Ordinary and Partial Differential Equations and Applications
Dr. D. N. Pandey
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 06
Existence and Uniqueness of Solutions of a System of Differential Equations

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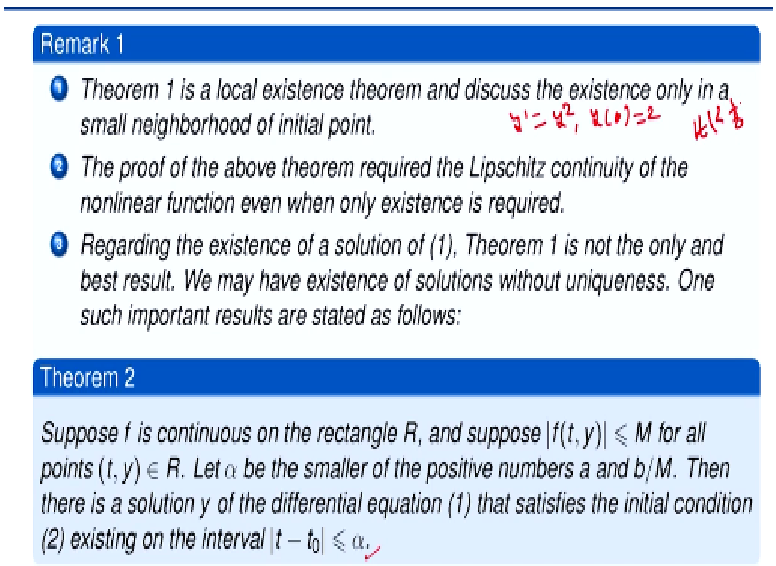

Hello friend, welcome to the lecture, here we continue our decision which we have started in previous lecture, so what we have seen in previous lecture that this existence and uniqueness theorem, which we have consider is a local existence theorem and only discuss the existence in a small neighbourhood of initial point and that we have seen using one example that is y dash = y square with the initial condition y0 = 2.

It means that we have consider this y dash = y square and y0 = 2, here we have shown that the solution actually exist in a large interval but if we use our existence and uniqueness theorem and then this interval is quite a small that is given as modulus of t < 1/8, so here this says that our theorem is a local existence theorem, which gives a solution only in a small neighbourhood of initial point.

And the proof of this existence and uniqueness theorem requires a Lipschitz continuity of the non-linear function even when we only required existence of the solution, so for this, we will give not only existence but also guaranteed the uniqueness of the solution but in some cases, we need only the existence, there this is a kind of a strict theorem, it is requiring a lot of more thing, so in this regard, the existence of a solution at this theorem 1 is not the only and best result.

We have one more result which is; which gives the existence of solution without uniqueness, without guarantee, without giving a guarantee for uniqueness and one such important results, we can give it right now that suppose, f is continuous on the rectangle R and suppose modulus of f (t, y) is bounded by some constant M for all point t, y in that rectangle R and let alpha be this one of the positive numbers a and b/M that we have already defined as a rectangle.

Then, there is a solution y of the differential equation 1 that satisfies the initial condition to existence on the interval modulus of t – t0 <= alpha, the only difference between this and the previous existence and uniqueness theorem that here we have not assume that f (t, y) satisfy the Lipschitz condition. So, here this Lipschitz condition is remove, so by removing this, we lost the guarantee of uniqueness.

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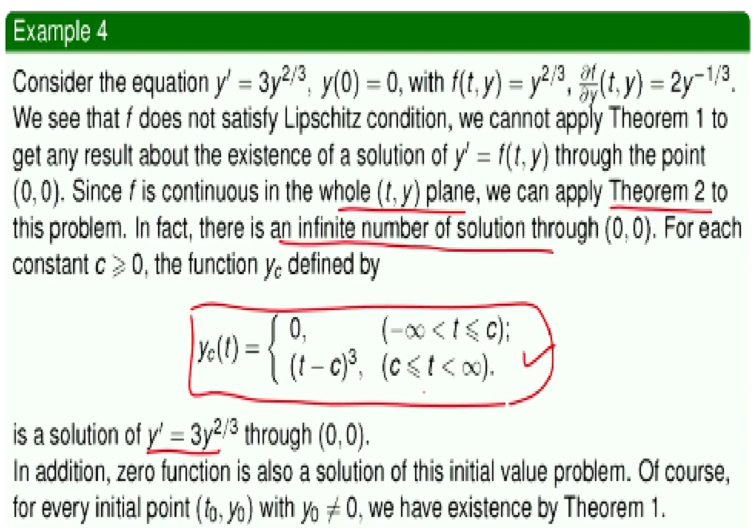

So, here we have only the guarantee of existence and solution may be unique may not be unique, so let us look at what example? So, consider the question y dash = 3y to the power 2/3, y0 = 0 here and initial condition is given as y0 = 0 and if you look at your f (t, y) is given as y to the power 2/3 and we can check that it does not satisfies the Lipschitz condition in several way and if you look at, you can find out the partial derivative dou f/dou y ty = 2y to the power -1/ 3.

