Ordinary and Partial Differential Equations and Applications

Dr. D. N. Pandey
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture - 57
Solution of Non-Homogenous Wave Equation

Hello friends, welcome to this lecture. So in this lecture we will discuss a very important method, method of characteristics to solve a non-homogenous problem or we can say that Riemann Green's Method to solve a non-homogenous problem.

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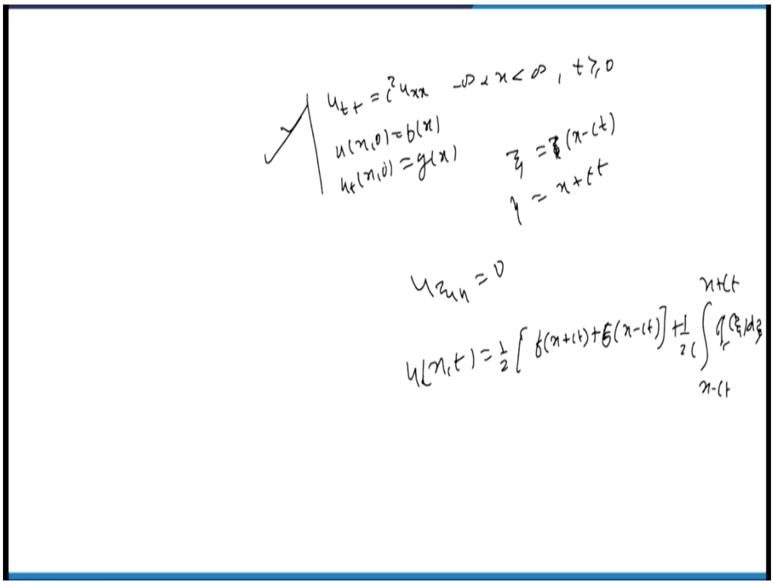

In this regard if you recall in the very first lecture of wave equation we have solved one dimension wave equation that is you can say that this problem i.e., u tt = c square 2 u xx, here x is lying between –infinity to infinity, t greater than or equal to 0, with the initial condition that the u(x, 0) = f(x) and u t(x,0) = g(x). And here we have solved this problem by considering that since it is a hyperbolic equation so here we have characteristic exist Xi = Xi(x – c*t).

Here you can write it as Xi = x – c*t and Eta = x + lambda * t. And with the help of this we have reduced this problem as u Xi Eta = 0 and then we saw after solving this we have obtained our solution say u (x , t) = ½*[f(x + c*t) + f(x – c*t)] + 1/(2*c) integral (x – c*t) to (x+ c*t)[g(Xi) * d(Xi)]. Here we have used method of --- to solve this one dimensional infinite string problem.

Now in this lecture we will again discuss the method of characteristics to solve the non-homogenous one dimensional wave equation. So now let us concentrate on this now. So here this method is popularly known as Riemann's Method or Riemann Green's Method. We will see what is this.

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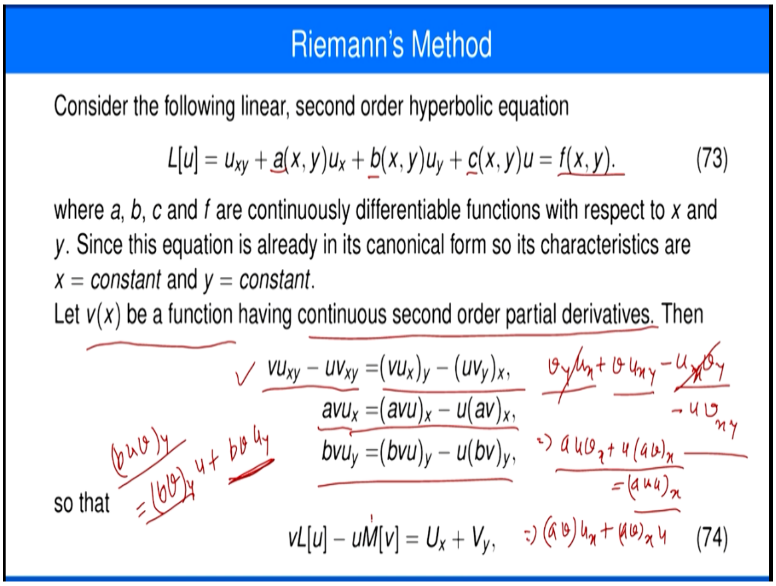

So here consider the following linear, second order hyperbolic equation i.e., L (u) = f(x, y) where L (u) is defined as u xy + a(x, y)*u y + c(x, y)*u = f(x, y) where a, b, c and f are continuously differentiable functions. So these a, b, c and f are continuously differentiable functions with respect to the variables x and y.

And since we assume this form as a canonical form because if you remember that any hyperbolic equation second order, linear hyperbolic equation can be reduced to this problem, this canonical form, i.e. L(u) = f(x, y). So we are assuming that here x and y are your characteristics. So here I am assuming that since this equation is already in its canonical form so its characteristics are x= constant and y=constant. So here x and y are your characteristics coordinates.

So now we try to solve this L (u) =f using Riemann's method. So for that here we assume v(x) be another function having continuous partial order partial derivatives. Now let us look at the following identities. So first thing is v*u(x, y) – u*v(x, y), this I can write as (v* u x) y - (u* v y) x. If you look at how these 2 are equal, if you simplify this it is what v y * u x + v* u xy – u x * v y – u* v xy.

If you cancel this v y * u x here, then what you will get v* u xy – u* v xy. That is what we have written here. So being said these are the identities we are using a *v * u x = (a*v*u) x – u*(a*v) x. Or you can say that here I am writing a *v * u x + u*(a*v) x= (a*v*u) x , that you can simplify this. If you are simplifying this it is what it is (a*v)*u x + (a*v)x *u so which is already written here.

So we are writing a *v* ux as a*v*u) x – u of partial derivative with respect to x of a*v. Similarly, in the same way we can write (b*u*v) y as (b* v) y * u + b*v*u y. So here I can write b*v*u y as (b*u*v) y - (b* v)y * u, so (b*v*u)y – u * (b* v)y . So here we are using these identities. So idea is that if we can write v* L[u] – u*M[v] = u x + v y. We just arrange it.

