfies the differential equation x dot = Ax so, if the trivial solution and then if xt star = 0 then the trivial solution and xt solution will coincide at t = t star and so by the uniqueness of the solution it will follow that xt = 0 for all t, so if xt is a non-trivial solution, then xt must be != 0 for any value of t. Now, we know that, we have already seen that the space V of all solutions one is a vector space.

So, now what we are going to do is; now the question is that how many solutions linearly independent solutions of the equation x dot = Ax we must find, so that we can write the general solution of x dot = Ax, that question is answered by this theorem. This theorem tells that the dimension of the space V of all solutions of the homogeneous linear system of differential equations one is n, okay.

So, this theorem tells us that if we can find an linearly independent solutions of the homogeneous linear system x dot = Ax, then we can write the general solution The general solution will be a linear combination of those an linearly independent solutions, so this theorem is a very important, let us prove this theorem, so what we do is; in order to prove this theorem, we need to show that the better space V as a basis, which has got an elements, okay

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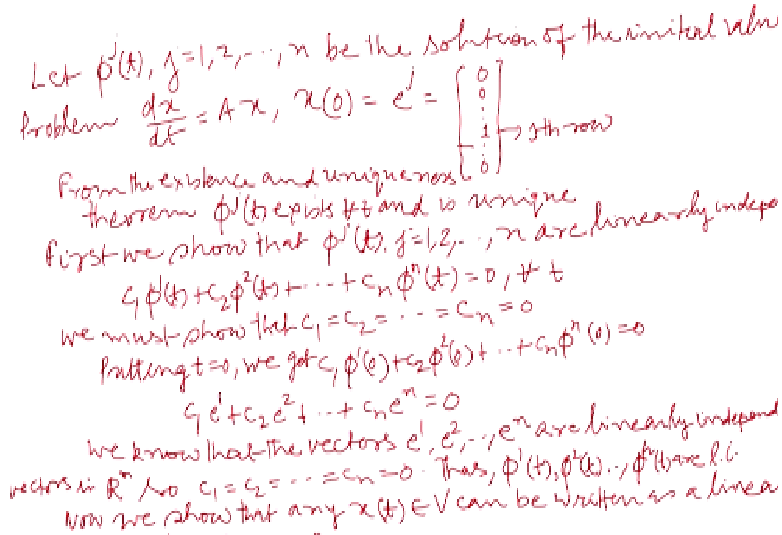


So, what we do is; let us say, let phi jt, j = 1, 2 and so on up to n, okay be the solution of the initial value problem, dx/dt = Ax, x0 = ej, where ej denotes the vector, column vector, 0, 0, 0 and so on 1; 1occurs in the gth row and so on up to 0, okay and now, from the existence and uniqueness theorem, phi jt exist for all t and is unique, okay, so let phi jt be the solution; phi jt, where j runs from 1, 2and so on up to n with the solution of the initial value problem.

So, this is the initial value problem, you can see; dx/dt = x; the value of x at 0 is the vector ej, okay, where one occurs in the j through, for example, if you write for j =1, then phi and t will be solution of dx/dt = x and x at 0 will be 1, then 0, 0, 0, ends on up to 0, so we shall actually show that these n functions phi jt form basis for the solution; or the vector space V, okay, so if we can show that this n function phi jt form a basis for the vector space V, then the dimension of V will be = n, okay.

From the existence and uniqueness theorem, it follow that phi jt exists and is unique. Now, what we do is; first we show that these n functions phi jt are linearly independent; first we show that phi jt, j = 1, 2 and so on up to n, these n functions are linearly independent, okay. So, for this, what we will do? Let us write the equation, c1 phi 1t + c2 phi 2t and so on cn phi nt = 0, where c1, c2, cn are constants.

So, if we want show that if phi 1t, phi 2t, phi nt are linearly independent then we must show that c1, c2, cn are all 0's, so we must show that the c1, c2, cn are all 0's, now since this equation, c1 phi 1t + c2 phi2t and so on cn phi nt = 0 is valid for all t, okay, let us put t = 0 in this equation, so putting t = 0, we get c1 phi 1 0+ c2 phi 20 and so on cn phi n0, okay, now phi n0 = e1, okay we get c1 e1, then c2 e2 and so on cn en = 0.