And we can also check that in direct thing that I have does not satisfy the Lipschitz condition and hence we cannot apply existence and uniqueness theorem 1 to get any result about the existence and uniqueness of the solution in fact that will not also give the condition that it has a solution also. So, here we observed that f is continuous in the whole t, y plane and we can apply the previous theorem that is theorem 2 to this problem.

In fact, these are infinite number of solutions 0, 0, so since f (t, y) is continuous we can apply the previous theorem; theorem 2, we can say that this f (t, y) is bounded by; if we define our rectangle in a proper way then we can say that this f (t, y) is continuous and in a close rectangle, it will achieve the maximum value that is some constant M, so here we guarantee the existence of a solution, we may not have the uniqueness.

And we have seen that in particular; in this particular problem, we have an infinite number of solution passing through 0, 0 and we can say that for every constant c > 0, the function yc defined by this yc t = 0; 0 is between – infinity to c, t – e to the power 3, where t is defined from c to infinity is a solution of this y dash = 3y to the power 2/3 and which passes through 0, 0 and this is a one parameter family of solution.

In fact, you can define 2 parameter family of solution of the same problems, so it means that here not only the solution exist but solutions are infinite in number, so in addition zero function is also solution of this initial value problem and of course for every initial point t0, y0 with y0 !=0, we have existence by theorem 1. So, if you look at this y0 is = 0, this initial condition is creating problem for this initial value problem.

If we remove this initial condition y0 by some nonzero number then we can apply our existence and uniqueness theorem and we can say that it has a unique solution, so if we consider y0 ! = 0, we have a unique solution in a rectangle, where y0 != 0 is a point but if we have; if we consider a rectangle where we have y0 = 0, then we will lose the uniqueness and in fact, we can see that we have one parameter family of solution given by this equation, y ct = this thing.

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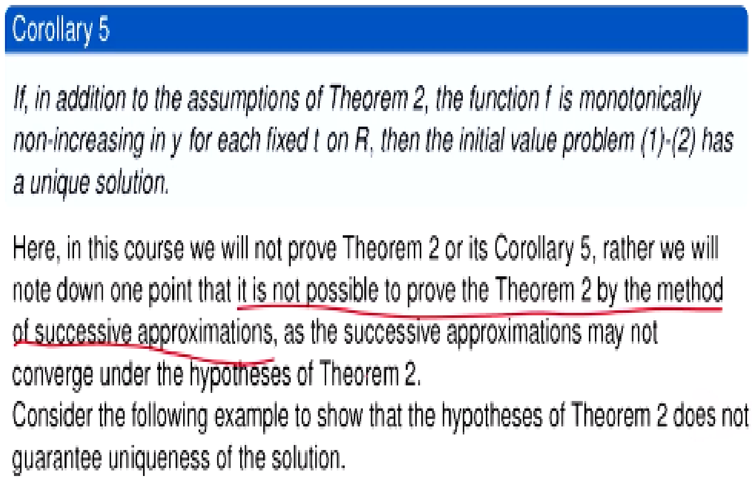

Or in fact, if you do a little bit more, we can define 2 parameter family of solution of the same problem, so moving on next we say that so, what we have done? We remove the Lipschitz condition on f and we lost, we have no guarantee that solution is unique or not but if you look at; if in addition to this assumption of theorem 2, the function f is monotonically non increasing in y for each fix t on R, then the initial value problem 1, 2 has a unique solution.

So, it means that to compensate the loss, if we put one more addition condition that f is monotonically non increasing in y for each team, then we regain the uniqueness of the solution. Here, we will not provide any solution of the theorem 2 and this corollary because it is little bit involved and it requires some more advance theory, so we are not giving this thing but we observed this thing that it is not possible to prove the theorem 2 by the method of successive approximation that one thing we have to point out.

That here, the proof of this theorem 2 and this corollary 5 is not given with the help of successive approximation because here as a successive approximation may not converge under the hypothesis of theorem 2, so we can look at the following example where we have seen, we will see that the condition given in theorem 2 is not sufficient to give surety for convergence of approximate; successive approximations.

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Example Consider the function f defined in the region D in the (t, y) plane, where D is given by $-\infty < t < 1$, $-\infty < y < \infty$, by

$$f(t,y) = \begin{cases} 0, & (-\infty < t \leqslant 0, -\infty < y < \infty) \\ 2t, & (0 < t < 1, -\infty < y < 0) \\ 2t - \frac{4y}{t}, & (0 < t < 1, 0 \leqslant y \leqslant t^2); \\ -2t, & (0 < t < 1, t^2 < y < \infty). \end{cases}$$

This function f is continuous and bounded by the constant 2 on D. The successive approximations to the solution y of $y' = f(t, y)$ through the initial point $(0, 0)$ are given by

$$y_0(t) = 0$$
$$y_{2k-1}(t) = t^2$$
$$y_{2k}(t) = -t^2 \quad (0 \leqslant t \leqslant 1; k = 1, 2, \ldots)$$

So, let us consider this example, consider the function f define in the region D in the t, y plane, where D is given by - infinity to 1 and – infinity to + infinity and f (t, y) is given as 0, when – infinity to t <= 0 and – infinity < y < infinity and 2t when 0 < t < 1 and – infinity < y < 0 and similarly, we can define in different, different region, the function f (t, y) and here we can say that f (t, y) is continuous and bounded by the constant 2 on D.