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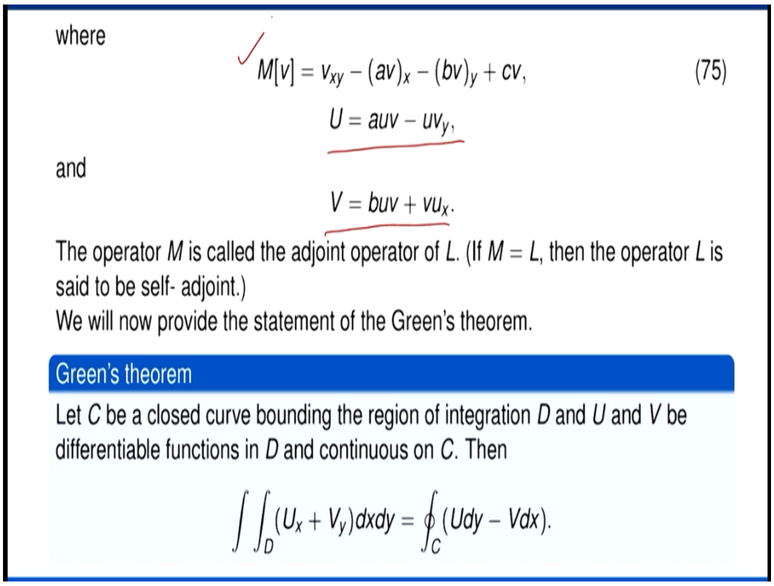

Now here what is M what is v let us see. Then I can write it M[v] as v xy – (a*v) x - (b*v) y + c*v. And here U= (a*u*v) – u * v y and V as (b*u*v) + v*u x. And here M is some operator defined in terms of v and we say that this is a operator which is known as the adjoint operator of L. So here let us again see how it is done.

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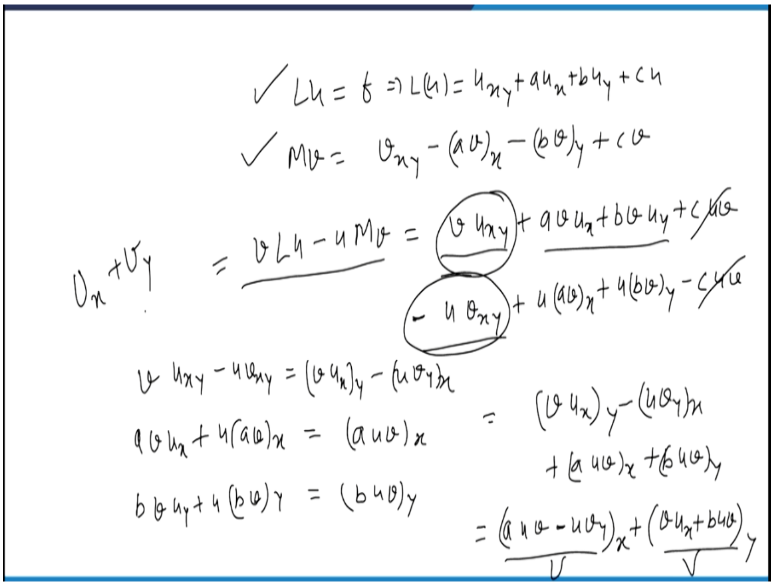

So here what we have done, here L u = f and here L (u) is defined as following u xy + a* u x + b* u y + c*u. So here we define your M v as v xy – (a*v) x – (b*v) y + c*v. So we need to find out v*L u – u* M v. So this we want to find out. So what is this, here we can write v*u xy + a*v*u x + b *v*u y + c*u*v – u*v xy + u*(a*v) x + u*(b*v) y – c*u*v. So this will be cancelled out.

So this term here v* u xy – u *v xy which we have approximated like this (v* u x) y – (u*v y)x. So this term minus this term we have approximated like this. So this how we write it here, so this a*v*u x, let me look at here. Here we have written a*v*u x as = (a*v*u) x – u*(a*v) x. So let me write here, say a* v* u x + u*(a*v) x. This we can write (a*u*v) x. And that we can see it here.

And then we can write b*v*u y + u*(b*v) y, and this we can write it as (b*u*v) y. So it means that all this thing I can write it like this, this I can write it (v*u x) y – (u*v y)x + (a*u*v) x + (b*u*v)y. So now let us collect the proof. So here a *u*v –u*v y) that will be with respect to x + (v*u x + b*u*v) y. So call this quantity as u and this quantity as v. So look at here u is a*u *v – u*v y, so a*u *v – u*v y and V as b*u*v + v*u x.

We have written that how that if you take L u as this, M v as this then you can write the v* L u – u* M v as u x + v y here. So this is some identity which is known as Lagrange Identity. It is similar to your ordinate differential equation.

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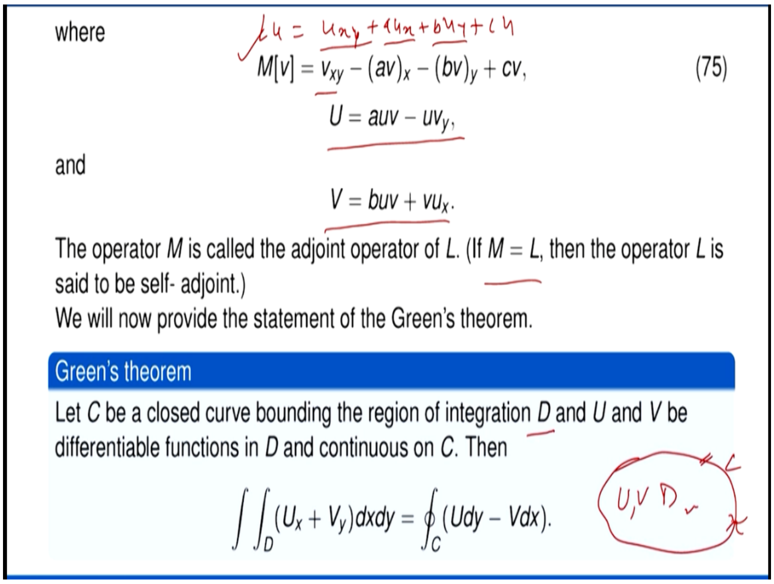

In ordinary differential equation we have written the similar kind of identity and these 2 identities are known as Lagrange's Identity and the operator M which we have defined here is known as adjoint operator. And in a similar way as we have defined for ordinate differential equation that if M = L then L is called as self-adjoint operator. So here if you look at what is the difference between L and M, in L u you have written as u xy + a*u x + b* u y + c*u.

