Ordinary and Partial Differential Equations and Applications

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Lecture – 58
Solution of Homogenous Diffusion Equation - I

Hello friends, welcome to this lecture. In this lecture, we will discuss the parabolic equation in particular we will discuss heat equation as a prototype of parabolic equation. So here if you recall, we classify our partial differential equation say.

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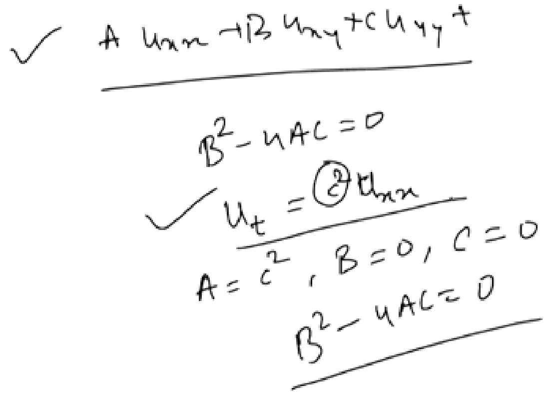

A * Uxx + B * Uxy + C * Uyy because classification is done only say linear second order terms. So here if B square - 4 AC = 0 you call these kind of equations as a parabolic equation. So we will discuss this equation Ut = c square Uxx or say some constant times Uxx and we say that here if we look at A is your constant C2 and B is 0 because there is no mixed order, mixed partial derivative is present and c is also 0.

Because we do not have double derivative corresponding to yy or other variable. So here we can say that B square - 4 AC is coming out to be 0. So we say that this heat equation which is written as Ut = some constant * Uxx is a parabolic equation and now we are looking at how to solve the heat equation and the other properties of heat equation.

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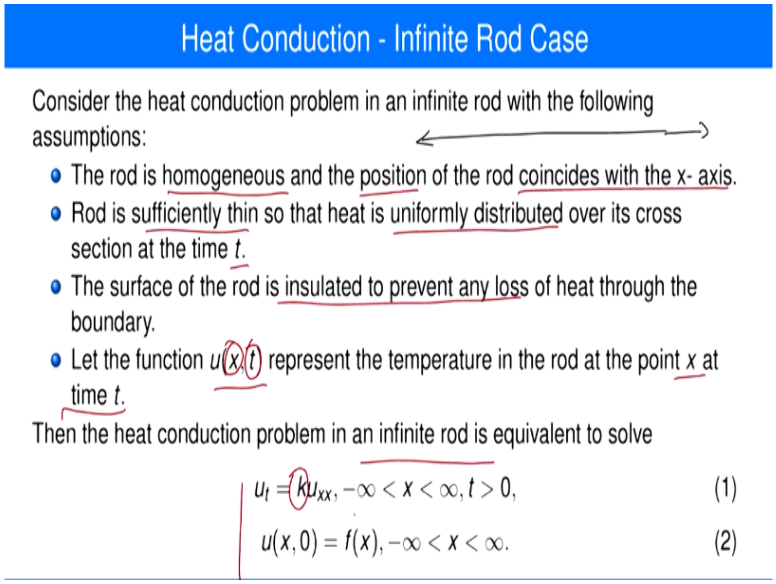

So first let us consider the heat conduction problem in an infinite rod with the following assumptions. So, here we consider an infinite rod and this is going to say so here we consider that the rod is homogeneous and the position of the rod coincide with the x-axis. So we will say that this rod is kept along with the x-axis and rod is sufficiently thin so that the heat is uniformly distributed over its cross section at the time t.

So first thing is that rod is homogeneous and position is coinciding with the x-axis. Second thing is it is sufficiently thin so that heat is uniformly distributed over its cross section in the time t and third is the surface of the rod is insulated to prevent any loss of heat to the boundary and let us say that this heat distribution is represented by U(x, t) so this U(x, t) represent the heat in the rod at point x and at time t.

So U(x, t) represent the temperature in the rod at the point x at time t. Now what we are considering a problem of heat conduction an infinite rod is which is = Ut = kUxx - infinity where x lying between - infinity to infinity and t is > 0 with the initial condition that U(x, 0) = f(x) and x lying between - infinity to infinity.

So here this constant k is a constant which depend on the material of the rod and here this represent U(x, 0) = f(x) represent that at t time t = 0 the temperature at the point x is given by f(x). Now, we want to find out using the equation 1 and 2 to find out the temperature at any point at time t. So suppose the Fourier transformation of U(x, t) with respect to space variable is given as U(alpha, t).

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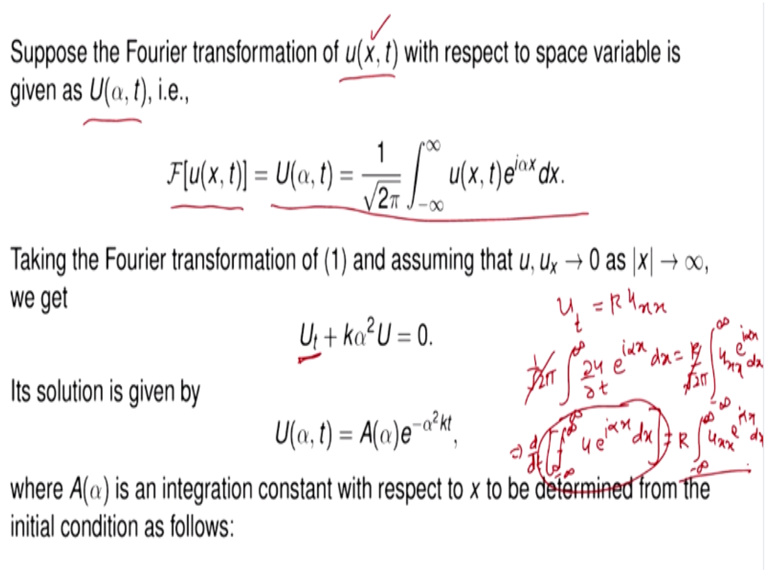