So, continuous to see that it is continuous, just look at the function f (t, y) and you can see that if we move from right hand side to left hand side, you will get the same value, so it is a continuous function and it is bounded by 2 on domain D and if you want to find out the successive approximation for this particular problem y dash = f (t, y) and we can check that the successive approximation is given by y0 t = 0.

And y2 K – 1t = t square and y2 Kt = -t square and it means that all the odd successive approximation is given by t square and all the even approximation is given by –t square and so that your solution is oscillating between the t square and -t square and since it is oscillating, this approximation will not converge to the solution which we are searching for.

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Thus the successive approximations alternate between t^2 and $-t^2$ and do not converge. Since the function $f(t, y)$ is continuous and bounded on D, therefore by Theorem 2, we have the existence of a solution. Also, since f is monotonically nonincreasing in y for every fixed t, Corollary 5 gives us the assurance of existence of a unique solution.

Thus we may conclude the following.

1. Assurance of merely existence of solution does not require Lipschitiz condition, only continuity of f is required in y.
2. Here approximations do not converge, but still we have a unique solution, thus continuity of f and uniqueness of solution do not imply the convergence of approximate solution.
3. The uniqueness results and convergence of successive approximation are two different independent phenomena.

So, here we can say that there is a successive approximation alternate between t square and –t square and do not converge. Since the function f (t, y) is continues and bounded, so our theorem 2 will be applicable and we can say that existence of a solution exist and not only this, we can observe that this function f is monotonically non increasing in y for every fixed t and so corollary 5 is also applicable and this gives us that we have a unique solution of this particular problem.

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Example Consider the function f defined in the region D in the (t, y) plane, where D is given by $-\infty < t < 1$, $-\infty < y < \infty$, by

$$f(t,y) = \begin{cases} 0, & (-\infty < t \leqslant 0, -\infty < y < \infty) \\ 2t, & (0 < t < 1, -\infty < y \leqslant 0) \\ 2t - \frac{4y}{t}, & (0 < t < 1, 0 \leqslant y \leqslant t^2); \\ -2t, & (0 < t < 1, t^2 < y < \infty). \end{cases}$$

This function f is continuous and bounded by the constant 2 on D. The successive approximations to the solution y of $y' = f(t, y)$ through the initial point $(0, 0)$ are given by

$$\begin{aligned} y_0(t) &= 0 \\ y_{2k-1}(t) &= t^2 \\ y_{2k}(t) &= -t^2 \quad (0 \leqslant t \leqslant 1; k = 1, 2, \ldots) \end{aligned}$$

$y_1 = \int_0^t f(s, y_0(s))\,ds = \int_0^t 2s\,ds = t^2$

$y_2 = \int_0^t f(s, y_1(s))\,ds$

So, not only this, so here we can see that this here, successive approximation do exist but they will not converge to a solution but it still this problem has a solution and not only that solution exists, it will have a unique solution. So, how to find out this y0 = 0 and 2K – 1t square, I will just show you this condition, y1 is going to be what? Since y0 is given as 0 here, so here we simply say it is 0 to t and f as y of s, y0s and ds, now this is y0 s, y0 is given as 0.

So, fs0 is going to be, it is 2t, so here it is equality is given, so it is given as 2t, so we can say that 0 to t and 2s ds, so it is given as say, t square, so here we can say that y1 is given at t square, now to find out y2; y2 is given as 0 to t and f of s and y1 s ds. Now, y1 s is t square, so it will lie here, so here we write f sys as 2t – 4y1 upon t, so what is this, let me do it here.

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$$y_2 = \int_0^t \left(2t_1 - \frac{4y_1}{t_1}\right) dt_1 \qquad y_1 = t^2$$

$$= \int_0^t \left(2t_1 - \frac{4 \times t_1^2}{t_1}\right) dt_1$$

$$= \int_0^t (2t_1 - 4t_1)\, dt_1 = \int_0^t -2t_1\, dt_1$$

$$= -t^2$$

So, here we have y1; y2 as 0 to t, it is 2t – 4y upon t, dt, so here, we have y1, so y1 is given as t square, so here we can write it 0 to t 2t -4 * t square upon t and dt, so here there is a small abuse of notation, I should write 2s - 4y1 upon s ds, so if you have no ambiguity let me write it here, so here we can write this as 0 to t 2t - and this I can write it 4t dt and this we can write it 0 to t – 2t dt, so that is given as - of t square.