Now, we know that the vectors e1, e2 and so on en are linearly independent vectors in Rn okay, so c1 must be = 0, c2 must be = 0 and so on cn must be =0 and thus phi 1t, phi 2t and so on phi nt are linearly independent, so thus the n functions, phi 1t, phi 2t and so on phi nt; phi nt are linearly independent. Now, we show that any xt belonging to V; V is the space of all solutions of the homogeneous system x dot = Ax.

Any xt belonging to V can be written as a linear combination of phi 1t, phi 2t and so on phi nt, so that will mean that the span of the vectors phi 1t, phi 2t and so on phi nt is = V and since we have already shown the linear independence of phi 1t, phi 2t and so on phi nt, it will follow that they form a basis for the vector space V.

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Let $x(0) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

Then let us construct the function

$$\phi(t) = c_1\phi^1(t) + c_2\phi^2(t) + \cdots + c_n\phi^n(t)$$

Then $\phi(t)$ is clearly a solution of $\dot{x} = Ax$.

$$\phi(0) = c_1\phi^1(0) + c_2\phi^2(0) + \cdots + c_n\phi^n(0)$$

$$= c_1e^1 + c_2e^2 + \cdots + c_ne^n = c_1\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + c_n\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = x$$

By existence and uniqueness theorem $x(t)$ and $\phi(t)$ must be same. Thus, $x(t) = c_1\phi^1(t) + c_2\phi^2(t) + \cdots + c_n\phi^n(t)$.

Hence $\phi^1(t), \phi^2(t), \ldots, \phi^n(t)$ form a basis of $\dot{x} = Ax$

$\therefore \dim V = n$.

Now, what we do is; let us say that x0, okay = c1, c2 and so on cn, xt is any vector belonging to the vector space V, okay, we want to show that xt is a linear combination of phi 1t, phi 2t and so on phi nt, so let us assume that at xt function at t = 0 is given by this column vector c1, c2, cn, then let us construct the function phi t = c1, phi 1t c2, phi 2t and so on cn phi nt. Now, since phi 1t, phi 2t and so on phi nt is a solution of x dot = Ax, okay.

Then, their linear combination c1 phi 1t + c2 phi 2t, and so on cn phi nt will also be a solution of the equation vector differential equation x dot = Ax, now what we notice is that let us put 0 in this, so then phi t is clearly a solution of x dot = Ax, now phi 0 = c1 phi 10+ c2 phi 20 and so on cn phi n0 which is equal to c1 e1 + c2 e2 and so on cn en which is = c1 times 100, c2 times 0100 and so on cn times 0 0 and so on 01.

This will give us c1, c2 and so on cn, which is = x0 by our assumption. So, now what do we notice? We see that phi t is a solution of x dot = Ax, okay xt is also a solution of x dot = Ax and phi 0 and x0 are same, phi 0 matches with x at t = 0, so this means that by the existence and uniqueness theorem, it follow that the solution of the equation x dot = Ax must be unique, therefore phi t must be same as xt.

So, by existence and uniqueness theorem, phi t must be = xt, xt and phi t must be identical and thus replacing phi ty xt here, xt is = c1 phi 1t+ c2 phi 2t and so on cn phi nt, so the functions and functions phi 1t, phi 2t and so on phi nt generate the whole space V that is a linear combination of those n functions gives us any vector xt belonging to V, so hence phi 1t, phi 2t and so on phi nt form a bases of x dot = Ax okay.

And which means that dimension of V = n, so dimension of the vector space V = n, so this means that in order to write the general solution of the vector space V of the vector differential equation x dot = Ax, we need to find n linearly independent solutions of these vector differential equation x dot = Ax. Now, our next theorem tells us how to check the linear independence of the n solutions of the differential equation x dot = Ax.

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Since dim V=n, we need to find n linearly independent solutions of (1) in order to write general solution of (1).