So here we are solving this 1 dimensional heat problem with the help of Fourier transform. So here since we have 2 variables U(x, t). So here we apply the Fourier transform with respect to the space variable that is x here. So here we simply say that let us say that U(x, t) obtain the Fourier transform of U(x, t) with respect to the variable x and it is denoted as U(alpha, t). It means that Fourier transform of U(x, t) which is denoted as U(alpha, t) = 1/root 2 pi * - infinity to infinity U(x, t) e to the power i * alpha x * dx.

So here we are taking the Fourier transform with respect to the space variable that is x here. So U(alpha, t) is denoted by this thing. So now using the Fourier transform, let us transform our equation 1 in a frequency domain. It means that apply Fourier transform on this Ut = k * Uxx so here we have if you take Ut = k * Uxx and if you apply the Fourier transform it takes 1/root 2 pi and here it is - infinity to infinity dou u/dou t * e to the power i * alpha x * dx = k * 1/root 2 pi * - infinity to infinity and Uxx and e to the i * alpha x * dx.

Now here since this is integration with respect to t so I can write this as d/dt of the whole thing that is 1/root 2 pi. 1/root 2 pi you can cancel it out. So we have say - infinity to infinity U * e to the power i * alpha x * dx = k * here it is - infinity to infinity and Uxx * e to the power i * alpha x * dx here. So this, I can write it as U alpha t. So we have written that it is d/dt of U alpha t so we can write this as Ut. Now, we have to handle this term. It means that we have to find out the Fourier transform of Uxx. So how to find out the Fourier transform of Uxx.

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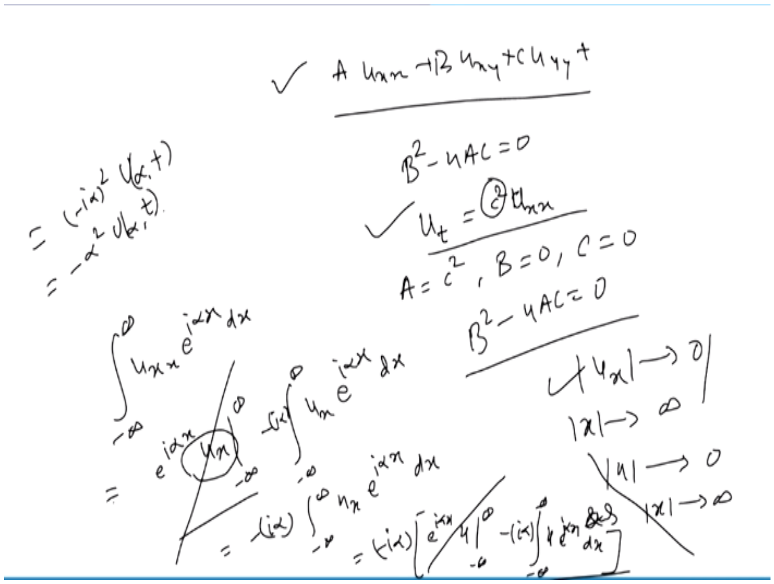

It is - infinity to infinity Uxx e to the power i * alpha x * dx here. So, here we apply the integration by part and we transform the derivative on e to the power i * alpha x. So we will write first integration of second that is Ux between - infinity to infinity - - infinity to infinity integration of this and e to the power i * alpha x derivative of this so i * alpha will come out dx. Now here if you look at the boundary conditions, boundary condition is what that here as x tending to infinity.

We assume that the derivative Ux is tending to 0. So it means that here Ux is tending to 0 as x tending to mod of x tending to infinity. So, using this boundary condition that at infinite length, the temperature distribution is going to be 0. So not only Ux we are assuming that u is also tending to 0 as s mod of x is tending to infinity. So these are the physical condition we are applying on this infinite rod case. So using these physical conditions, this part is simply gone.

So this I can write it - i * alpha - infinity to infinity * Ux e to the power i * alpha x * dx. Now still we have to apply 1 more time and when you apply we have - i * alpha and when you write it here it is first integration of second so e to the power i * alpha x u - infinity to infinity - again it will come - i * alpha - infinity to infinity U and e to the power i * alpha x * dx. Now here again we use the same thing that u is tending to 0 as mod of x tending to infinity.

So this will simply move and what we will have? It is - i * alpha whole square - infinity to infinity * U * e to the power i * alpha x * dx and which is nothing but U alpha t here. So I can write this as - i * alpha square * U alpha t. So this is what. This is your - alpha square U alpha t and k is already there. So we can write that it is k alpha square U. So we can write it here. I think there is some problem here. Yeah.

It is - alpha square U alpha t that is what it is written and using k you transform here you will have Ut + k * alpha square * U = 0. Here U is your U(alpha, t) here and if you look at this is a differential equation in terms of variable t here. So, here this is an ordinary differential equation given in terms of t and we know this is first order and we can easily solve and our solution is given by U(alpha, t) = A(alpha) e to the power - alpha square * kt.

Now here this A alpha is an integration constant which is constant with respect to the variable x and we can determine from the initial condition as follows because the constant we can find out using the initial condition or what is the initial condition that U(x, 0) = f(x), but since this equation is given in terms of frequency domain or you can say that alpha domain. So we transform our initial condition also in frequency domain.

