Ordinary and Partial Differential Equations and Applications
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Lecture – 52
Laplace and Poisson Equations

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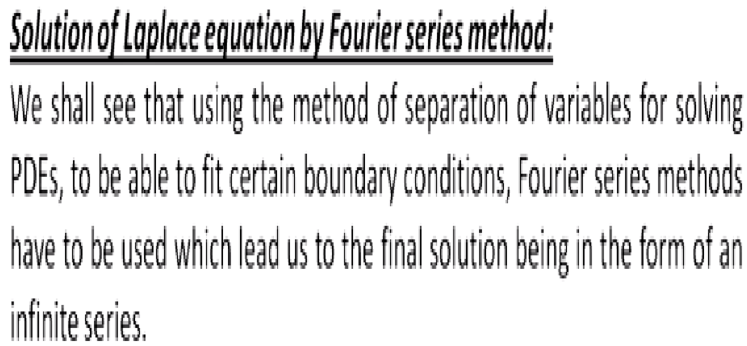

Hello friends, welcome to my lecture Laplace and partial equation, first we will consider the solution of Laplace equation by Fourier series method, we shall see that using the method of separation of variables for solving partial differentiation equations to be able to fit certain boundary conditions Fourier series methods have to be used which lead us to the final solution being in the form of infinite series.

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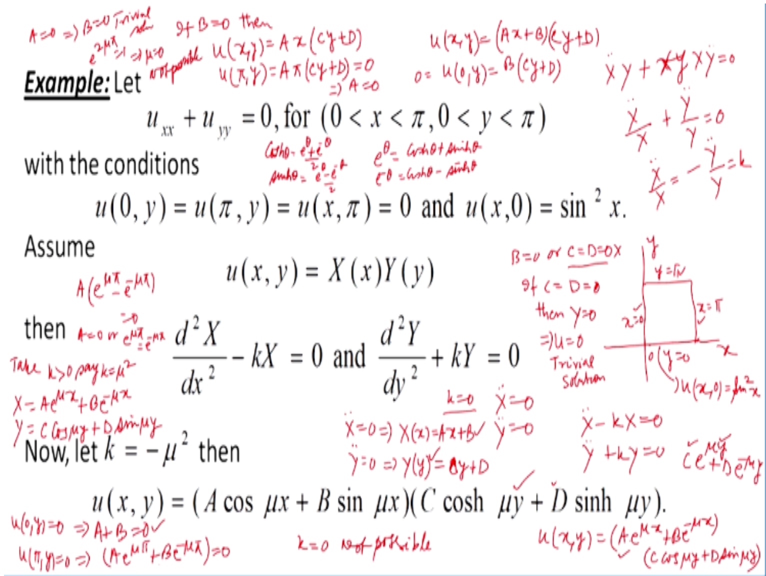

Let us consider the Laplace equations in 2 dimensions; uxx + uyy = 0, where 0 is < x < pi, o is < y < pi and the boundary conditions are, when x is 0 for all y, u0y = 0, when x =pi y is for all y, u pi y = 0, then ux pi = 0 and ux0 = sin square x and let us consider the Laplace equation in two dimensions; uxx + uyy = 0, for 0 < x < pi 0 < y < pi with the boundary conditions; u0y = u pi y= ux pi = 0 and ux0 = sin square x.

So, you can consider this rectangular plate, this is x axis, this is y axis, here we have y = 0, x = 0, here x = pi and we have y = pi here, so at this end y = 0 okay, we have ux0 = sin square x, so here we have ux0 = sin square x this end, this end and this end, these 3 ends are kept at 0, if you call u as the temperature, these 3 ends are kept at 0 temperature, we have to find the steady state temperature in the rectangular plate at any point xy.

We have 0 < x < pi, 0 < y < pi, okay, so what we will do; we will solve this partial differential equation by using the method of separation of variables, so let us put uxy = xx * yy in the given second order differential equation then we have x double dot * y + x * y double dot x * y double dot = 0, okay. So, uxx becomes x double dot y, uyy becomes xy double dot = 0 and when we divide by X * Y, this equation I get x double dot over x + y double dot over y = 0.

Or, I can write it as x double dot over x = - y double dot over y, okay, so now left hand side is a function of x only and right hand side is the function of y only, okay, both are equal, so they must

= constant let us say k, so we get 2 ordinary differential question of second order, x double dot – kx =0 and y double dot + ky =0, okay. Now, there are 3 cases, k = 0, let us take k =0, sp k = 0 means, x double dot = 0 and k = 0 means, y double dot = 0.