So here if you look at u xy = v xy and a*ux is now written as –(a*v) y and (b*u*y) is –(b*v) y derivative + this term is no change, so it means there is a sign change in first order derivatives and no other change. So this is only for second order, if it is higher order then first third and all odd terms will have this kind of change and even order will not have any kind of change. So here we simply say that if M= L then the operator L is said to be self-adjoint.

Now we recall a very important theorem which is Green's theorem, and we will see how this Green's theorem is very useful in solving our problem. So what is Green's theorem, let C be a closed curve bounding the region of the integration D and U and V be differentiable functions in D and continuous on C.

So let us say that C is what, C is some kind of you can say domain, so C is this and D is domain closed by this curve C. And u and v are differentiable functions inside your domain here. So u and v are some functions which is differentiable on this and on boundary it is continuous. So how this double integration which is done inside your region is now turned to the integration on the boundary.

So here your area integration is now reduced to line integration. So this is the power of the Greens theorem and here you can write u x + v y dx*dy taken on the region D is now converted into line integral U dy – V dx and this is the orientation that it is clockwise here. So then (u x + v y) dx*dy integration over the domain D is now line integral U dy – V dx over the contour C and this having the direction anticlockwise. So it is like this. So now let us see how this Green's theorem is utilized to find out the solution of non – homogenous wave equation.

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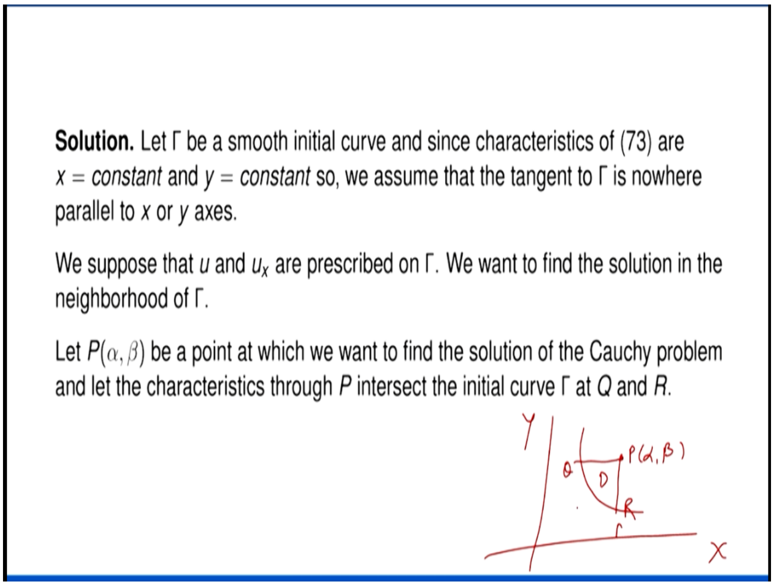

So here let us start with let gamma be a smooth initial curve and since characteristics are x = constant and y = constant so we assume that the tangent to gamma is nowhere parallel to x or y axes. So we consider this, this is your x and this your y and suppose your gamma is given like this. And so here it is your gamma and so here we are assuming the initial (()) (13:57) that if you take the tangent at any point of gamma, tangent is parallel to x axis and y axis.

So now we want to find out the solution. We suppose that u, u x and u y, in fact u and normals of u are prescribed on gamma. And we want to find out the solution in the neighborhood of this gamma. So let us say that we have a point P whose coordinates are alpha, beta and here we want to find out the solution if the non-homogenous problem. So this initial problem is known as Cauchy problem.

Here we want to find out the solution of this initial curve problem. So let us say you find out the characteristics parallel to your axes. So here we say that this is P and let us say this Q and this is R. So it means that the characteristics drawn from P will cut the initial curve in 2 points i.e., P, Q and R. So now we want to find out the solution at this particular point. So let us say that D be the region bounded by the closed contour PQRP, it is PQRP, so it is the contour, and now D is the region bounded by this.

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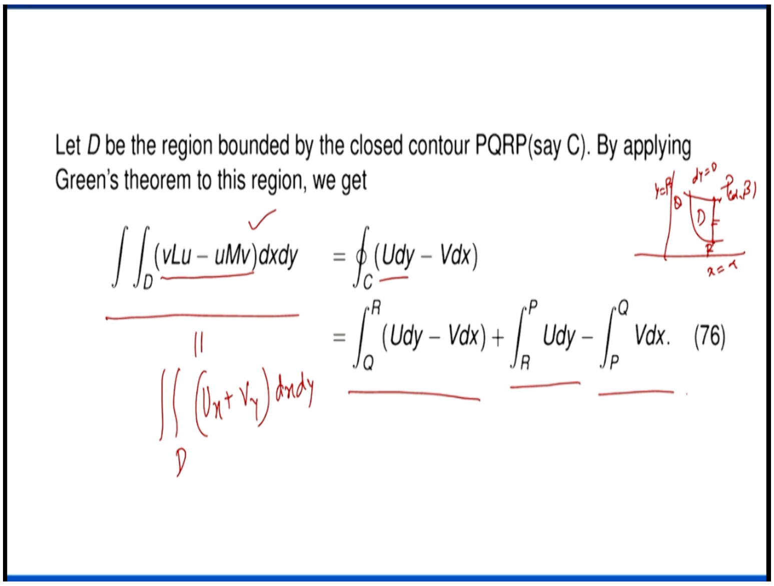

So here by applying Green's theorem to this region we get, so here we already know that v*Lu –u*Mv by Lagrange identity we can write that v * Lu – u*Mv, this is nothing but your U*dy – V*dx. Let me write it here, this is what it is double D and it is (u x + v y)*dx*dy. So this is where we have utilized the following thing, here we have utilized this property that here v*L[u] – u*M[v] = u x + v y. So here we have written u x + v y.