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$$U(\alpha,0) = \mathcal{F}[u(x,0)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,0)e^{i\alpha x}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx := F(\alpha), \quad (say).$$

Hence

$$U(\alpha,t) = F(\alpha)e^{-\alpha^2 kt}$$

$G(\alpha)$ $g(x)$

$U(\alpha,t) = F(\alpha) \times G(\alpha)$

By convolution theorem, we get

$$u(x,t) = f(x) * \mathcal{F}^{-1}(e^{-\alpha^2 kt}).$$

Recall

$$f * g = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(u)g(x-u)du,$$

So let us write down (alpha, 0) as Fourier transform of U(x, 0) that is 1/root 2 pi - infinity to infinity u(x, 0) e to the power i * alpha x * dx. Now let us denote this quantity as F(alpha). So we are writing 1/root 2 pi - infinity to infinity U(x, 0) is f(x) e to the i * alpha x * dx. Now let us denote this as F(alpha). So it means that Fourier transform of f(x) is given by F(alpha). So using this initial condition what you say here U(alpha, t) = A(alpha) e to the power - alpha square kt.

Now put t = 0. Then your U(alpha, 0) is coming out to be F(alpha) so we can write this is F(alpha) = A(alpha) and this is what. So A(alpha) is coming out to be F(alpha) and so we can write U(alpha, t) as F(alpha) * e to the power - alpha square kt. So now we have solved our problem in frequency domain, but we want our solution in x and t domain so we can find out our U(x, t) by taking the inverse Fourier transform of U(alpha, t).

So let us take the inverse Fourier transform of U(alpha, t) and if you look at here it is I can write this is as a Fourier transform of F(alpha) * e to the power - alpha square kt. So if I say that this is e to the power - alpha square kt is a Fourier transform of some function say g(x) then I can write this U(alpha, t) as what? It is F(alpha) * g(alpha).

So here we can use our convolution theorem to write down our solution U(x, t) as F convolution F inverse Fourier transform of e to the power - alpha square kt and what is this convolution means. Here we define convolution like this. F convolution g = 1/root 2 pi - infinity to infinity f(u) * g(x - U) dU. So here this convolution is defined like this. Now to write down this solution properly, I need to find out the inverse Fourier transform of e to the power – alpha square kt.

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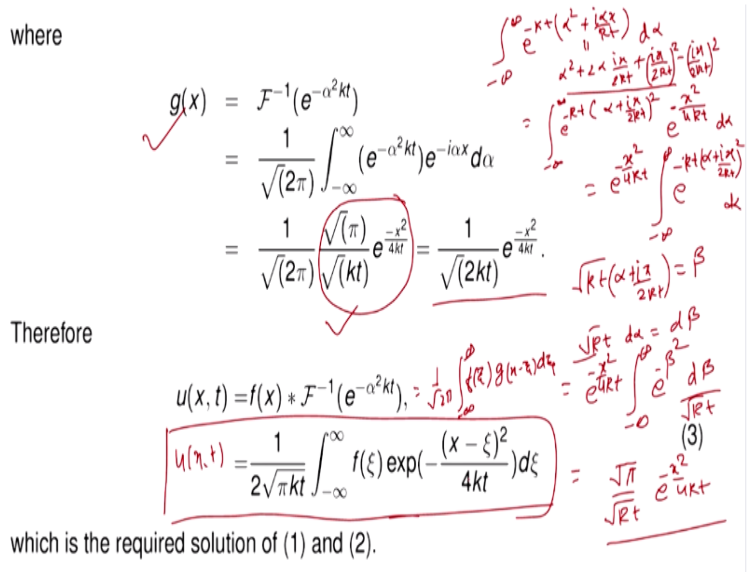

So for that we define g(x) as inverse Fourier transform of e to the power - alpha square kt. So this I can write it as 1/root 2 pi - infinity to infinity e to the power - alpha square kt * e to the power - i * alpha x * d alpha and we can simplify this expression and we can say that it is coming out to be 1/root 2 pi * this quantity is coming out to be pi/root kt * e to the power - x square/4 * kt and how we can find out. Let us write down here.

Let me write it here. - infinity to infinity I can write e to the power - kt out. Then we have alpha square and then we have + and we have what? We have taken alpha out. So it is i alpha x/kt. So we can write it like this and d alpha. So to solve this let us make the perfect square here. So we can write here that since I can write this as i x* 2k. So I can write this as alpha square + 2 * alpha ix/2 kt, right. So if I want to make a perfect square, I need to add something.

I need to subtract something. So I am adding this ix/2 * kt whole square and similarly I am subtracting ix/2 kt whole square, right. So this I can write it like alpha square + 2 alpha * ix/2 kt + ix/2 kt whole square - ix/2t whole square * d alpha. So I can write this as - infinity to infinity * e to the power - kt. Now this part I am writing as alpha + ix/2 * kt whole square and what is left here. It is e to the power this - and - +. And then i square will come out.

So it is - x square/kt and 1 kt will cancel out so we have 4 kt * d alpha. Now since this integration is with respect to d alpha so I can write this as e to the power - x square upon 4 kt * - infinity to infinity e to the power - here we have kt and this is alpha + ix/2kt whole square and d alpha. So let us make this quantity like kt (alpha + ix/2 kt) some beta. So by assuming this in fact root of this then what you will get, then root of kt d alpha = d beta and if you look at the limit, limit is coming out to be -.