X double dot = 0 means, x = ax + b, y double dot = 0 means, y = cy + d; cy; okay cy + d, so what we have? So, this means that uxy = ax + b * cy + d, okay. Now, let us use the boundary conditions when x is 0, y0 = 0, so u0y = B times cy + d, okay; u0y = 0 gives B =0 or c = d = 0, okay so this gives you; B = 0 or c = d = 0. Now, if c = d = 0, then y = 0; y = 0 means, u = 0, okay so trivial solution.

And therefore, we shall not consider c = d = 0, let us take B = 0; if B =0, then what we will get? If B =0, then uxy = a times x + * cy + d, okay, now again u pi y = 0, so u pi y = 0 means, again either a = 0 or c = d = 0; c = d = 0 is not possible, so a must be 0 and when a is = 0, a = 0, b = 0, okay both give x = 0, this X = 0, so again we get a trivial solution, so k = 0, is not possible. Let us now consider the case k + mu square; k is > 0, okay.

So, take k > 0, say k = mu square, then what we will get? D square x over dx square – mu square x = 0 means x will be = a e to the power of mu x + b e to the power –mu x and here k = mu square means, the auxiliary equation will be m square + mu square = 0, so m will be +- ik, so will have complex roots, so y will be = c times cos mu y + d times sin mu y, okay what we will get? So, uxy will be = xx * yy, so a e to the power mu x + b e the power –mu x * c cos mu y + d sin mu y, okay.

Now, let us see u0y = 0, so uoy = 0 gives; put x =0, so we get a + b * c cos mu y + d sin mu y, okay, so either a + b, 0 or c and are = 0; c = d = 0 will give uxy = 0, so we cannot take c = d = 0, so we consider a + b =0, now let us put x =pi, so u piy = 0 means, a e to the power mu pi + b e to the power – mu pi times c cos mu y + d sin mu y = 0, again c = d = 0 is not possible, so we take this = 0.

Now, from this equation b = -a, if you put b = -a in this, what we will get? a times e to the power mu pi – e to the power –mu pi = 0, okay. Now, if a = 0, then b =0, then we get trivial solution, otherwise e to the power mu pi = - e = e to the power –mu pi, okay. So, I mean to say a = 0, or e to the power mu pi = - e to the power; sorry; = e to the power – mu pi, okay, a =0 gives; a = 0 you put here, you get b =0 and thus then we get the trivial solution, okay.

If you take e to the power mu pi = e to the power – mu pi, then you get e to the power 2 mu pi = 1, and which gives mu = 0 and mu = 0 is not possible because we are considering k is > 0, k = mu square, so this is not possible. So, what we do; we consider the case k < 0, where k is – mu square. When k is – mu square, you put here – mu square d square x over dx square + mu square x = 0 will give you a cos mu x + b sin mu x, the solution, okay.

And here, when we k = – mu square, you get here the solution of this equation; d square y over dy square – mu square yx c cos hyperbolic mu y + d sin hyperbolic mu y, we can write it as c cos c e to the power mu y + de to the power –mu y also, okay this part, this part can be written also as c e to the power mu y + d e to the power mu y, we can write either one, either this one or this one.

Because we know that e to the power theta = cos hyperbolic theta+ we have cos hyperbolic theta = e to the power theta + e to the power – theta/2 and sin hyperbolic theta = e to the power theta – e to the power – theta/ 2, okay. So, corresponding to e to the power mu y, you can write; if you add here, e to the power theta becomes cos hyperbolic theta + sin hyperbolic theta and e to the power – theta becomes cos hyperbolic theta – sin hyperbolic theta, okay.

So, e to the power mu y, you can replace y cos hyperbolic mu y + sin hyperbolic mu y and e to the power – mu y and you can replace by cos hyperbolic mu y - sin hyperbolic mu y a and then collect the coefficients of cos hyperbolic mu y and sin hyperbolic mu y a, they are some constants, so we can replace them by new constants and we have the solution also in this form, okay.

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$$u(0,y)=0 \Rightarrow A=0$$

then $$u(x,y)=\sin \mu x(E\cosh \mu y+F\sinh \mu y)$$

$$u(\pi,y)=0 \Rightarrow \mu=n, n=0,\pm1,\pm2,\ldots$$

then $$u(x,y)=\sin nx(E_n\cosh ny+F_n\sinh ny)$$

$$u(x,\pi)=0 \Rightarrow \frac{F_n}{E_n}=-\frac{\cosh n\pi}{\sinh n\pi}$$

Now, $$u(x,y)=\sum_{n=1}^{\infty}\sin nx(E_n\cosh ny+F_n\sinh ny)$$

where $$\frac{F_n}{E_n}=-\frac{\cosh n\pi}{\sinh n\pi}$$

So, now what we do what, let us use the boundary conditions we are given, u0 y = 0; when u0y =0, we will have here, put x = 0, so this will reduce be a +; this will be 0, so a times c cos hyperbolic mu y + d sin hyperbolic mu y, this is 0 for all y, okay so that will mean that either a = 0 or c = d = 0; c = d = 0 will give this part corresponding to this Y as 0 and therefore, uxy will be 0 for all x and all y, so which will be a trivial solution.