The following theorem yields a test for the linear independence of solutions of (1) . It turns out that the n solutions $x^1, x^2, \ldots\ldots, x^n$ of (1) are linearly independent if and only if there values

$$x^1(t_0), x^2(t_0), \ldots\ldots, x^n(t_0)$$

at an appropriate $t_0$ are linearly independent vectors in $\Re^n$.

So, let us go to the theorem, here we have said that since dimension of V is = n, we need to find n linearly independent solutions of the system one that is x dot = Ax in order to write the general solution of the equation x dot = Ax, so the following theorem yield as a test for the linear independence of solutions of one. It turns out that the n solutions x1, x2, xn of one are linearly independent if and only if their values x1 at t0, x2 at t0 and xn at t0 at an appropriate t0 are linearly independent vectors in Rn.

So, you can see the; to check the linear independence of the n solutions, it is very easy what you do, you choose a suitable t0, a convenient t0, and then evaluate the values of the n functions; x1, x2, xn at t = t0 and check for the linear independence of the n vectors; x1 t0, x2 t0, xn t0 in Rn, so let us look at the theorem.

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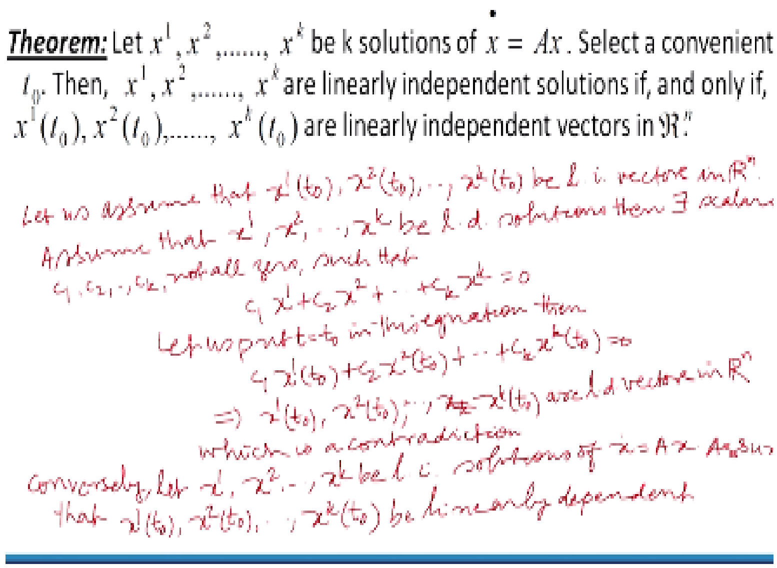


This theorem says that if x1, x2, xk be any k solutions of the vector differential equation x dot = Ax, then select a convenient t0, it follow that x1, x2, xk are linearly independent solutions if and only if x1 t0, x2 t0, and so on xk t0 are linearly independent vectors in Rn, so let us assume that x1 t0, x2 t0 and so on xk t0 be linearly independent vectors in Rn, okay let us assume that x1 t0, x2 t0 and so on xk t0 be linearly independent vectors in Rn.

Then we have to prove that x1, x2, xk are linearly independent solutions, so we shall prove it by contradiction method, so assume that x1, x2 and so on xk be linearly dependent; linearly dependent solutions, then why the definition of linear dependence of vectors, then they are adjusted to scalars; c1, c2 and so on ck not all 0 such that c1, x1 + c2 x2 and so on ck xk is = 0, this equation holds at for all t, okay, c1; that is c1x1 tx1 is a function of t, xt is a function of t.

So, c1 x1t, c2 x2t and so on ck xkt = 0, now let us put t = t0 in this equation, then we get c1 x1t0 + c2 x2 t0 and so on ck xk t0 = 0, okay which implies that the k vectors x1t0, x2t0 and so on xk t0 are linearly dependent vectors in Rn, so which is a contradiction because we have assumed that x1 t0, x2 t0, xk t0 are linearly independent, so which is the contradiction and thus it follow that if x1t0, x2 t0, and so on xk t0 are linearly independent vectors in Rn.