Now we apply Greens theorem and we can write this as integration U*dy – V*dx. So here again let me write it here, this is your P, this is Q, this is R, so here we have D, integration on the region D is now thrown away to its boundaries, so it is integration C (U dy- V dx). Now this boundary can be truncated into this term. This is QR. So (U*dy – V*dx) on Q, R + R, P here its U dy – V dx, plus it is on PQ.

In this line RP, your x is fixed. So x is here alpha and here it is y = beta. So x is fixed, it means that if you take the dx then it is 0. So on RP your dx component is gone. So here we have only U dy. On RP we have only term U dy and similarly on PQ your y = beta, so it means that dy = 0. So we have only one term left i.e., - P, Q (V dx). So here integration v* Lu – u*Mv*dx*dy which is given as double integration U x + V y*dx*dy.

Then using Greens theorem, we write line integral U*dy – V*dx and then we truncate using this equation for C we have written as along QR + RP + PQ. So along QR we have this, along RP dx component is missing because dx=0 and along PQ we have only dx component because dy =0. So now this equation is now written as sum of 3 integrations, from Q to R (U*dy – V*dx) + R to P (U*dy) – P to Q (V*dy).

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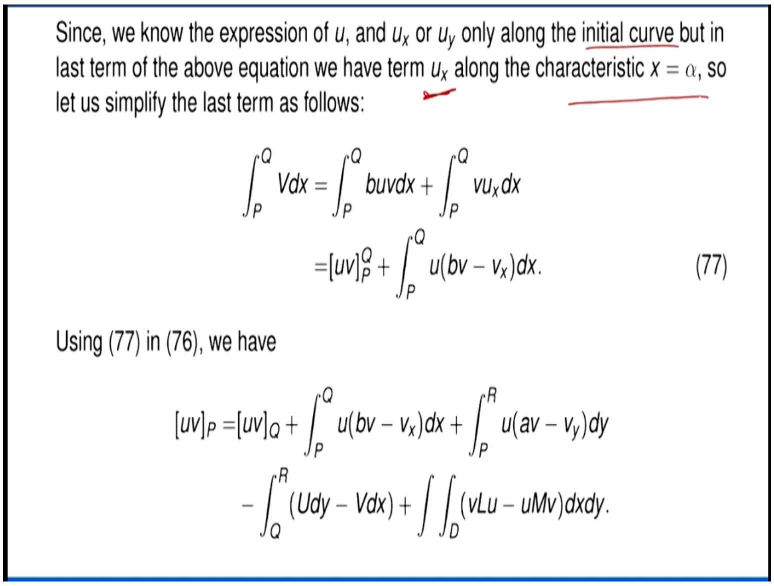

Now since we know that the equation of u and u x or u y only along the initial curve but in the last term of the above equation we have term u x along the characteristic x = alpha.

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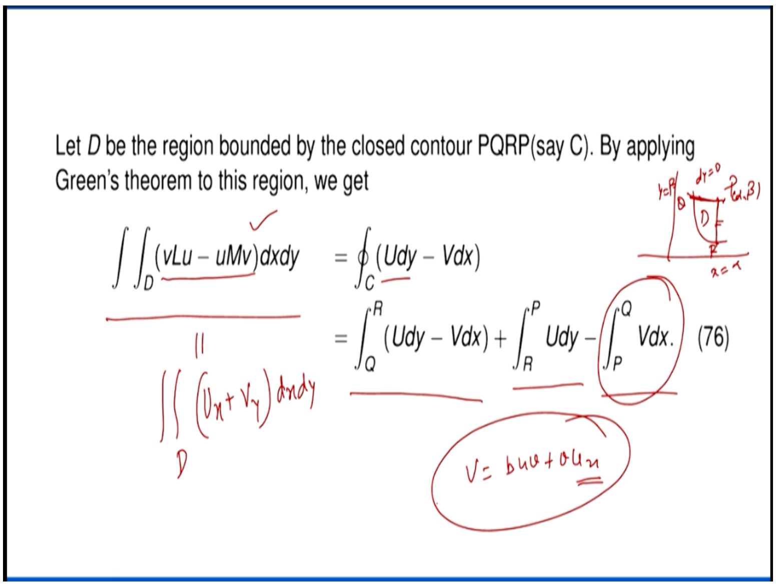

If you look at here, in here what is the equation for V, V equation is given as this V = b* u*v + v*u x. So here b*u*v +v * u x. So here this u x we want to know on these characteristics which we don't know because we don't know what is the value of U x along this PQ. So what we try to do, we just simplify this expression so that we need only the function u and its normal only on the characteristics means only on the initial curve not on characteristics. So here we simplify this P to Q V dx in the following way.

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Since, we know the expression of u, and u_x or u_y only along the initial curve but in last term of the above equation we have term u_x along the characteristic $x = \alpha$, so let us simplify the last term as follows:

$$\int_P^Q V dx = \int_P^Q buv dx + \int_P^Q v u_x dx$$
$$= [uv]_P^Q + \int_P^Q u(bv - v_x) dx. \qquad (77)$$

Using (77) in (76), we have

$$[uv]_P = [uv]_Q + \int_P^Q u(bv - v_x)dx + \int_P^R u(av - v_y)dy$$
$$- \int_Q^R (Udy - Vdx) + \int\int_D (vLu - uMv)dxdy.$$

So let us say P to Q V dx = b*u*v dx + v”* u x dx. So we can write it here [u*v] P to Q, we try to do solving this problem, the integration by part so that this (()) (19:26) is shifted on this v because so far we do not know what is v. We have just taken v as some twice differentiable continuous function, that is all.