So this whole thing I can write it e to the power - x square upon 4 kt limit will be as it is and here what you will get e to the power - beta square and d beta/root kt. This quantity will come down and we already know that - infinity to infinity e to the power - beta square d beta is your root pi. So I can write this as root pi and this is root kt will be here * e to the power - x square/4 * kt and that is what we have written here.

So it means that this quantity - infinity to infinity e to the power - alpha square kt * e to the power - i alpha * x * d alpha and if you solve this we have this quantity that is root pi upon root kt * e to the power - x square/4 kt. Now by simplifying we can write it is 1/under root 2 kt e to the power - x square upon 4 kt. So our g(x) is coming out to be this quantity. Now our solution is what f convolution g(x).

Now using the formula that F convolution g is given as 1/2 root pi - infinity to infinity f(u) * g(x - u) du. So here we can write this as 1/root 2 pi and here we have 1/root 2 pi here and 1 root 2 kt here. So 2 can be taken out and we can write 1/2 root pi * kt - infinity to infinity f(psi) * exponential of - in place of x I am writing here we are writing x - u. So here u is replaced by psi so I can write - x - psi whole square/4 *kt t * psi.

So here we simply say that this is what we can simply write f convolution g(x) as 1/root 2 pi - infinity to infinity f(psi) now in place in argument of g we write x - psi * d psi. So we already have g(x). So we can define g(x) - psi and we have written like this. So our solution U(x, t) is given as 1/2 root pi * kt - infinity to infinity psi * exponential of x- psi whole square/4 kt d psi.

So it means that this is our solution and once we know our initial condition then we can find out our temperature resolution at point x at time t by this formula.

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When $k = 1$ and

$$f(x) = \begin{cases} 0, & x < 0; \\ a, & x > 0. \end{cases} \tag{4}$$

$$u(x,t) = \frac{a}{2\sqrt{\pi t}} \int_0^{\infty} \exp(-\frac{(x-\xi)^2}{4t}) d\xi. \tag{5}$$

Put $\eta = \frac{\xi - x}{2\sqrt{t}}$, we have

Now let us say that we consider a particular example in which your F(psi) is given by this then initial temperature is given by f(x) = 0 when x is < 0 and a constant value a which when x is > 0 and here we are assuming for simplicity that our k = 1 here. Then our simplified solution is given by U(x, t) as a/2 root pi * t 0 to infinity exponential of - x - psi whole square/4t * d psi. Now here if you put this term, x - psi/2 root t as, say eta then we can simplify our U(x, t) as follows.

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$$\begin{aligned} u(x,t) &= \frac{a}{\sqrt{\pi}} \int_{-x/2\sqrt{t}}^{\infty} e^{-\eta^2} d\eta, \\ &= \frac{a}{\sqrt{\pi}} \left[\int_{-x/2\sqrt{t}}^{0} e^{-\eta^2} d\eta + \int_0^{\infty} e^{-\eta^2} d\eta \right] \quad \frac{\sqrt{\pi}}{2} \\ &= \frac{a}{2} + \frac{a}{\sqrt{\pi}} \int_0^{x/2\sqrt{t}} e^{-\eta^2} d\eta, \\ &= \frac{a}{2} \left[1 + erf\left(\frac{x}{2\sqrt{t}} \right) \right], \end{aligned}$$

$$erf(y) = \int_0^y e^{-\eta^2} d\eta$$

where erf(x) is the error function.

You can write a/ root pi - x/r root infinity e to the power - eta square d eta which we can simplify this range of integral we can write it - x/2 root t to 0 + 0 to infinity e to the power - eta square d eta. So this quantity is coming out to be root pi/2. So here using this term can be written as a/2 + a/root pi 0 to x/2 root t e to the power - eta square d eta. Now this expression is given error function of x upon 2 root.

Here error function of (y) is written as 0 to y e to the power - eta square d of eta. So using the expression for error function we can write down our U(x, t) is given by a/2 1 + error function of x/2 root t. So here it is a very particular case when we assume that k = 1 and f(x) is taking this value. So this is a very particular example of this, but in general if f is given some value then our solution U(x, t) is written as 1/2 root pi kt - infinity to infinity f(psi) exponential of - f(x) - psi whole square/4 kt d psi. So this is the solution.

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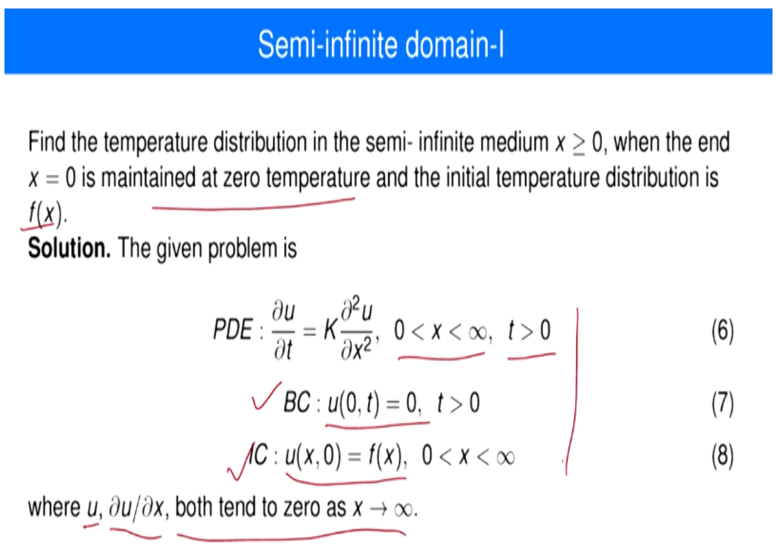

We will move to same infinite domain. So it means that here your length is only infinite or semi-infinite and 1 end is kept at say at 0 temperature so we look at our problem like this that find the temperature distribution in the semi-infinite medium x >= to 0 when the end x = 0 is maintained at 0 temperature and the initial temperature distribution is given as f(x). So problem is what that dou U/dou t = K dou 2 U/dou x square x is between 0 to infinity, t is > 0.