Then, the n functions x1t, x2t, and so on the k functions x1t, x2t and so on xk t are linearly independent solutions of the vector differential equation x dot = Ax, now let us prove the converse. So, conversely let us assume that x1, x2 and so on xk be linearly independent solutions of the vector differential equation x dot = Ax, we have to prove that the k vectors x1t0, x2t0, xk t0 are linearly independent.

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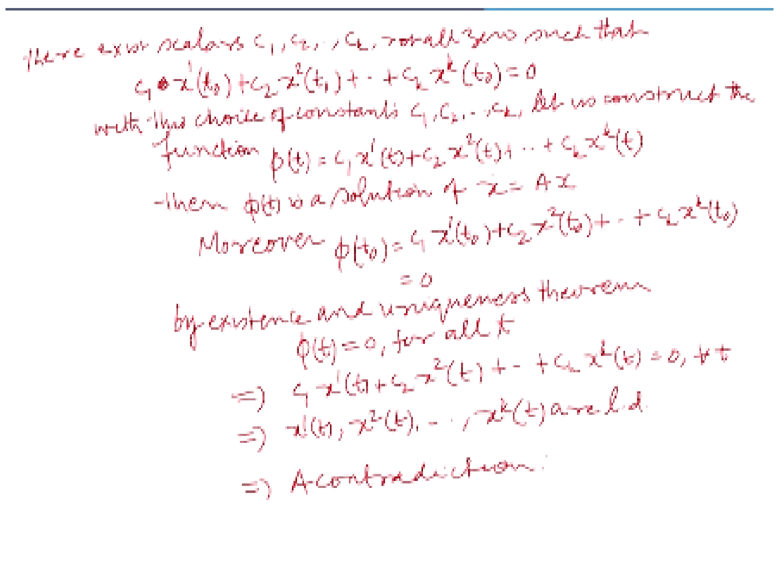


So, let us assume that the k vectors x1t0, x2t0, and so on xk t0 be linearly dependent, okay, so again we shall have to arrive at a contradiction in order to prove the result, so by the definition of linear dependence, there exist scalars, c1, c2 and so on ck not all 0 such that c1 times x 1t0 + c2 times x2 t0 and so on ck times xk t0 = 0, okay. Now, what we do is; so with these choice of constant c1, c2, ck; with this choice of; let us construct the function phi t = c1 x1t, c2 x2t and so on ck xkt, okay.

Then, since x1t, x2t, xkt are solutions of the vector differential equation x dot = Ax, okay, it follow that phi t is a solution of x dot = Ax, so then phi t is a solution of x dot = Ax, more over phi t0 = c1 x1t0, c2 x2t0 + ck xkt0, okay = 0, is it, c1 x1t0, c2 x2t0 + ck xkt0 = 0, so phi t0 = 0, so what happens is that phi t is the solution of the differential equation x dot = Ax such that phi t0 =0 and this means that phi t must be 0 for all values of t, okay.

So, by existence and uniqueness theorem, phi t and the trivial solution must be the same, so phi t =0 for all t, okay and which implies that c1 x1t + c2 x2t and so on ck xkt = 0 for all t and this tells us that the k functions x1t, x2t, and so on xkt are linearly dependent, okay and so we get a contradiction, so this proves the theorem. The k functions x1t, x2t, xkt are linearly independent if only if at an appropriate t0, the k vectors; x1t0, x2t0 and so on xk t0 are linearly independent.

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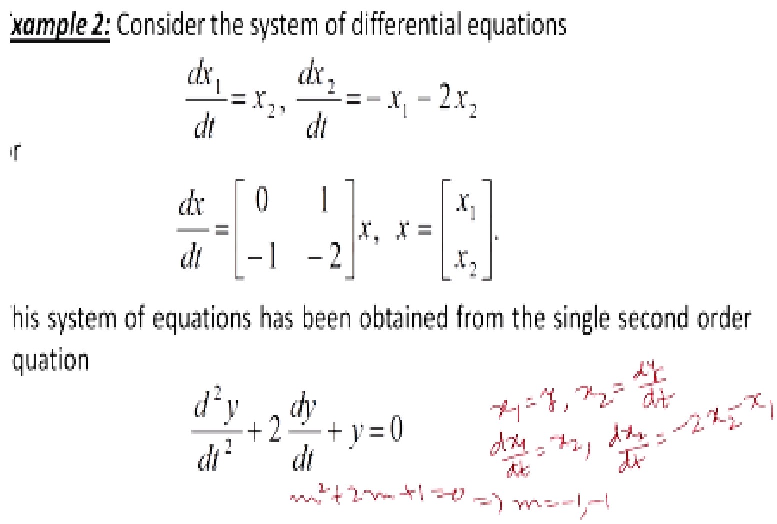