Let us shift our derivative on v. So here we can write it this as v * integration of u over PQ + P to Q and differentiation on v so v x * u and this term is already there, so uv, so we can write this as integration P to Q V dx as [u*v] P to Q + P to Q (u*(b*v –v x) dx). So here now derivative is not on u, it is on x. So using this relation (77) now look at again this (76) here it is [u*v] from P = [u*v] Q, this I can write it here as [u*v] at Q – [u*v] at P + this term.

So now take that side. So we have [uv] P = [uv]Q + P to Q(u(b*v –v x)dx) + P to R(u(a*v – v y)dy).

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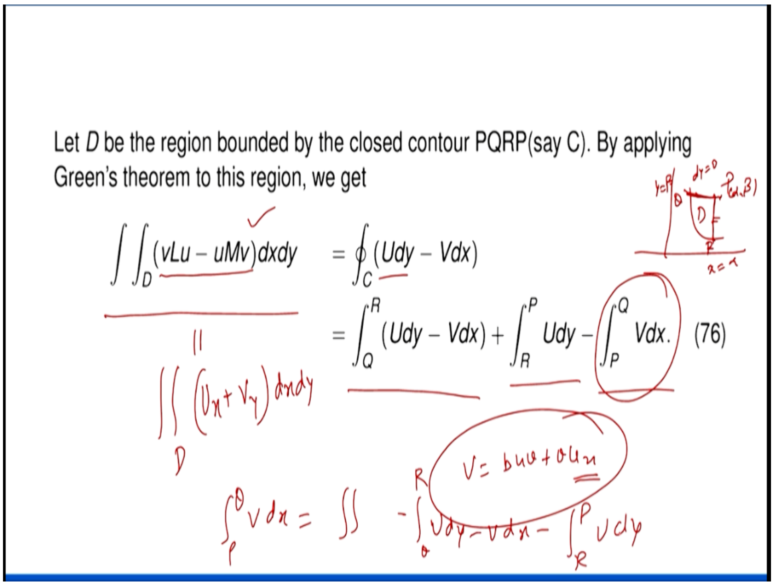

Here we are not changing anything. So what we try to do, from this you just take out the value here, so you have P to Q V*dx equal to this whole thing. So here we have double, minus Q to R U *dy –V*dx – R to P U*dy. So that is what we are writing here. So it is [u*v] P = [u*v] Q + P to Q (u*(b*v –v x)) that is the term which we have obtained from 77 and then P to R(u*(a*v – v y)dy), we adjust writing – Q to R(U*dy – V*dx) that is on the initial curve plus double integration.

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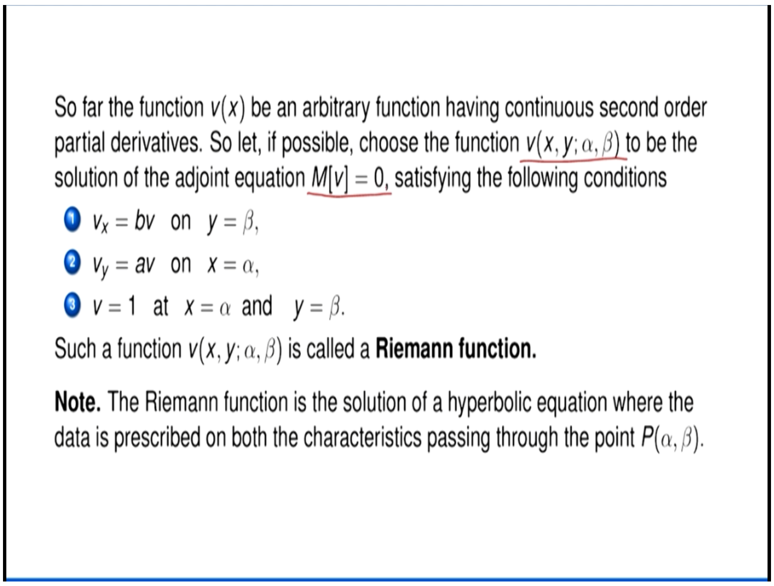

Now so far the function v(x) be an arbitrary function having continuous second order partial derivative. We don't have any information on V. So let us if possible now choose function v(x, y; alpha, beta) to be the solution of the adjoint equation M[v] =0. Now we are putting condition on M[v] so that our problem is reduced to a simpler problem by which we can find the solution for the given point P.

So let us say that v x = bv   y=beta. So on Y= beta, we want that v y must be equal to a v along PR. So what is PR is this. So along x = alpha we want u y must be equal to av and similarly along PQ we want that your v x must be equal to by and x = alpha we want v y = av. And also that this Mv is 0, so it means that v is the solution of adjoint equation.

So let's write it here v y = av on x=alpha and v=1 at x= alpha and y = beta. So it means that at point P, at this point we want that v has to be 1 and if we do this then our solution is reduced to [uv] p = [uv]q this term is gone, this term is also gone, what we have is very simpler value [uv]Q – Q to R(U*dy –V*dx) + double integration v*Lu dx*dy. Now this will give you the solution of U at the point P.

So such a function which satisfy all these conditions that it is a solution of adjoint equation and satisfy these initial values along the characteristics v = beta and x = alpha is your characteristic lines. And such a function v(x, y; alpha, beta) is called a Riemann function or Riemann's Green function. And we already know that the Riemann function is the solution of a hyperbolic equation where the data is prescribed on both the characteristics passing through the point P(alpha, beta).

So this is one characteristics, this is another characteristics. So your data is defined on these characteristics. So we can say that here the solution if data is prescribed along the characteristics we will get a unique solution. So such a v is going to be a unique solution.