On the condition is that point x = 0 is kept at 0 temperature that is U(0, t) = 0 and initial condition is that u(x, 0) = f(x) when x is lying between 0 to infinity and here as we have assumed before that our temperature and its dou u/dou x corresponding to x both tend to 0 as x tending to infinity. So this is the whenever we consider any infinite, semi-infinite problem we always assume these kind of condition that U and its partial derivatives are say 10 * 0 as mod of x tending to infinity.

So here since we are considering only the positive site so it means that x tending to infinity U and dou U/dou x both are tending to 0. So here I cannot apply the full Fourier transform. So here we will apply say Fourier transform define for only 0 to infinity. So we have 2 choices, 1 is Fourier sine transform and another one is Fourier cosine transform.

And so which Fourier sine transform we should apply or Fourier cosine transform we should apply that is our question. So let us handle some result and say that based on that result we can indentify that based on the initial condition and boundary condition we can define that whether we should apply Fourier cosine transform or Fourier sine transform.

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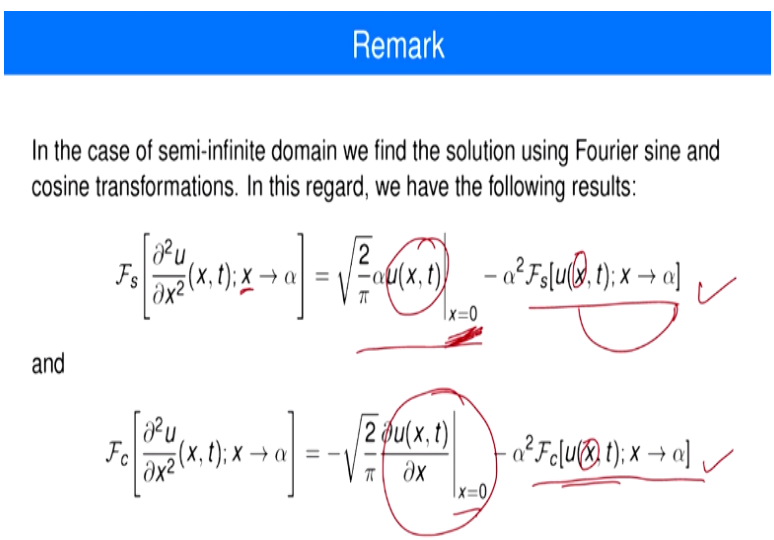

So here let us the following remark. In the case of semi-infinite domain, we find the solution using Fourier sine and Fourier cosine transform and in this regard we have the following result because if you look at this term can be handled very easy that it is d/dt U(alpha, t) and but we have to handle this term by indication by part. So we have to handle when you do integration by part then we have a boundary term that is say.

What happen when we have what is the value of U or Ux at the end point lower end point that is x = 0. So here because we have only this condition that U * dou u/dou x both tend to 0 as x tending to infinity, but we do not know the behaviour of U and Ux at the end point x = 0. So depending on this, we will choose Fourier cosine transform or Fourier sine transform. So let us first consider the result that Fourier sine transform.

dou 2 U/dou x square with respect to the space variable x is given as root 2/pi * alpha U(x, t) given at x = 0 - alpha square Fourier transform of U(x, t) and Fourier cosine transform of dou 2 u/dou x square is given as - root 2/pi dou u/dou x given at x = 0 - alpha square Fc Fourier cosine transform of u(x, t). This simple simply represent that we are finding the Fourier sine transform of u(x, t) with respect to the variable x.

And this represent that we are finding the Fourier cosine transform of U with respect to the variable x and we are considering the alpha a frequency domain here. So here if you look at these 2 formula then if you apply Fourier sine transform then I need the value of u at x = 0, but if you apply Fourier cosine transform then I need the value of dou u/dx at the point x = 0. So this will give you the idea that I should apply Fourier sine transform or Fourier cosine transform.

For example, we have this result that U(x, 0) = f(x). So here the point at x = 0, here the condition U(0, t) = 0 is given. So here this will give you that here you should apply the Fourier sine transform because in Fourier sine transform only we need u at x = 0, but if you apply Fourier cosine transform.

Then I need to have the value of dou u/dou x at x = 0 which is not given here. So in this particular example boundary condition is given in terms of u. We will apply Fourier sine transform and if this boundary condition is given in terms of ux then we will apply Fourier cosine transform. So first let us have a simpler proof of these 2 results.
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Result-I

$$\mathcal{F}_s\left[\frac{\partial^2 u}{\partial x^2}(x,t); x \to \alpha\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin\alpha x \frac{\partial^2 u}{\partial x^2}dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\partial u}{\partial x}\sin\alpha x\right]_0^\infty - \sqrt{\frac{2}{\pi}}\alpha\int_0^\infty \cos\alpha x\frac{\partial u}{\partial x}dx$$

If we assume that $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$. Then the right hand side term of the above equation becomes

$$-\sqrt{\frac{2}{\pi}}\alpha\int_0^\infty \cos\alpha x\frac{\partial u}{\partial x}dx = -\sqrt{\frac{2}{\pi}}\alpha\left\{[u(x,t)\cos\alpha x]_0^\infty + \alpha\int_0^\infty u(x,t)\sin\alpha x dx\right\}$$

So let us look at the first result. First result is that Fourier sine transform of dou 2 u/dou x square is given as root 2 upon pi 0 to infinity sine alpha x dou u/dou x square dx and then we apply say integration by part. So, we write first sine alpha x * integration of this dou 2 u/dou x square that is dou u/dou x between 0 to infinity - root 2/pi alpha derivative of this sine alpha x that is cos alpha x * alpha and integration of dou 2 u/dou x square that is dou u/dou x.