So, we consider a = 0, so u0y = 0 gives a =0 and then and then what will happen is that this equation will reduce to ux y = b sin mu x * c cos hyperbolic mu y + d sin hyperbolic mu y, this b can be multiplied inside bc can be replaced by new constant and bd can also replaced by new constant and we can write sin mu x times e cos hyperbolic mu y + f sin hyperbolic mu y. Now, let us use the boundary condition, u pi y = 0.

When u pi y = 0, what we will get? Sin mu pi times e cos hyperbolic mu y + f sin hyperbolic mu y, okay = 0, so again e and f are both 0's or sin mu pi =0, e and f both; if we take 0, then uxy will be 0 for all x and y, so we take sin mu pi = 0 and sin mu pi = 0 gives you mu pi = n pi, okay, so we can cancel pi and we get mu = n, here n takes values 0, + -1, + -2 and so on okay. So, thus we get ux/s sin nx corresponding to each value of n, we will have a constant cnf, with enf, we can write them as en and fn.

Now, here we shall be considering only positive integral values of n not n = 0 or negative integer values of n; n = 0 if you take, then what we will get; mu will be 0 but k = mu square, okay and k is < 0, so mu can ever be 0, and therefore n = 0 is not admissible, when if you take n = -1, -2 and so on, then we know that sin – theta, okay = -sin theta, okay, so that negative value of n will not get any another solution only the constants will be change.

They will be replaced by new constants, so negative value does not give any new solution, therefore we only consider n = 1 to 3 and so on, so for each value of n = 1 to 3 and so on, un, xy; this could be un, xy; un, xy is = sin nx en cos hyperbolic ny + fn sin hyperbolic ny. Now, we know that ux pi = 0, this is given to us, the condition ux pi = 0, okay, so this condition we use and so put y = pi here, when y = pi, we get sin nx times en cos hyperbolic ny; n pi, fn sin hyperbolic n pi = 0 which means that this is = 0.

So, we can write en cos hyperbolic n pi = - fn sin hyperbolic n pi and therefore, fn over en will be = - cos hyperbolic n pi over sin hyperbolic n pi, now let us write this equation uxn in order to include the boundary condition; ux0 =sin square x, we need to consider linear combination, this sigma n = 1 to infinity sin nx en cos hyperbolic n pi; ny + fn sin hyperbolic ny, here we can make use of this relation, fn over en = -cos hyperbolic n pi over sin hyperbolic n pi here.

And then it will become; okay so let me write; what we have here, this is uxy, if you use this uxy = sigma n = 1 to infinity, en; sorry, sin nx times, we want to replace fn by en, so we take en outside and then we have cos hyperbolic ny – fn over; + fn over en, so fn over en will be -cos hyperbolic n pi over sin hyperbolic n pi multiplied to sin hyperbolic ny, okay, so this is what we have and therefore, we just had to evaluate this constant en.

For that we make use of the condition, ux0 = sin square x, so you when you put y = 0 in this what you get? Ux0 = sigma n = 1 to infinity sin nx * en and you put y = 0; y = 0 means cos hyperbolic 0, cos hyperbolic 0 is 1 and here what we will get? Sin hyperbolic 0 is 0, so we will get en * 1, okay and this is Fourier, half range Fourier sin series, okay so what we do? We multiplied both sides by sin mx and then integrate over 0 to pi.

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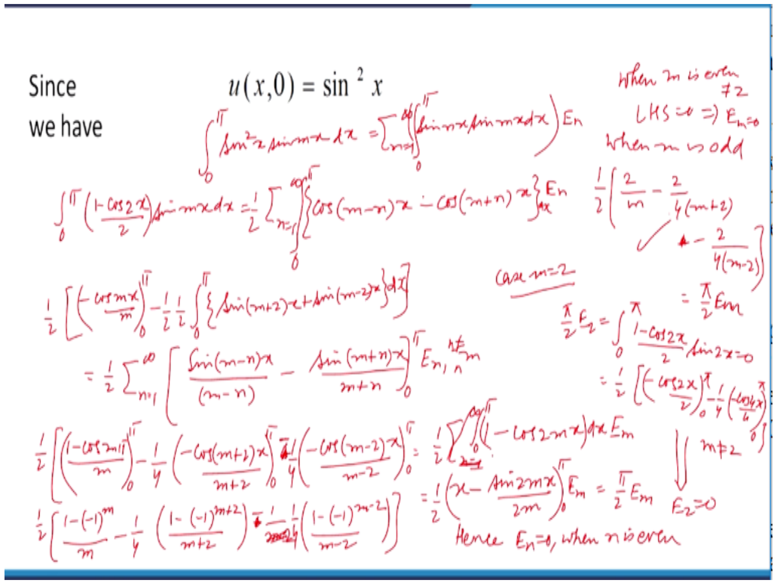