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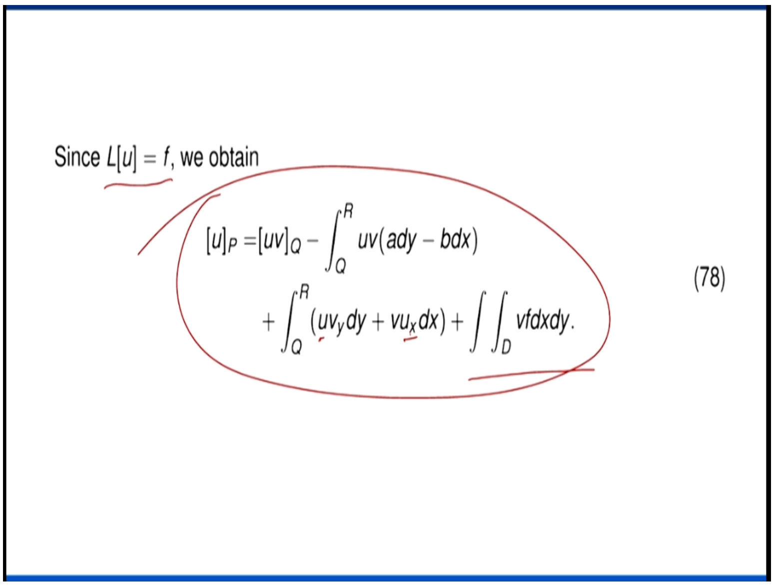

So now once we have this v and the thing that Lu = f then we can write on our equation, last equation in the following form that [u] at P = [u*v] at Q – Q to R[u*v(a*d*y-b*d*x)] + Q to R[u*v y*dy + v* u x* dx] + double integration v*f*dx*dy, f is known , v is known and uv is known and u and u x is known too on the initial curve. So by (78) you can find out the solution u at point P using this formula. But here we have used only the condition u and u x on your initial data.

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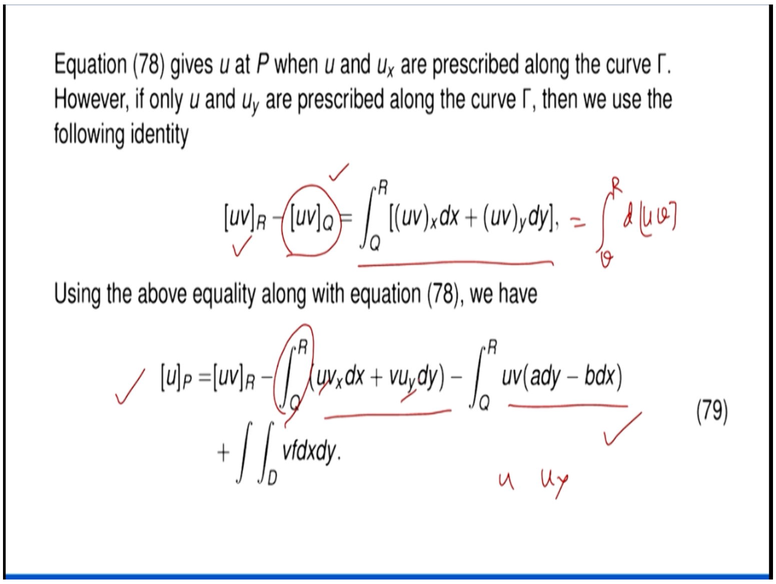

Now it may happen that on initial curve you know only u and u y rather than u and u x. So if you know u and u y only we cannot use the equation (78) to find out the solution at P. Here we use one small identity i.e., [u*v] R – [u*v] Q = Q to R [((u*v) x) * dx + ((u*v) y)* dy]. In fact it is what, it is simply along QR d [uv]. So this we can write it [u*v] R – [u*v] Q. So using this if you simplify here you can use the value of [u*v] Q from this.

You can find out the value of [u*v] Q, put it back and you can have this formula. I am not going to simplify this. You can simplify, take out the value of [u*v]Q in terms of [u*v]R and this integral and when you put it back it will give you the following thing [u]P =[u*v] R – Q to R (u*v x dx +v*u y dy) this is unchanged, the only thing we have changed here, this expression and this thing. This will be changed a bit, rest everything is the same.

So [u] P = [u*v] R – Q to R (u*v x dx +v*u y dy) – Q to R (u*v)*(a*dy – b*dx) + double integration v*f*dx*dy. So this is the equation (79) will give you the solution of u at P when u and u y is prescribed on initial data i.e. the initial curve is this. So here u and u y is used. But what happen if u, u x and u y are given along the initial curve. So I cannot use (78), I cannot use (79) alone.

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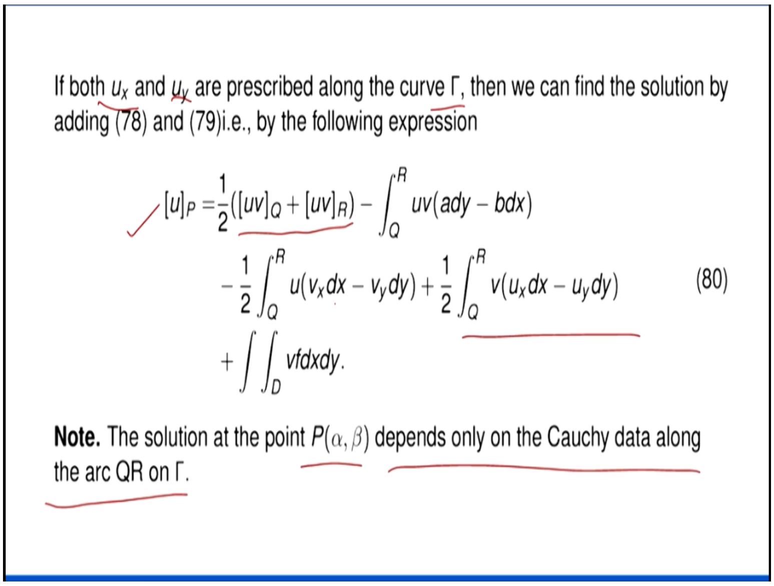

So what we do, we sum this (78) and (79) and have the following thing that if both u x and u y are prescribed along the curve gamma, then we can find the solution by adding the previous 2 formula and we have the following expression that [u]P = ½ *([u*v] Q + u*v] R) - Q to R (u*v)*(a*dy – b*dx) – ½* Q to R u*(v x *dx – v y * dy) + 1/2 * Q to R v*(u x * dx – u y* dy) + double integration v* f* dx* dy. The solution at the point P(alpha, beta) depends only on the Cauchy data along the arc QR.