Now here we already assume that this dou u/dou x is tending to 0 as x tending to infinity so this at point infinity as x tending to infinity this point is gone. Now at x = 0 since sine of 0 is 0, so this boundary values will be cancelled out. So then the right hand side of the above equation is now reduced to - square root of 2/pi alpha 0 to infinity cos alpha x dou u/dou x * dx. Now again we apply integration by part and we can write - root 2/pi alpha * in a bracket we are looking at the expression for this.

So here it is what? This cos alpha x integration of dou u/dou x that is u(x, t) + alpha 0 to infinity differentiation of cos alpha x that is - sine alpha x will be there, so - - it will come out to be plus and u(x, t) integration of dou u/dou x and this is what this is basically the Fourier sine transform of U(x, t) and again since we have assumed that u is tending to 0 as x tending to infinity so at this point your this point is gone to be 0. So only thing remain at x = 0 u(x, t) cos of alpha x value. So here this boundary value at x = 0 is no given by this.

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Also, assuming that $u(x,t) \to 0$ as $x \to \infty$, above equation reduces to

$$\sqrt{\frac{2}{\pi}}\alpha u(x,t)\bigg|_{x=0} - \alpha^2 \mathcal{F}_s[u(x,t); x \to \alpha]$$

Hence,

$$\mathcal{F}_s\left[\frac{\partial^2 u}{\partial x^2}(x,t); x \to \alpha\right] = \sqrt{\frac{2}{\pi}}\alpha u(x,t)\bigg|_{x=0} - \alpha^2 \mathcal{F}_s[u(x,t); x \to \alpha] \qquad (9)$$

Result-II Similarly, it can be shown that if

$$u(x,t) \to 0 \text{ and } \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty$$

So we can simplify that it is under root 2/pi alpha U(x, t) at x = 0 - alpha square Fourier sine transform of U(x, t). So we can say that using this value I can write Fourier sine transform of dou 2 u/dou x square = under root 2/pi alpha U(x, t) at x = 0 - alpha square Fourier sine transform of U(x, t) and in a similar way we can prove the second result that if we take U(x, t) tending to 0 and dou u/dou x is tending to 0 as x tending to infinity.

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then

$$\mathcal{F}_c\left[\frac{\partial^2 u}{\partial x^2}(x,t); x \to \alpha\right] = -\sqrt{\frac{2}{\pi}}\frac{\partial u(x,t)}{\partial x}\bigg|_{x=0} - \alpha^2 \mathcal{F}_c[u(x,t); x \to \alpha] \qquad (10)$$

Therefore, in case of semi-infinite domain problem the choice of the sine or cosine transform will depend on the form of the boundary condition at the lower limit of the variable selected for exclusion.

Thus, to handle the term $\partial^2 u/\partial x^2$ in PDE, we require

$u|_{x=0}$ in the case of sine transform

$\frac{\partial u}{\partial x}\Big|_{x=0}$ in the case of cosine transform.

$\frac{\partial^2 u}{\partial x^2}$ u, u_x

we can write Fourier cosine transform of dou 2 u/dou x square = - root 2 upon pi * dou u/dou x define at x = 0 - alpha square Fourier cosine transform of U(x, t) at x tending to alpha. So therefore if you summarize this in case of semi-infinite domain problem the choice of sine or cosine transform will depend in the form of the boundary condition at the lower limit of the variable which we want to exclude.

Here our problem point is dou 2 u/dou x square, how to handle this. So this we want to exclude so here we want to see the availability of u and ux at the lower end point. So here thus to handle the term dou 2 u/dou x square in PDE we require that if U is given at x = 0, then we apply Fourier sine transform and if dou u/dou x at x = 0 is given then we apply Fourier cosine transform. So using this observation in our problem, we will apply Fourier sine transform. Here we apply Fourier sine transform because boundary condition is given in terms of u at x = 0.

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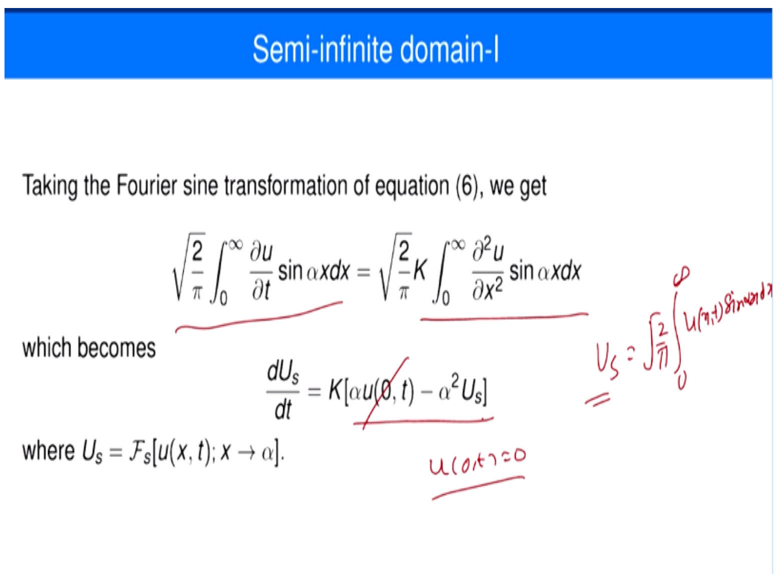

So applying Fourier sine transform we have this. So taking the Fourier sine transform the equation 6 we get under root 2/pi 0 to infinity dou u/dou t sine alpha x * dx = root 2/pi K * 0 to infinity dou 2 u/dou x square sine alpha x * dx. Now we have this value available for us and we can write dUs/dt where Us we are given as root 2/pi 0 to infinity so U(x, t).