So, what we have? We have this one, okay, so we have ux0 is sin square x; so sin square x * sin mx integral over 0 to pi dx, okay, sigma m = 1 to infinity sin nx 0 to pi * sin mx dx multiplied to en, so we multiply both sides by sin mx and then integrate over the interval 0 t0 pi, now this I can write 0 to pi, sin square x can be written as 1 – cos 2x/ 2 * sin nx dx = sigma n = 1 to infinity.

We can multiply and divide by 2, so 1/ 2 2 sin nx ; sin nx dx, we can write that as cos m – n * x – cos m + n * x * en, now we integrate this over 0 to pi dx, so we will integrate this, so we will have 1/2, the integral of sin mx will be - cos mx divided by m, 0 to pi and then we have -1/2, sin mx * cos 2x, so again 1/2 integral 0 to pi, we 2 sin mx cos 2x, so we have sin m + 2 * x + sin m – 2 and 2x dx, this is the left hand side, okay = 1 over 2 sigma n = 1 to infinity, integral of cos m – n * x will be sin m – n over m – n * x over m – n, assuming that n is != m – sin m + n * x divided by m + n0 pi * en.

So, here we are taking n! = m; n != m, so that the division by m – n is defined, right, now this is how much? 1/2; we have 1 – cos n pi e/ m, okay and -1/4, here we will have -cos m +2 * x divided by n + 2, 0 to pi and here you will have - cos m -2 * x divided by m – 2, 0 to pi, so for this to be able to divide sin function by m -2, we need to consider m! = 2, so m is != 2 here, all right, now this is what, okay.

So, 1/2 sigma n = 1 to infinity sin m – n, okay now when we put the limits, sin m – n * pi is 0, sin m + n * pi is 0, the whole thing is 0, when you put 0 for x and 0 for y, these again 0, okay, so what we get? So, we can say that sigma 1/2 sigma n = 1 to infinity okay, this quantity becomes 0, so we can say that we have to consider m = n, okay, this quantity and the bracket become 0, we cannot determine this thing en.

So, let us put m = n, when we put m = n, what we get? 1/2 sigma n = 1 to infinity m = n, okay, m = n means, 1 – cos 2mx cos 2mx divided by 1 – cos 2x okay, integral 0 to pi dx and em because we are taking only m, so this will be only this, okay, 1 over 2, 0 to pi 1 – cos 2 mx dx * em, this is what we will get and we are first calculating for m! = 2, so we will have integral here, x – sin 2mx divided by 2m 0 to pi, this is 1/2 here.

So, em, what do we notice at x = pi, we have pi/2 * em, this term become 0 to pi and both the terms becomes 0 at x =0, so we have 1/2 pi * em m is !=2 and what will happen here, we have here 1/2, 1 – cos m pi, so 1 - -1 to the power m divided by m and here we will have -1/ 4 1 - -1 to the power m + 2 divided by m +2, okay and here what do we have? 1 over m -2 okay, 1 - - 1 to the power m -2 over m -2 okay, this is what we get.

When m is any integer other than 2, so here what will happen is that you can see, if m is even what we get? M is even then this is 1 – 1, so 0, m is even means m + 2 even, so this is also 0, when m is even, 1 - 1 to the power -2 is 0, so this is also 0, so let us take m to be odd, okay, m = 2 case, we will resolved later, so let us take m = odd, so when m is odd, when m is even, ! = 2, okay we get left side is 0.

So, em = 0, okay, when m is odd, okay, we can get, this is 1/2; 1 - - 1, so we get 2 upon m -; this will be 1/4 2 upon m + 2, so 2/4 m +2, okay and here we will get 2 upon m – 2, this 1/4 have to multiplied to both, here also okay because this is 1/4 we are integrating this, this integration we are doing, so – and this will be also - because we have – sign here; - sign here, so that - times this – times this will have, so this is - we will have okay, so we have – here, okay.