Now, let us consider the system of differential equations, dx1/dt = x2 and dx2/dt = -x1 -2x2, you can see we can write this in the vector differential equation form, dx/dt = now, coefficient of x1 is 0 here, so 0 coefficient of x2 is 1 and then here coefficient of x1 is -1, coefficient of x2 is`-2, so the matrix A is 0 1, -1, -2, now this system of equations has been obtained from the single second order differential equation, d square y over dt square + 2 dy over dt + y = 0.

You can see, here if you take x1 = y, x2 = dy/dt, then what you get? Dx1/dt = dy/dt, which is = x2 and dx2/dt will be = d square/ d2 square which is = - 2 dy/dt is x2 – y, so –x1, so you can see we have dx1/dt = x2 dx2/dt = -x1 - 2x2, so this system of differential equations has been obtained from the single second order differential equation d square y/ dt square + 2dy/dt + y= 0, let us find the solution of this differential equation; system of differential equation.

So, what we will do is; we will find first the solution of this second order differential equation which is a differential equation of second order with constant coefficients, so we will do is; we can write the auxiliary equation here, we have m square +2m+1 = 0 which gives us m = -1-1, so 2 values are m are same.

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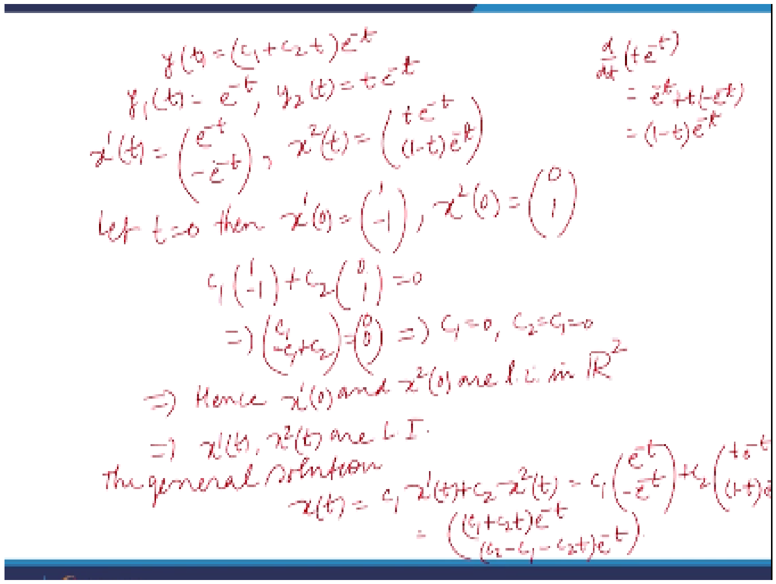


And therefore, the solution; general solution of this second order differential equation will be yt = c1 + c2t * e raise to the power –t, so we can 2 solutions of this second order differential equation as y1t = e to the power –t and y2t = t raise to the power –t, okay. Now, so we can get solutions of; we can get 2 solutions; x1t = e to the power -t the then – e raised to the power -t and x2t = e to the power -t and then we differentiate t e to the power –t d over dt of t e to the power –t =; y derivative of t is 1.

So, we get e power -t and then we get t times –e raised to the power –t, it should be get 1 - t * e raise to the power –t, so we get 1- t * e raised to the power -2-t, okay. Now, we can see the linear independence of these 2 solutions of the differential equation x dot = Ax, okay, what we do is; we choose a suitable value of t, just take t = 0, so then x10 is = 1-1 okay, x20 is = 0 and here we will get 1, okay.