So, please here, we let us write it t1 here, so that we should not have any confusion, so here we have that y2 is given as –t square and we have seen that y1 is given as t square and similarly, you can calculate y3, y4 and so on and we can observe that y0 t is 0 and y; all the odd approximation given by t square and all the even given by –t square, so here approximation; approximate solution will not converge to any solution.

And so we cannot apply theorem 1 to find out the solution of this particular problem but we can apply theorem 2 and corollary to find out that it has a solution and not only the solution, it will have a unique solution, so we may conclude the following thing, that assurance of merely existence of solution does not require Lipschitz condition, so it means that Lipschitz condition is required only for uniqueness.

If we drop the uniqueness Lipschitz condition that may not guarantee whether the solution exists or not, so we can say that if you want only for existence then we do not require Lipschitz condition, we need only continuity, only continuity of f is required in y, so only if we require only the existence, continuity of f is sufficient, now here approximation do not converge but it still we have a unique solution.

So, it means that the approximation solution converge and having a unique solution, these two are two different thing, so it means that continuity of f and uniqueness of solution do not imply the convergence of approximate solution, so in particular and in previous problem, we have a unique solution, we have so; but still this successive approximate solution does not converge, so we can say that their uniqueness result and convergence of successive approximation are 2 different independent phenomena.

So, it means that here it is not related, we in previous class; in previous lecture we have seen that Lipschitz continuity is required for uniqueness of the solution but here, we have shown that we have a unique solution but it still we do not require Lipschitz condition; Lipschitz continuity and we also do not have the convergence of successive approximation, so we say that convergence of successive approximation and uniqueness are 2 different phenomena, okay.

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Existence Theory for Systems of First- Order Equations

We now wish to consider the extension of existence and uniqueness theorem to system of first order equations of the form

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}) \tag{5}$$

where **y** and **f** are vectors with n components and t is a scalar.
Let **f** be continuously differentiable with respect to t and with respect to the components of **y** at all points of D and suppose that there exists a constant $K > 0$ such that the norms of $\partial \mathbf{f}/\partial y_j$ satisfy

$$\left| \frac{\partial \mathbf{f}}{\partial y_j}(t, \mathbf{y}) \right| \leq K \quad (j = 1, 2, \ldots, n) \tag{6}$$

norm

Now, let us move forward for a system of first order equation and we try to discuss the existence and uniqueness theorem which we have developed for a scalar differential equation for the system of first order equation. So, here we are looking of a system of first order question, we now wish to consider this extension of existence and uniqueness theorem 2 system of first order equation of the form, y dash = f (t, y).

So, here, this represent what? Here, y and f of vectors with n components and t is a scalar, so I can write this as the following, so here, we have y dash = f (t, y), so here, this y is vector of dimension n cross 1, similarly, your f is also vector of n cross 1 and we can write this as y1 dash = f1 t, y1, y2 and yn here, similarly, y2 dash = f2 t, y2, sorry, y1, y2 and yn and so on, so it means that this y dash = f (t, y) represent this system of first order ordinal differential equation.

So, here we can write this y1 to yn as y here, so it is n cross 1 here, similarly, we can write f as f1 and so on to fn, so here if we use a vector notation as this, then we can write down this system of first order equation as y dash = f (t, y), so that represents the system of first order ordinal differential equation and here, we assume this condition that let fb a continuously differential with respect to t and with the respect to the components of y at all points of D.

And suppose that they exist a constant K > 0 such that the norms of dou f/ dou y j satisfy this following condition. So, here to get that existence and uniqueness theorem, we talk about the first theorem, which we have discussed which is known as Picard iteration scheme, so to generalise that we assume that f is a continuously differentiable function, so here we assume this first thing with respect to t and with respect to the component yf.

So, it means that here, dou f/ dou yj ty is <= K, here this represent norm, I hope that you are aware about this function norm, it is a generalised distance function satisfying certain condition, so we are using this as a norm here.

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Then $\mathbf{f}$ satisfies Lipschitz condition in D as for

$$G(\sigma) = \mathbf{f}(t, \mathbf{z} + \sigma(y - z)) \qquad 0 \leq \sigma \leq 1$$

and consider $\mathbf{f}(t, \mathbf{y}) - \mathbf{f}(t, \mathbf{z})$. We have

$$\mathbf{f}(t, \mathbf{y}) - \mathbf{f}(t, \mathbf{z}) = \mathbf{G}(1) - \mathbf{G}(0) = \int_0^1 G'(\sigma) d\sigma$$