Then, what will have here? 2 over 4 okay, so this is also 1 over 4 and with negative sign, so -2 over 4 m -2 = pi/2 * en, okay, these regarding m is odd, okay this we can simplify in get the value of em for odd, let us look at the case m = 2, the case m = 2 separately, so m = 2 means, pi/2 * e2, okay, m = 2, let us put here in the left hand side, then we will have integral 0 to pi 1 – cos 2x/2 * sin 2x, okay

So, this will be = 1/2 integral of sin 2x will be – cos 2x/2, 0 pi and then sin 2x cos 2x is 1/2 sin 4x, so we will have -1/4 –cos 4x/4 0 pi okay and what we get? 0 pi, if you put here, cos 2 pi is 1, cos 0 is also 1, so they will be same okay, they will be same and therefore, this value will be 0 and similarly this will be 0, so e2 = 0, so this implies e2 = 0, so hence en is 0, when n is odd, when n is even sorry, n is even.

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$$\Rightarrow \qquad E_m = \begin{cases} -\dfrac{8}{\pi m(m^2-4)}, & m \text{ is odd} \\ 0, & m \text{ is even} \end{cases}$$

Hence,

$$u(x,y) = \frac{8}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n(n^2-4)} \left(\frac{\sinh n(\pi - y)}{\sinh n\pi} \right) \sin nx$$

And en is = this quantity, em gives you; you can find the value of em from here, for m is = what? and so that you can get n, okay, so em comes out to be this okay and you can then put the value of em here okay, in this series, where we have written this series, 0 to pi, we wrote; in this series we have written okay, n = 1 to infinity sin nx en times this, so here en become 0, when n is even only when n becomes odd, we get the value of em.

And em = -8 over pi m * m square – 4, so that value you put in that series, we get the result okay and then this can be simplified; sin hyperbolic n pi * cos hyperbolic n y – cos hyperbolic n pi over sin hyperbolic n pi that can be written in this form; sin hyperbolic n pi – y, so this is how we get the solution of this.

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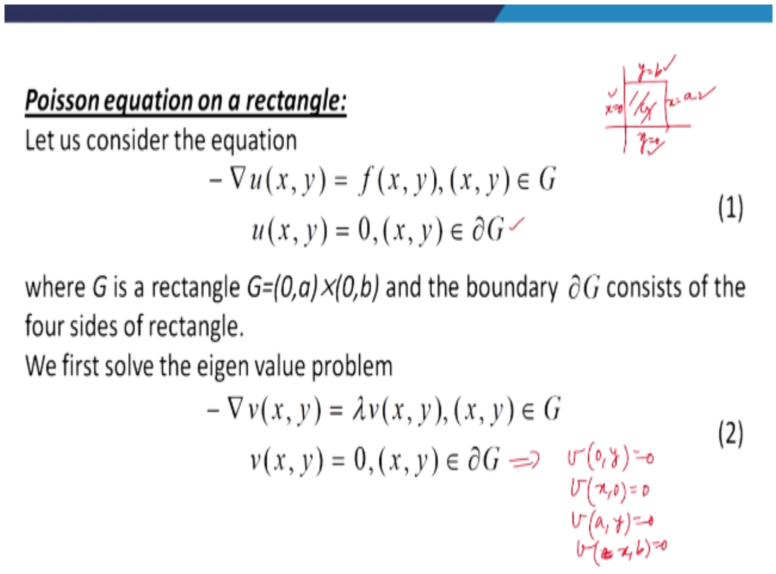

Now, let is consider Poisson equation on a rectangle, let us consider the equation; Poisson equation, - del u x y = fx y, where xy belongs to G, G is the rectangle 0a cross 0b, this is y = 0, this is y = b, here x = 0, x = a, okay, so we are considering the Poisson equation on a rectangle, okay, G denotes the interior of this rectangle, this is G, okay and this boundary consisting of 4 sides is of the rectangle is del G, okay.

So, boundary consist of the 4 sides of the rectangle, now first we will solve the Eigen value problem – del v xy = lambda v xy, where belongs to G and we will consider the Dirichlet boundary conditions that is bx value 0, on the boundary of G that is on the 4 sides, so v xy = 0 on the boundary of G means, v 0y = 0, this one, v0 y = 0, then v x0 = 0 corresponding to this side and then va y = 0 corresponding to this side and v0; vx v = 0 corresponding to this sides.

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We consider the case of Dirichlet boundary conditions so $\lambda > 0$.