And sine of alpha x * dx. So we are denoting Us as this value. So we can write this as dU/dt = K * and here we have using our result 1. We have alpha U 0 to t - alpha square Us, but our U(0, t) is coming out to be 0. It is already assumed that we have in this boundary condition. So this part is gone and we can write this as dUs/dt + K alpha square Us = 0.

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Using the boundary condition (7), we obtain

$$\frac{dU_s}{dt} + K\alpha^2 U_s = 0 \tag{11}$$

By taking the Fourier sine transform of initial condition (8), we get

$$U_s = F_s(\alpha) \text{ at } t = 0 \tag{12}$$

Equation (11) can be written as

$$\frac{d}{dt}(U_s e^{K\alpha^2 t}) = 0 \tag{13}$$

On integrating, we get

$$U_s e^{K\alpha^2 t} = cons.$$

So using the boundary condition we obtain that we have dU/dt + K alpha square Us = 0 and now we can find out this K by using the initial condition (8) that is Us = Fs(alpha) at t = 0 here. So here our equation is dUs/dt = K * this quantity. Now this boundary condition is already given as U(0, t) = 0. So using the boundary condition we obtain dUs/dt + K alpha square * Us = 0 and we also given initial condition.

So applying Fourier sine transform of the initial condition we can write Us = Fs alpha at t = 0. So this is the initial condition is given to us. So we have ordinary differential equation 11 along with the initial condition 12 that Us at t = 0 is given as Fs(alpha). Now we can solve this equation number 11. I can write this equation 11 as d/dt of (Us * e to the power K * alpha square t) = 0 and this can simply solve by writing Us * e to the power k alpha square t = constant and how to find out the constant there we can use initial condition that Us at t = 0 is given as Fs alpha.

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Using equation (12), we note that $F_s(\alpha) = constant$. Therefore,

$$U_s e^{K\alpha^2 t} = F_s(\alpha)$$

$$U_s = F_s(\alpha) e^{-K\alpha^2 t} \tag{14}$$

By taking the inverse Fourier sine transform of equation (14), we get

$$u(x, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\alpha) e^{-K\alpha^2 t} \sin \alpha x \, d\alpha$$

So using this we can write that your constant value is coming out to be Fs(alpha). So we can write Us * e to the k * alpha square t = Fs(alpha) and we can write thus our solution is given as Us = Fs(alpha) e to the power K alpha square t. So this is the Fourier sine transform of U(x, t). So once we know our Fourier sine transform of U(x, t) then we can find out U(x, t) by taking the inverse Fourier sine transform of this equation.

So U(x, t) we can write under root 2/pi 0 to infinity Fs(alpha) e to the power - K alpha square t * sine alpha x * d alpha. So by solving this we can find out the solution in terms of x and t. So we can write our solution as U(x, t) = under root 2/pi 0 to infinity Fs(alpha) e to the power - K * alpha square t * sine alpha x * d alpha. So here we found the solution of semi-infinite reason when the initial condition is given in terms sorry boundary condition is given in terms of U.

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semi-infinite domain-II

Find the temperature $u(x,t)$ in semi- infinite rod $0 \leq x \leq \infty$, determined by the PDE

$$\begin{aligned} u_t &= k u_{xx}, \quad 0 < x < \infty, \quad t > 0 \text{ subject to} && (15) \\ IC: u(x,0) &= f(x), \quad 0 \leq x \leq \infty, && (16) \\ BC: u_x(0,t) &= -u_0, \quad \text{when } \ x > 0, \ \text{ and } \ t > 0, && (17) \end{aligned}$$

where u, $\frac{\partial u}{\partial x}$, both tend to zero as $x \to \infty$.

Now consider another problem of the same semi-infinite domain, but this time your boundary condition is given in terms of Ux. So here problem is what? Find the temperature distribution U(x, t) in semi-infinite rod lying x between 0 to infinity determined by the following PDE that is ut = k* Uxx where x is lying between 0 to infinity t is > 0 subject to the initial condition that U(x, 0) = f(x).

And the boundary condition is given that Ux(0, t) = - u0 when x is > 0 and t > 0 and here again we have the same physical condition that u and dou u/dou x both tend to 0 as x tending to infinity. Now if you look at the boundary condition, boundary condition is given in terms of Ux, so we apply Fourier cosine transform to handle this problem.