By the chain rule, letting $\mathbf{f}_{y_j} = \frac{\partial \mathbf{f}}{\partial y_j}$ $(j = 1, 2, \ldots, n)$, we have

$$\mathbf{G}'(\sigma) = \mathbf{f}_{y1}(t, \mathbf{z} + \sigma(y - z))(y_1 - z_1) + \mathbf{f}_{y2}(t, \mathbf{z} + \sigma(y - z))(y_2 - z_2) + \cdots + \mathbf{f}_{yn}(t, \mathbf{z} + \sigma(y - z))(y_n - z_n)$$

$$|f(t, \mathbf{y}) - f(t, \mathbf{z})| = \int_0^1 |G'(\sigma)| d\sigma \leq K\{|y_1 - z_1| + |y_2 - z_2| + \ldots + |y_n - z_n|\} = K|\mathbf{y} - \mathbf{z}|$$

i. e. a function $\mathbf{f}$ satisfying an inequality of the form (6) for any $(t, \mathbf{y})$, $(t, \mathbf{z})$ in D

So, if we have these condition, then we can say that then f satisfies the Lipschitz condition in D, so it means that here we are assuming that the partial derivative of f exist with respect to the component yj and it is bounded, then we can say that it satisfies, then it guarantees that the function f satisfies the Lipschitz condition and how we can guarantee that here we can write, we can define a new function g sigma, which is given as f (t, z + sigma y – z).

And if you look at this is defined as one variable of sigma, here if you put sigma = 0 then it is nothing but f (t, z) but if you take sigma as 1, then it is nothing but f (t, y), so with the help of G sigma, we define this value, this ft y – ft z as G1 - G0, so this G1 – G0 is given by 0 to 1, G dash sigma D sigma, so using mean value theorem, we can write G1 - G0 as 0 to 1 G dash sigma D Sigma.

And by the chain rule, we can find out G dash sigma as this, so here we use a notation, fyj = dou f/ dou yj for j = 1 to n and we can write G dash sigma as fy1, here we are using of finding the derivative of G sigma with respect to sigma, so here we can write f y1 t z + sigma y - z and multiplied by y1 – z1 + fy2, so here partial derivative of with respect to y2 augmentally same y2 – z2 and so on.

So, here we can write norm of fty – ftz is = 0 to 1 G dash sigma D sigma and G dash sigma is given by this quantity, so we can write this as since fy1, this is bounded by K, this is bounded by K and all these are bounded by K, so we can take it out and what is left here is modulus of y1 – z1 + modulus of y2 – z2 and so on and this we can say that this is nothing but a norm of y – z, here we are using one norm of y.

So, here we define what is one norm of y, so one norm of y is given as; if y is given as y1 to yn, then one norm of y1 is given as modulus of y1 + modulus of y2 and so on and modulus of yn, so here we are using one norm and we can say that modulus of f ty – f tz is <= K times y – z and which says that that function f ty satisfies the Lipschitz condition in the variable y, so it means that if we assume only this condition that partial derivative exist with respect to component of y.

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Theorem: Let f and $\frac{\partial f}{\partial y_j}$ $(j = 1, 2, \dots, n)$ be continuous on the box $B = \{(t, y) : |t - t_0| \le a, |y - \eta| \le b\}$, where a and b are positive numbers, and satisfying the bounds

$$|f(t, y)| \le M, \quad \left|\frac{\partial f}{\partial y_j}\right| \le K \quad (j = 1, 2, \dots, n) \qquad (7)$$

for (t, y) in B. Let α be the smaller of the numbers a and $\frac{b}{M}$ and define the successive approximations

$$\begin{cases} \phi_0(t) = \eta, \\ \phi_n(t) = \eta + \int_{t_0}^{t} f(s, \phi_{n-1}(s))ds \end{cases} \qquad (8)$$

Then the sequence $\{\phi_j\}$ of successive approximations converges uniformly on the interval $|t - t_0| \le \alpha$ to a solution $\phi(t)$ of (5), that satisfies the initial condition $\phi(t_0) = \eta$.

And it is bounded then f satisfies the Lipschitz condition, which is verified here, so now we define the theorem that let f and dou f/ dou yj be continues on the box, B(t, y), where, modulus of t – t0 is <= a and norm of y – eta < or b, here eta represent the initial condition y at t0, where a and b are positive numbers and satisfying the bound, modulus of f (t, y) is <= M, here, sorry, it is norm of f (t, y) <= M and norm of dou f/ dou yj <= K, 40 y in b.

And here, let us take the alpha be the smaller the number a and b/M and define the successive approximation as we define in theorem 1, phi 0 t as eta and phi n t as eta + t0 to t, f (S, phi n – 1S dS), then this sequence phi j of successive approximation converge uniformly on the interval modulus of t – t0 <= alpha, to a solution phi t of 5, that satisfies initial condition phi t0 = eta and the solution is given as we have done in case of theorem 1.

The only thing is that here modulus is replaced by norm, so here as we have not given emphasis on this, this represent norm in vector case, so it needs a theorem 1, you can exactly write it here, the only understanding we have to assume that this is not a modulus sign, this is a norm in n dimension, so similarly, we can define the uniqueness of solution.