Let us assume

$$v(x, y) = p(x)q(y) \qquad (3)$$

Since $v(x, y) = 0$ on $(x, y) \in \partial G$, we have

$$p(0) = 0,\ p(a) = 0,\ q(0) = 0 \text{ and } q(b) = 0$$

$v(0,y)=0$
$v(a,y)=0$
$v(x,0)=0$
$v(x,b)=0$

Substituting (3) into (2) we get

$$-\frac{p''(x)}{p(x)} - \frac{q''(y)}{q(y)} = \lambda$$

$$-\frac{p''(x)}{p(x)} = \lambda + \frac{q''(y)}{q(y)}$$

So, these are 4 boundary conditions, which come from this Dirichlet condition, v xy = 0. Now, we consider the Dirichlet boundary conditions; in the case of Dirichlet boundary conditions, this lambda is strictly positive, now let us put the v xy the solution or we let us find of this equation, okay, so let us put v xy = px * qy function of x * a function only y, since vxy = 0 on xy belongs to G we have okay.

Now, you can see v0y = 0 gives you p0 = 0, okay, vay = 0 gives you pa = 0 and vx0 = 0 gives you q = 0, vxv = 0 gives v qv = 0, now substitute in this vxy =px * qy into equation 2, this equation, okay, what we have this; - p double dash x over px – q double dash y over qy = lambda.

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Since $-\dfrac{p''(x)}{p(x)}$ depends only on x and $-\dfrac{q''(y)}{q(y)}$ depends only on y,
both terms must be constants.
We get

$$-\frac{p''(x)}{p(x)} = \mu, \; -\frac{q''(y)}{q(y)} = \upsilon, \; \mu + \upsilon = \lambda.$$

Thus, we obtain two eigen value problems:

$$-p''(x) = \mu p(x), \; p(0) = 0, \; p(a) = 0$$

$\Rightarrow$ the eigen values $\mu_j = \left(\dfrac{j\pi}{a}\right)^2$ and eigen functions

$$p_j(x) = \sin\left(\frac{j\pi x}{a}\right), \; j = 1, 2, \ldots$$

$(D^2+\mu)p(x)=0$
$m^2 = -\mu$
$p(x) = A\cos\sqrt{\mu}x + B\sin\sqrt{\mu}x$
$p(0)=0 \Rightarrow A=0$
$p(x) = B\sin\sqrt{\mu}x$
$p(x) = \sin\sqrt{\mu}x$
$\sin\sqrt{\mu}a = 0$
$\sqrt{\mu} = \frac{n\pi}{a}$
Eigen value $=\mu$

If we assume that – p double dash over x over px depends only on x - q double dash y over qy depends only on y, so both must be a constant, okay, why? Because you can see like this, this quantity you bring to the other side, okay, lambda is a constant, so you can write – p double dash x upon p dash x = lambda +; okay, now this is the function of x alone, this is a function of y alone, okay, lambda is the constant.

So, left hand side depends on x, right hand side depends on y and therefore, both must be = a constant, this must be a constant and this must be a constant, okay, this is a constant means, q double dash y; -q double dash y over qy is a constant, so let us take this = mu, this = nu, okay, then we can see that mu + nu = lambda, okay, so mu + nu = lambda, now this leads us to 2 Eigen value problems; -p double dash x = mu px – q double dash y = nu qy, okay.

And the boundary conditions on these Eigen value problem are p0 =0, pa = 0, we know that when you solve this problem, p double dash x; -p double dash x = mu px, you get these equations; d square + mu; d square + mu px = 0 okay, so px; so this means that m square = - mu, this gives us 2 complex roots because mu, we are taking here as positive oaky, so px will be = a cos root mu * x + B sin root mu * x, okay, so what will happen?

P0 =0 will give; p0 will give A =0 , so px will be = B sin root mu * x, okay and therefore, B can be taken as 1 without any loss of general things; px will be = sin root mu * x, so eigenvalues are;

eigenvalue is mu, okay, apart this Eigen value problem, - p double dash x = mu px eigenvalues mu, so eigenvalues = mu okay and okay, we use this other condition pa = 0; pa = 0 means, sin root mu a = 0, so root mu = n pi/ a, okay.

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Further, $$\int_0^a p_j(x)\,p_k(x)\,dx = 0, \quad j \neq k$$

i.e. the eigen functions are orthogonal and complete on the interval [0,a].

Now, $$-q''(y) = \upsilon q(y),\ q(0) = 0, q(b) = 0$$

$\Rightarrow$ the eigen values $\upsilon_k = \left(\frac{k\pi}{b}\right)^2$ and eigen functions

$$q_k(y) = \sin\left(\frac{k\pi y}{b}\right),\ k = 1,2,\ldots$$

Further, the eigen functions satisfy

$$\int_0^b q_j(y)q_k(y)\,dy = 0, \quad j \neq k$$

So, mu = n pi/ a whole square, so we get in place of n, I am writing j, so j pi/ a whole square, we get and the Eigen functions are sin root mu is n pi or you can say j pi/a, so sin j pi x/ a, we get; the sin functions are satisfied the orthogonality property that is integral 0 to a pj x, pkx, dx = 0, whenever j is != k, they are orthogonal also they are complete that means 0 to a pj square x dx is non zero, we can calculate its value.