So here we have utilised what, we have utilised only the value u and u x and u y on the initial data. So by the equation number (80) you can find out the solution P if initial data is u, u x, u y defined on the initial data. If only u and u y is defined then you can use equation number (79), if u and u x is given only then you can use equation number (78). So we have now 3 formulas to give you the solution of your problem where the initial data is given along the initial curve.

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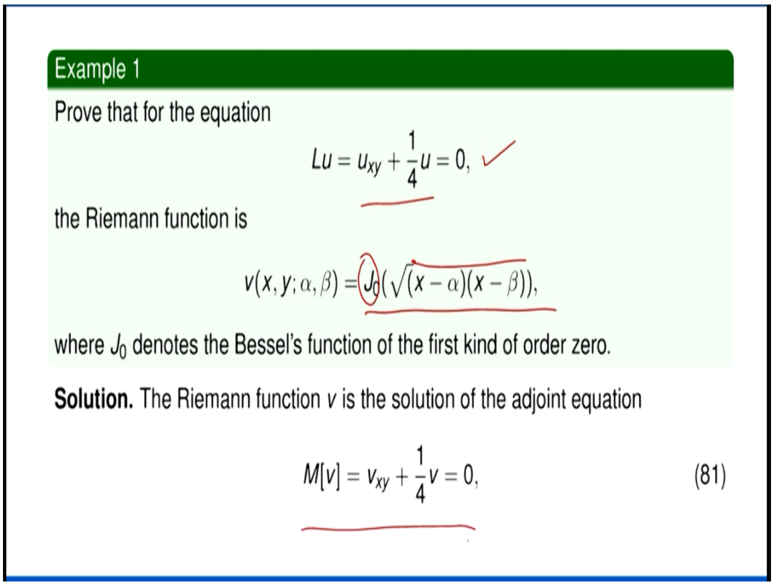

So now let us consider one simple example based on this so that we can understand what we have just explained. So now, the example is this, prove that for the equation Lu = u xy + ¼ *u = 0, the Riemann's function that is v is J0*(square root of (x – alpha) *(x – beta)). Here J0 is the 0^{th} order Bessel function of the first kind. Here the problem is only to find out your Riemann's function, it's not the problem of solving the entire problem.

So here we have initial function homogenous equation and we want to find out the Riemann's Green function. So here if you have Lu = 0, then look at the adjoint problem and since we don't have any first order partial derivative the corresponding adjoint function is the same. In fact, it is a self-adjoint problem. Here M[v] = v xy + ¼*v = 0.

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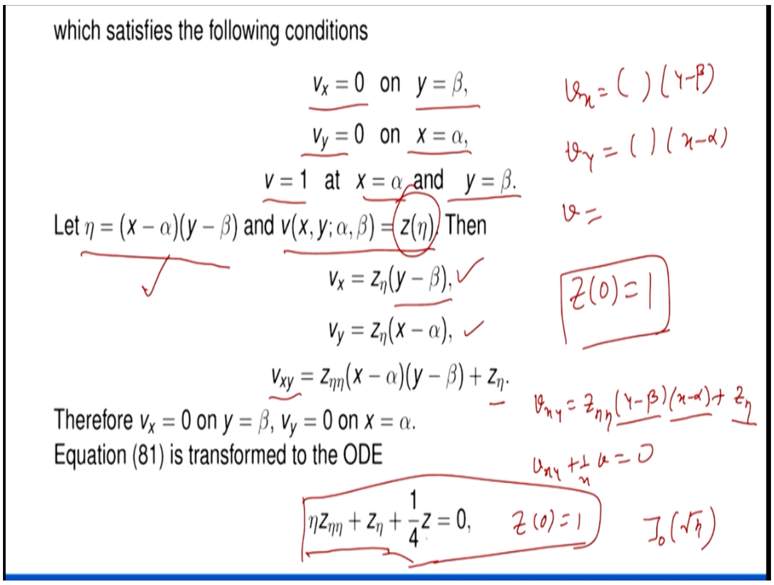

Now we have the initial condition along with this. So here we don't have the first order derivatives means a= 0 b=0 so here we can say that v x =0 which we have assumed in terms of a and b, so v x=0 on y=beta and v y =0 on x=alpha. And v=1 at x=alpha and y=beta. So we simply assume that v x = something * (y – beta) and this v y = something * (x – alpha).

Here we try to see that, let us use that if v is some function of y- beta and x-alpha, then we can achieve this. So let us say that let Eta = (x-alpha)*(y-beta) and let us assume that v = z (Eta). So when you do this, then v x is what, v x = z Eta(y –beta), so it will be 0 at y=beta. v y= z Eta(x- alpha), so it will be 0 at x = alpha. So let us say that using Eta as (x –alpha) * (y-beta) let us assume that v is some z (Eta).

Now we want to find out the z (Eta).So to find out z (Eta), we have conditions. So we can say that, condition is what, that we add 1. Add x= alpha and y= beta. So it means that z (0) =1, that is first condition we have. And what equation it will satisfy we have to find out. So let us find out v xy. So v xy is what z EtaEta *(x-alpha)*(y-beta) + z Eta.

So that you can find out, so here we differentiate with respect to y, so v xy = z EtaEta (y – beta), then if you differentiate again you will get what, we differentiate, sorry z EtaEta (y – beta) and you will get (x-alpha) also + when you differentiate y-beta with respect to y you will have z Eta only. So here z Eta Eta(y – beta)* Eta y. Eta y is x-alpha.and + z Eta. So v xy is this. So here v xy + ¼*v =0. So this is the equation which we have to satisfy.

So here we have xy is what z EtaEta. This is what, this is nothing but Eta we have assumed, so Eta * z EtaEta + z Eta + ¼*z = 0 and z(0) = 1.This is the initial condition which we know, that z Eta which is nothing but your Riemann's Green function must satisfy this ordinary differential equation along with the conditions that z(0) = 1.