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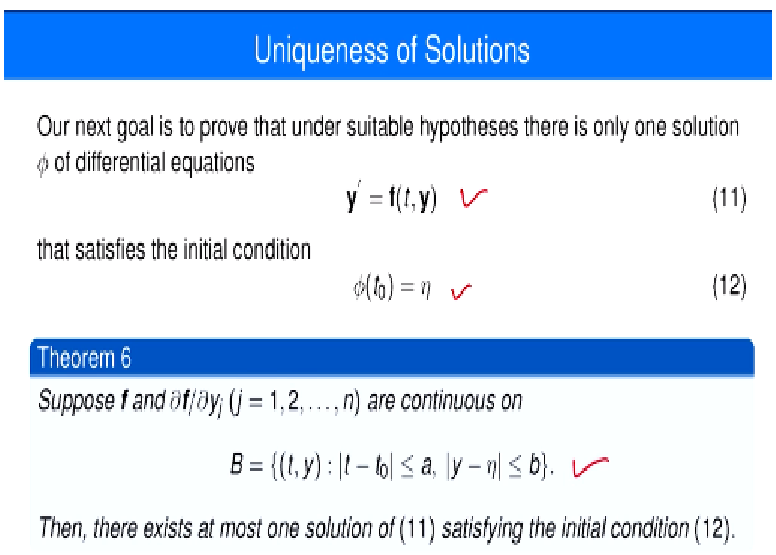

So, our next goal is to prove that under suitable hypotheses, there is only one solution phi of differential equation, y dash = f (t, y) with the initial condition, phi t0 = eta and here, the condition is Lipschitz condition, so suppose f and delf/ del yj, j from 1 to n are continuous on this rectangle b, then they exists at most one solution of one satisfying the initial condition, the 12.

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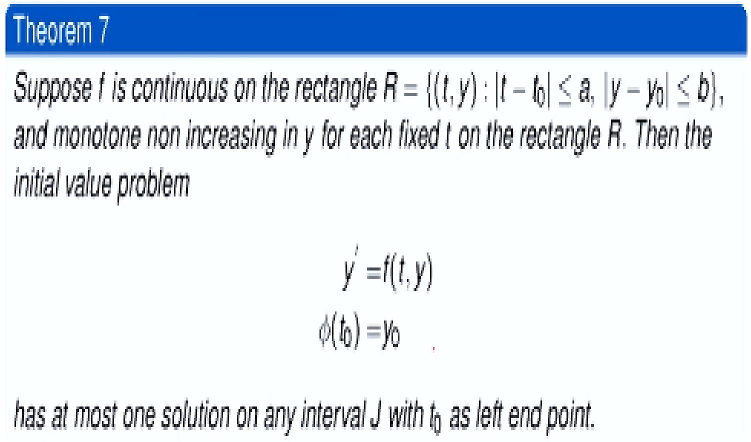

So, here we can say next theorem that suppose f is continuous on the rectangle R to define like this and monotone non increasing in y for fixed t on the rectangle R, then the initial value problem has at most one solution on any interval j with to as left end point. So, here we have discussed this theorem that is existing theory for system of first order equation that if f is continuously differentiable and satisfy these equation number 6 that partial derivative exist and bounded by K, the solution exist.

And not only the solution exists, it will have a unique solution that is given by uniqueness of solution, so the same condition is also true for having existence of unique solution of this and if we assume only the continuity part, then we have one solution of the system.

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Example 8

Let $g(x, u) = (a_{11}u_1 + a_{12}u_2, a_{21}u_1 + a_{22}u_2)$. Then

$$\begin{aligned}
\|g(x,u) - g(x,v)\| &= \|a_{11}(u_1 - v_1) + a_{12}(u_2 - v_2), a_{21}(u_1 - v_1) + a_{22}(u_2 - v_2)\| \\
&= |a_{11}(u_1 - v_1) + a_{12}(u_2 - v_2)| + |a_{21}(u_1 - v_1) + a_{22}(u_2 - v_2)| \\
&\le |a_{11}||u_1 - v_1| + |a_{12}||u_2 - v_2| + |a_{21}||u_1 - v_1| + |a_{22}||u_2 - v_2| \\
&= [|a_{11}| + |a_{21}|]|u_1 - v_1| + [|a_{12}| + |a_{22}|]|u_2 - v_2| \\
&\le \max |a_{11}| + |a_{21}|, |a_{12}| + |a_{22}|[|u_1 - v_1| + |u_2 - v_2|] \\
&= \max |a_{11}| + |a_{21}|, |a_{12}| + |a_{22}|\|u - v\|.
\end{aligned}$$

Hence,the Lipschitz constant is

$$L = \max\{|a_{11}| + |a_{21}|, |a_{12}| + |a_{22}|\}$$

So, here we will take just one quick example and here, we look at the problem y dash = g (t, u), g (t, y) here and here y is given as y1 and y2 and so it is in 2 dimension and we want to find out that here this g (x, u), which is defined as a11 u1 + u12 u2, a2 1 u1+ a22 u2, then this function g is actually satisfying the Lipschitz condition, here we are not applying the result that partial derivative exist with respect to the component.