Similarly, the other eigenvalue problem; this one; -q double dash y over qy = v with boundary conditions q0 = 0, qv = 0 has gives us the eigenvalue = vk =; Eigen value nu k = k pi over b whole square and qk y Eigen functions; qk y = sin k pi y over b, these Eigen functions again satisfy this orthogonality property and they are complete on the interval 0, b.

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and are complete on [0,b].

Thus, the eigen function $v_{jk}(x,y) = p_j(x)q_k(y)$ and eigen values

$$\lambda_{jk} = \mu_j + \upsilon_k, \text{ for } j = 1,2,\ldots \text{ and } k = 1,2,\ldots$$

with the inner product

$$\langle f, g \rangle = \iint_G f(x,y)g(x,y)dxdy.$$

We have

$$\langle v_{jk}, v_{j'k'} \rangle = \int_0^a \int_0^b v_{jk}(x,y)v_{j'k'}(x,y)dxdy = 0 \text{ for } (j,k) \neq (j',k').$$

For any function $F(x,y)$,

$$\int_0^a \int_0^b (p_j(x)q_k(y))(p_{j'}(x)q_{k'}(y))dxdy = \left(\int_0^a p_j(x)p_{j'}(x)dx\right)\left(\int_0^b q_k(y)q_{k'}(y)dy\right) = 0 \cdot 0$$

Thus the Eigen function; vjk xy is pjx * qky and eigenvalues are lambda = mu + nu, we have the eigenvalue is here, this is the Eigen value lambda, okay, you can see here, lambda is the Eigen value for this Eigen value problem and lambda is the sum of the Eigen values of the 2 Eigen value problem, so lambda = mu + nu, so lambda = mu + nu gives you; lambda jk = mu j + nu k, for j = 1 to n and so on and k = 1 to n and so on.

Now, the inner product of f and j is defined as over the; double integral over g fxy gxy dy dx dy, so inner product of vjk with j dash k dash is 0 to a, 0 to b, vjk xy vj dash k dash xy dx dy, now vjk is what? Vjk = pjx qky, so we have 0 to a, 0 to b pjx * qky, this is what we have for vjk and similarly for vj dash k dash, we have pj dash x * qk dash y, okay * dx dy. We can then separate the integrals, 0 to a, pjx * pj dash because they are orthogonal know, they are orthogonal.

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the Fourier coefficients F_{jk} are given by

$$F_{jk} = \frac{\langle F, v_{jk} \rangle}{\langle v_{jk}, v_{jk} \rangle}$$

$$= \frac{\int_0^a \int_0^b F(x,y) p_j(x) q_k(y) dxdy}{\left(\int_0^a p_j^2(x) dx \right)\left(\int_0^b q_k^2(y) dy \right)}$$

and then

$$F(x,y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} F_{jk} v_{jk}(x,y).$$

Okay, did that property we satisfy, so we have pj dash x dx * 0 to b qky * qk dash y dy, so they are 0's, okay 0 * 0 whenever j is != j dash, k is != k dash, now for any function fxy, the Fourier coefficients fjk are given by fjk = f vjk over vjk vjk, f vjk is the inner product with vjk, so 0 to a, 0 to b, f xy, vjk are pjk qky, so qky df dy and this vjk, vjk is the inner product of vjk with vjk and that we have already seen, okay.

This is vjk, vjk, so here in place of j dash, k dash, we will write jk, okay, so we will have 0 to a pj square x dx and 0 to b qk square y dy, so this is what we have and then fxy is given by sigma j = 1 to infinity, k = 1 to infinity fjk vjk xy.

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Solution of the Poisson problem:

The solution $u(x,y)$ can be represented in terms of its Fourier series

$$u(x,y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} u_{jk} v_{jk}(x,y)$$

where the coefficients u_{jk} need to be determined. Substituting this into (1), we get

$$-\nabla u = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} u_{jk} \left(-\nabla v_{jk}(x,y) \right)$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} u_{jk} \lambda_{jk} v_{jk}(x,y) = f(x,y)$$

Solution of the Poisson problem, let us we have to find the solution of the Poisson equation, uxy can be represented as sigma j = 1 to infinity, k = 1 to infinity ujk vjk xy by its Fourier series, we have to determine the values of ujk, so ujk have to be determined substituting this uxy series of uxy; Fourier series of uxy * this equation – del u = this one, we have this one equation, -del u = fxy, we have this okay, -del u = sigma j = 1 to infinity, k = 1 to infinity ujk –del vjk xy.