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Solution. Since $\partial u/\partial x$ is given at the lower limit, we take the Fourier cosine series of the given PDE to obtain

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\cos\alpha x dx = k\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos\alpha x dx$$

$$= k\sqrt{\frac{2}{\pi}}\left[\frac{\partial u}{\partial x}\cos\alpha x\right]_0^\infty + k\alpha\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial x}\sin\alpha x dx$$

Since $\partial u/\partial x$ tends to zero as $x \to \infty$. Therefore

$$\frac{d}{dt}(U_c) = -k\sqrt{\frac{2}{\pi}}\left(\frac{\partial u}{\partial x}\right)_{x=0} + k\alpha\sqrt{\frac{2}{\pi}}(u\sin\alpha x)_0^\infty - k\alpha^2\sqrt{\frac{2}{\pi}}\int_0^\infty u\cos\alpha x dx$$

As u tends to zero as $x \to \infty$, so we have

$$\frac{d}{dt}(U_c) = \sqrt{\frac{2}{\pi}}ku_0 - k\alpha^2 U_c$$

So applying since dou u/dou x is given at the lower limit we take the Fourier cosine series transform of the given PDE to obtain the following equation under root 2/pi 0 to infinity dou u/dou t cos alpha x *d x = k * under root 2/pi 0 to infinity dou 2/dou x square cos alpha x * dx. Now we already have the either we use our result 2 or we can simplify this by taking say applying integration by part.

And we can write k/root 2/pi dou u/dou x cos alpha x 0 to infinity + k alpha under root 2/pi 0 to infinity dou u/dou x * sine alpha x * dx. So here we applied integration by part and we can write it like this. Now we already know that dou u/dou x tend to 0 as x tending to infinity so at this boundary point we have this value 0 and we have only term left corresponding to x = 0. So we can write - k * under root 2/pi dou u/dou x at x = 0 + k alpha again.

We simplify this problem using integration by part we can write k alpha under root2/pi u sine alpha x at 0 to infinity - k alpha square under root 2/pi 0 to infinity u cos alpha x * dx. Now here if you look at this boundary point, at x tending to infinity u is tending to 0. So this part is gone and at x = 0 this sine alpha x is 0. So this boundary condition will be vanished. So what we have d/dt of Uc = - k/root 2/pi dou u/dou x at x = 0 and this value is given as - of U0.

So - - will give you + so we have under root 2/pi K of U0 and this quantity is what? This quantity is given as Fourier cosine transform of U(x, t). So we can write this as - k alpha square Fourier cosine transform of U(x, t) that is Uc. So here we have d/dt (Uc) = root 2/pi k u0 - k alpha square Uc.

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$$\frac{d}{dt}(U_c) + k\alpha^2 U_c = \sqrt{\frac{2}{\pi}} k u_0$$

which can be written as

$$\frac{d}{dt}(e^{k\alpha^2 t} U_c) = \sqrt{\frac{2}{\pi}} k u_0 e^{k\alpha^2 t}$$

On integration, we get

$$e^{k\alpha^2 t} U_c = \sqrt{\frac{2}{\pi}} k u_0 \int e^{k\alpha^2 t} dt + cons. \qquad (18)$$

Taking the cosine transformation of the IC, we get

$$U_c = \mathcal{F}(c) \quad \text{when} \quad t = 0$$

So here we have a simple ordinary differential equation, non homogenous ordinary differential equation in terms of Uc and we can simplify this by writing d/dt of e to the power k alpha square t * Ut = under root 2/pi * k u0 e to the power k alpha square t. Since we can find out the integration factor by e to the power k alpha square t we will multiply and we can write it like this and then we simply integrate.

And this equation is now reduced to this that e to the power k alpha square t Uc = root 2/pi k * U0 integration of e to the power k alpha square t * dt + some constant integration constant we have to apply. Now to handle this integration constant, we look at the initial condition and take the Fourier cosine transform of the initial condition and we have Uc= F(c) when t = 0. So using this condition, we can find our constant.

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Using this condition in equation (18), we get

$$\mathcal{F}(c) = \sqrt{\frac{2}{\pi}}\frac{u_0}{\alpha^2} + cons.$$

which gives

$$e^{k\alpha^2 t}U_c - \mathcal{F}(c) = \sqrt{\frac{2}{\pi}}\frac{u_0}{\alpha^2}\left[e^{k\alpha^2 t} - 1\right]$$

Hence,

$$U_c = \sqrt{\frac{2}{\pi}}\frac{u_0}{\alpha^2}\left[1 - e^{-k\alpha^2 t}\right] + \mathcal{F}(c)e^{-k\alpha^2 t} \tag{19}$$

Taking the Fourier inverse cosine transformation of (19), we get

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos \alpha x U_c(t) d\alpha$$

And it is coming out to be that constant is given as F(c). So this is F(c) = under root 2/pi k u0 and integration of this will be e to the power k alpha square t/k alpha square. So we can write it here under root 2/pi U0/alpha square + constant. So we can find out the value of constant and we can put it back. so we have e to the power k alpha square t * Uc - F(c) = under root 2/pi U0/alpha square e to the power k alpha square t – 1.

And we can simplify and we can write Uc as under root 2/pi U0/alpha square * 1 - e to the power - k alpha square t + F(c) e to the power - k alpha square t and we can find out our solution U(x, t) by taking inverse Fourier transform of the expression given in terms of equation number 19 and our solution is given as U(x, t) = under root 2/pi 0 to infinity cos alpha x Uc(t) d alpha. So by putting this expression for Uc(t).

We can find out the solution for this semi-infinite region when boundary condition is given in terms of Ux. So that is all of this lecture. So in this lecture what we have considered. We have considered the heat equation in infinite region and in semi-infinite region when we have 2 different kind of boundary condition is given.

If boundary condition is given in terms of u will apply Fourier sine transform and if boundary condition is given in terms of Ux then we solve our problem by applying Fourier cosine transform. So with this I end our lecture and in next lecture we will continue our discussion of solution of heat equation and finite domain and x term. Thank you for listening us. Thank you.