Here, we can use other method to show that it has a; it satisfy the Lipschitz condition for that just look at norm of gxu – gxv, now what is gxu? gxu is given by this, -gxu also put it here, so it is norm of u11, u1 – u1 1 + u12 u2 – u2, a21, u1 – u1 + a22, u2 – u2, so here we can simply write that it is <=; here, we define the norm y as one norm, so norm 1 as y1+ y2, modulus of this, so using one norm, we can write this as, this is nothing but modulus of u11, u1 – v1 + u12 u2 – v2 + here, we are using one norm, modulus of a21, u1 – v1 + a22, u2 – v2.

So, and this is <=; same quantity, you can write it and here if we write it a11 + a21, then multiply u1 – u1 + this quantity multiplied by u2 – v2, so if we take the maximum of this quantity, this quantity or you can say maximum of modulus of u11+ u21, a12 a22, then we can write it modulus of u1- u1+ u2 – u2 and if you look at this is nothing but the one norm of u – v, so it means that we can write that here, this <= maximum of a11+ a 21, a12 + a22.

Then, norm of; so one norm of gxu – gxv is <= K, which is denoted by this maximum value times u norm of u – v, so here, what we have shown here is a following thing that gxu – gxv, 1 norm is <= K times u – v one norm and one norm is defined like this that is these sum of the absolute value of the component. So, here we have shown that that g satisfy the Lipschitz condition in one norm.

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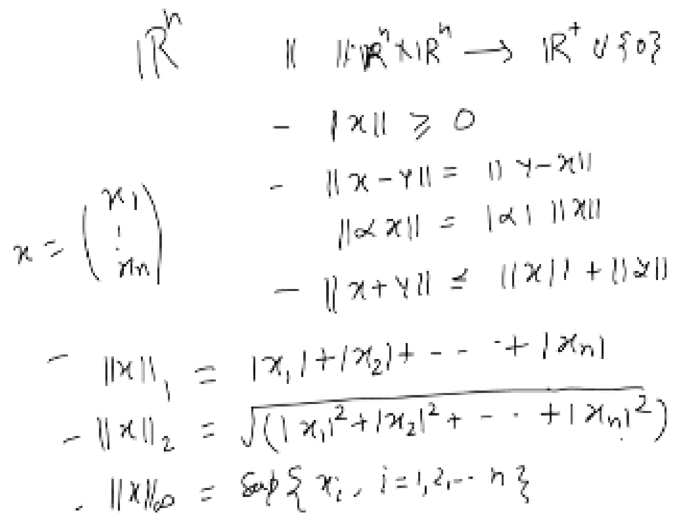

And here, we have note down one important thing that in Rn, here let me write some result that here in Rn, all; we can define several norm, so let me write it few norms, so first of all what is norm here? A norm is basically a function from Rn cross Rn to say R + union single term 0 and it satisfy certain properties that is norm of x is >= 0, this is one property, another properties that norm of x – y is same as norm of y - x or you can say that norm of alpha x is given as modulus of alpha norm of x here.

And it satisfy one more important property that norm of x + y is <= norm of x + norm of y, so it is a generalise distance function we can say and in Rn, we can define several norms and the common norm is a norm 1 that is if we look at x as x1 to, say xn, then one norm is given as modulus of x1 + modulus of x2 and so on, modulus of xn, so this is one norm and you can define 2 norm like this that it is under root of x1 square + x2 square and so on and infinity norm which we can say that it is supreme of xi and I is from 1, 2 and n.

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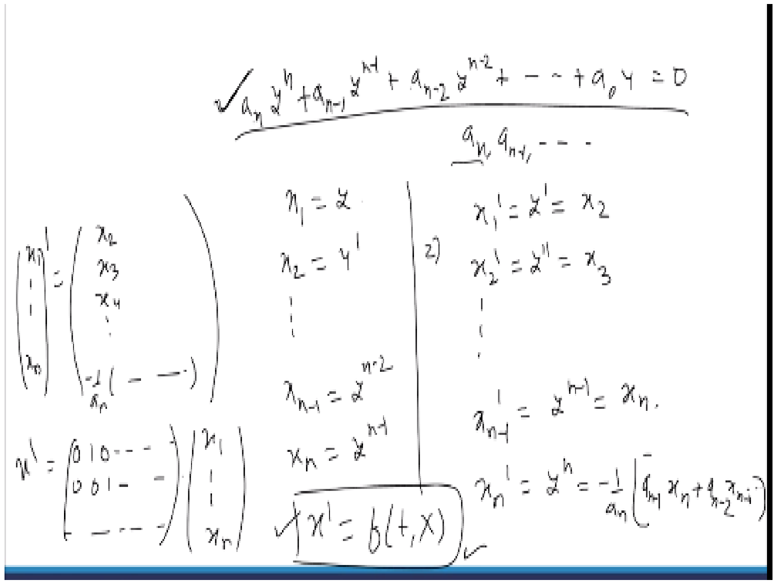

So, here we can say that we can define several norms; 1, 2, infinity here, it is the true for all; right and here we can also discuss that why this system is important, why we have written only for system of first order equation because if you look at any nth order differential equation, say yn + an yn an – 1 yn – 1 an -2 yn -2 and so on. So, here we can say that if you look at any this nth order differential equation, here this an maybe functions.