So, 1-1 and 0, 1, we have to show that these 2 are linearly independent, so c1 times 1-1 + c2 times 01 = 0 gives us what you have? c1 +0, so c1 and then here –c1 + c2, so - c1 + c2 = 0; 0 is 00 okay, so this implies c1 = 0 and c2 = c1, since c1 is 0, c2 is also 0, so we get c1 and c2 both are zeros and hence, x10 and x20, the 2 vectors are linearly independent in R2. So, by the theorem, which we have just now done, if k vector is x1, x2, xk at t = t0 are linearly independent.

Then, the k functions x1, x2, xk are linearly independent, so x1t, these 2 solutions; x1t, x2t are linearly independent and so, we can write the general solution of the given system; general solution is xt = c1 times x1t + c2 times x 2t which is = c1 times e raise to the power –t – e raise to the power -t + c2 times t e raise to the power -t and 1 –t e raise to the power –t, so this gives you c1 + c2t * e raise to the power –t.

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Hence,

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} te^{-t} \\ (1-t)e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} (c_1 + c_2 t)e^{-t} \\ (c_2 - c_1 - c_2 t)e^{-t} \end{bmatrix}$$

is the general solution of the given system.

And then, we will get here what? C2 – c1- c2t * e raised to the power – t, so this also a solution of the coefficient x dot = Ax for all values of t, so these are general solution of the given system of equations.

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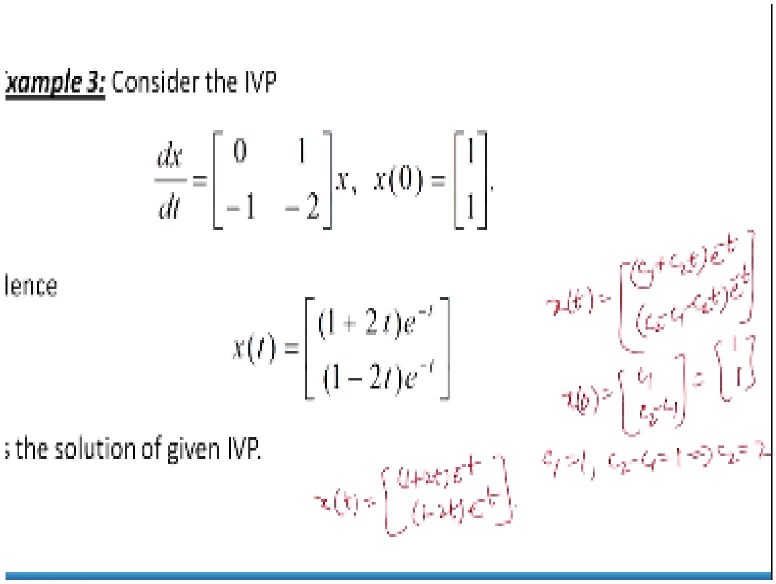


Now, here what I have done is; I have taken the initial value problem, the same better differential equation I have taken as in the case of example 2; dx/dt = 0, 1-1 2, only what I have done is; I have given the initial condition that is at t = 0, x is give 1 1 vector, okay, so we know that xt =; we have found xt =; okay c1 + c2t * e raised to the power –t, c1 + c2t * e raise to the power -t and then c2 - c1 – c2t * e raise to the power –t.

So, these general solution of the equation, x dot = Ax, if you want the solution of this initial value problem, then let us put t = 0, so x0 = what I will get? This is 0, this is 1, so we get c1 and here, we get c2 - c1, okay. Now, at (()) (51:39) be 1 1, so what I get? c1 is = 1 and c2 - c1 = 1 implies c2 = 2, so we xt =; c1 is 1, c2 is 2, so 1 + 2t e raise to the power - t and then here c2 is 2, c1 is 1, so 2-1 is 1, 1-2t * e raised to the power –t.

So, this is how we get the particular solution of the initial value problem by satisfying the condition at 0 = 1 1 in the general solution, so that is how we solve this homogenous system of linear differential equations. Now, in our next lecture, we shall discuss the method of eigenvalues and eigenvectors by which we can find the general solution of the homogeneous system of linear system of differential equations. Thank you very much for your attention.