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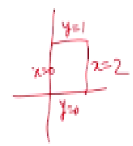
and obtain that $u_{jk}\lambda_{jk}$ are the Fourier coefficients of the function $f(x, y)$ hence

$$u_{jk} = \lambda_{jk}^{-1}\frac{\langle f, v_{jk}\rangle}{\langle v_{jk}, v_{jk}\rangle}.$$

Let us solve the Poisson equation for the case $f(x, y) = 1$.

Let us assume $a = 2, b = 1$.

then



$$\mu_j = \left(\frac{j\pi}{2}\right)^2, p_j(x) = \sin\left(\frac{j\pi x}{2}\right), \quad j = 1,2,...$$

and $$\upsilon_k = (k\pi)^2, q_k(y) = \sin(k\pi y), \quad k = 1,2,...$$

And this sigma j = 1 to infinity k = 1 to infinity –v del vjkxy is lambda jk vjk xy by the eqaution this one, this equation, okay, so we have this and this is = fxy. Now, ujk lambda k are jk are the Fourier coefficients of the function fxy, they are the Fourier coefficients of the function f xy and ujk * lambda jk are given by the coefficient of; Fourier coefficient of the inner product with vjk over inner product of vjk with vjk.

So, ujk can be determined as lambda jk inverse * this, now let us solve the Poisson equation for the case fxy = 1, we take xa = 2, v = 1, so we are taking this rectangle, at this is x = 0, this x = 2, this y =0 and y = 1, okay and f = 1, so in that case mu j will be j pi over 2 whole square, pja will be sin j pi x over 2, we are taking a = 2, v = 1 and then vk will be k pi whole square qk pi will be sin k pi y, okay.

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Therefore the eigen values

$$\lambda_{jk} = \left(\frac{j^2}{4} + k^2\right)\pi^2; \; j,k = 1,2,...$$

and the eigen functions

$$v_{jk} = \sin\left(\frac{j\pi x}{2}\right)\sin\left(k\pi y\right).$$

$$\langle v_{jk}, v_{jk}\rangle = \left(\int_0^2 p_j^2(x)\,dx\right)\left(\int_0^1 q_k^2(y)\,dy\right) = 1$$

Now,

$$\langle v_{jk}, v_{j'k'}\rangle = \frac{1}{2}$$

And then the eigenvalues lambda jk are mu j + qk, so j square pi square/ 4 k square pi square, eigen functions vjk are pjx * qky, so this into this, now vjk * vj dash k dash, okay; vjk, vjk we can find, vjk vj dash k dash is = 0, whenever j is != j dash, k dash is != k dash, so inner product of vjk with vjk, this is = integral 0 to a; 0 to a means 0 to 2, pj square x dx, okay and then 0 to 1, okay qk square y dy, so we can put the value of pjx as sin j pi x over 2, qky as sin k pi y and evaluate this.

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and

$$\langle F, v_{jk}\rangle = \begin{cases} \frac{4}{j\pi}\cdot\frac{2}{k\pi}, & \text{if } j \text{ and } k \text{ are odd} \\ 0, & \text{if } j \text{ and } k \text{ are even.} \end{cases}$$

This comes out to be 1/2 and then f vjk will be =; f vjk will be = inner product of f with vjk; inner product of f with vjk, this one, integral over 0 to a, 0 to b, fxy pjx qky dx dy, this we can find, this comes out to be 4 over j pi * 2 over k pi, if j and k are odd, 0, if j and k are even, we know the values of vjk; vjk is pjx * qky and f is f xy = 1, okay. Fxy = 1, so we will get the value of this.

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Hence,

$$u_{jk} = \frac{16}{jk\pi^4\left(\frac{j^2}{4} + k^2\right)}, \quad \text{both } j \text{ and } k \text{ are odd}$$

which implies

$$u(x,y) = \sum_{j=1,3,5,...} \sum_{k=1,3,5,...} \frac{16 \sin\left(\frac{j\pi x}{2}\right) \sin(k\pi y)}{\pi^4 jk\left(\frac{j^2}{4} + k^2\right)}.$$

And then, ujk can be determined, ujk is = lambda jk inverse f vjk over vjk, vjk, okay both j and k are odd, which implies uxy = this expression; uxy is = this, we have this expression for any function fxy, we had this, so for the function uxy, we can find j = 1 to infinity, k = 1 to infinity, these series, so we get this; uxy = this expression, so this how we solve this problem, with this I would like to end my lecture, thank you very much for your attention.