So, here an’ s; an – 1 and so on all these are maybe function or maybe constant values, so we can say that this can also be written in system of first order equation, the only thing we here, we have observed that x1 is let us say y and x2is y dash and so on we can define xn -1 as y; xn is y and x2 as y dash, so xn -1 is yn – 2, so xn is given as yn -1, so here we can write this as say, x1 dash is basically y dash, so which is nothing x2.

So, x2 dash is y double dash that is nothing but x3 and so on we can define xn -1 dash is = yn -1 that is given as xn and xn dash is given as yn and this is; this you can write it -1 upon an provided that this an is nonzero and you can write down this as this an -1, yn -1 is given as xn + an - 2 and xn -1 and so on, so here we can simply say that this I can write it here as x1 to say xn dash is =; now here, what we have written x1 dash = x2, so it is x2, x3, x4 and so on.

Here, we have written -1 upon an and this thing and if you further simplify this, you can write this as x1 to xn and here we can write down, 0, 1 and so on, 0, 0, 1 and so on, so I request you to complete this that any nth order differential equation with coefficient maybe constant, maybe function can be written as x dash = this thing and we can write that it is a particular case of this quantity.

So, it means that if we know the existence and uniqueness theorem for this system of first order equation, then we know existence and uniqueness theorem for this, so it means that here we have assumed that our function f (t, x) is continuous with respect to x, so here we say that f (t, x) is nothing but coefficient here. So, here if coefficients are continuous then by existence and uniqueness theorem for system of first order equation, we say that it will also have a solution.

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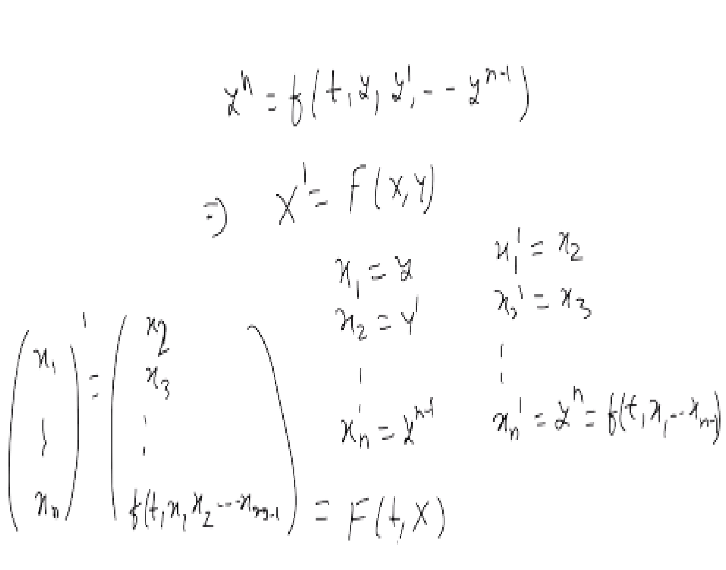

Similarly, you can now consider any nth order equation, yn = f (t, y, y dash up to yn – 1) to again system of first order equation that is x dash = f (x, y), idea is same that you assume x1 as y, x2 as y dash and so on, so here we have xn as yn -1 and you can write down x1 dash is = x2, x2 dash = x3 and so on xn dash is = yn that is given as f and here it is what? T, it is x1 to xn – 1, so you can simplify this.

And you can write x1 to xn here =; here, we can define this as x1, sorry, x2, x3 and so on, at last, here we have x1, x2 and so on and this you can call as f(t, x), so it means that any nth order non-linear differential equation as well as the linear differential equation with the constant coefficient or variable coefficient can be written as system of first order equation that is x is = f(t, x), so it is sufficient to discuss the existence and uniqueness theorem for system of first order ordinary differential equation.

So, in this lecture we have discuss several things that how the existence and uniqueness theorem given for scalar equation, a scalar differential equation can be generalise what system and with this, we may cover all the ordinary differential equation, whether it is a single, scalar, first order, second order or nth order, so here we have said that if it satisfy the Lipschitz condition, then we have a existence and uniqueness both.

If it satisfies only the continuity then it has a solution, it may not be unique but if it satisfy one more condition that the function F is satisfying the continuity as well as the monotone non decreasing condition that it will have a unique solution. So, here with this we stop and we will discuss new things in next lecture. Thank you very much.