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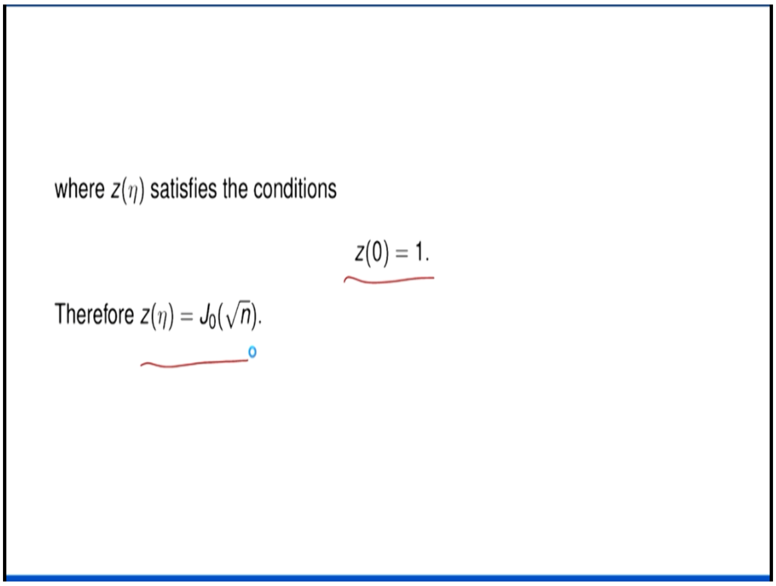

So we want to show that this is nothing but J0 *(square root of Eta). So our claim is that this solution of this problem is your J0 of root Eta. So we can show, so here we want to show that the solution is actually your z0 of u Eta.

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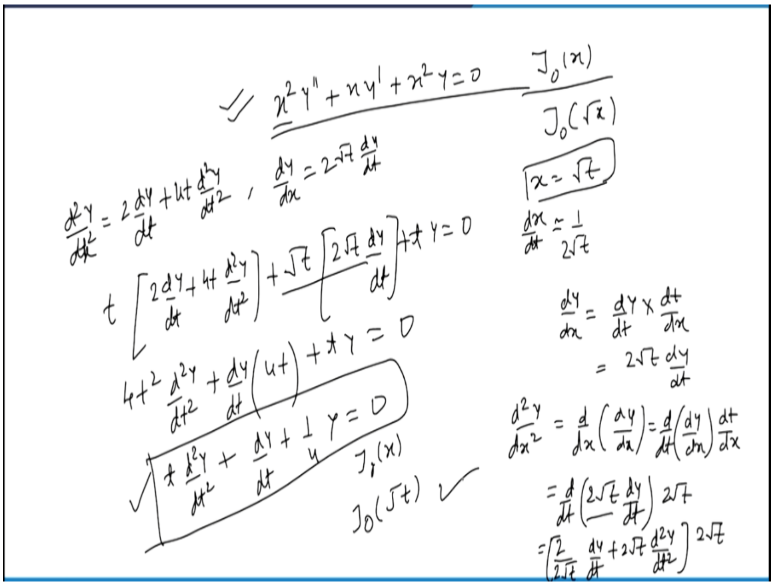

So for that let us show here that if we have x square 2 say z xx + let me use this notation, x square 2 * y" + x*y' + x square 2*y=0 and we already know that solution of this is J0(x). So solution is J0(x). Now we want to show that what is J0 (root x) will satisfy because we claim that the last problem, this problem has solution J0 (root of Eta). So we simply say that, let us say that you take x as root t and try to see that this will be converted into previous problem.

So here you can find out dx/dt as 1/ root (t)*2. And we can write it here dy/dx as what, dy/dx = dy/dt * dt/dx. So dt/dx is 2* root (t), so 2* root (t)* dy/dt you will get. Similarly, you want to find out d square 2y/ d x square2, so it what d/dx of (dy/dx) and this is what d/dt of (dy/dx) and dt/dx. So this is what d/dt (2* root (t) * dy/dt) and it is 2* root (t).

So here if you differentiate this you will get first thing (2 and it is root(t) if you differentiate it is 2* root(t), dy/dt + 2* root(t) * d sqaure2 y/dt square 2) into 2*root(t). So you can write it here d square 2 *y/dx square 2, when you multiply here you will get 2 * dy/dt + 4*t * d square 2*y/ dt square 2, that you will get. So this is d square 2*y/ dx square 2. And dy/dx is, it is what, dy/dx is we have already 2*root (t) * dy/dt.

So let us use these values. Since x square 2 is what, x square 2 is t only so t[2* dy/dt + 4*t* d square 2*y/dt square 2] + x, x is our root(t) and dy/dx is 2* root(t)*dy/dt + x square2 is again t, so y=0. So if you simplify what you will get it is 4* t square 2 * d square 2 *y/dt square 2 + collect the coefficient of dy/dt, it is 2t here and if you look at here it is again 2*t, so it is 4*t + ty =0. Now divide by 4*t.

So if you divide by 4*t, it is t* d square 2 *y/dt square 2 + dy/dt + ¼ *y=0.So here when you divide by 4*t we have the following equation t* d square 2 *y/dt square 2 + dy/dt + ¼ *y=0 and if you look at it is what, it is a transformation of the earlier equation whose solution is J0(x). So it means that here the solution will be J0(x but here what is x, we have assumed that x is root (t). So solution of this problem is J0 (root (t)).

So solution of this equation t* d square 2 *y/dt square 2 + dy/dt + ¼ *y=0 is your J0(root(t)) and that is what we claim here, that the solution of Eta * z EtaEta + z Eta + ¼*z = 0 is J0(root(Eta)) and it satisfies the initial condition that z(0) = 1.So here we said that the z(Eta) is coming out to be J0(root(Eta)) and what is z here, z is a Riemann Green's Function for the problem here.

So what we have proved here corresponding to the equation Lu = 0 where Lu is defined as u xy + ¼*u = 0, the corresponding Riemann Green function is J0 (root(Eta). So now what is Eta here, J0 (root of ((x-alpha)*(x-beta)). So here we have obtained the Green's function of this. So once we know the Green function and if the problem Lu = f is given you can solve the problem Lu = f in terms of v here. So with this I stop our discussion.

Next class we will discuss our some more problems based on heat equation. So with this I end our lecture. Thank you very much for listening. Thank you.