Ordinary and Partial Differential Equations and Applications
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Lecture – 05
Existence and Uniqueness of Solutions of Differential Equations- III

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Consider the following first order nonlinear differential equation

$$\frac{dx}{dt} = f(t, y) \qquad (1)$$
$$y(t_0) = y_0 \qquad (2)$$

where f, is a given function of t and y, defined and continuous in some neighborhood of the point (t_0, y_0). We can define a sequence of associated approximating solution as follows.

$$y(t_0) = y_0,$$
$$y_{j+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_j(s))ds, \quad j = 0, 1, 2, 3 \ldots \qquad (3)$$

Hello friends, welcome to the lecture, in this lecture, we will continue our study of existing and uniqueness theorems for ordinary differential equation. So, if you recall in previous lectures, we have discussed the following first order ordinary differential equation that is; dy/dt = f (t, y) yt0 = y0, where f is a given function of t and y, define and continuous in some neighbourhood of the point t0, y0.

And we have discussed the existing and uniqueness of the system 1 and 2 with the help of defining a hydrated sequence of solutions, so we can define a sequence of associated approximating solution as follows, so we have taken y t0 = y0 and yj + 1t = y0 + t2 t0 fs yjs ds, where j is learning from 0, 1, 2, 3 and so on. So, what we have done in previous class that once we have defined this sequence of associated approximated solution and the condition that fdy is satisfying the Lipschitz condition in rectangle R.

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Then, existence and uniqueness theorem is given as follows:

Theorem 1

If $f(t, y)$ is a continuous function of t and y in a closed and bounded rectangle R and satisfies the Lipschitz condition in R. Then the successive approximations y_j, given by (3), converge(uniformly) on the interval $J = \{t : |t - t_0| \leqslant \alpha\}$, to a solution y of the differential equation (1) that satisfies the initial conditions (2).

Example 1

Given that

$$\begin{aligned} y' &= t(1+y) \\ y(0) &= -1 \end{aligned} \tag{4}$$

We have shown that this sequence will converge to a solution and that solution is continuous and satisfy the system 1 and 2, so this we have already discussed in previous lecture and we have discussed the following theorem which is given as follows; that if fty is a continuous function of t and y in a close and bounded rectangle R and satisfies the Lipschitz condition in terms of Y in R, then the successive approximation yj, which we have defined earlier converges uniformly on the interval j.

And j is defined as t; all the t, so is that modulus of the t – t0 is <= alpha, where alpha is defined as minimum of A and B/m to a solution y of the differential equation 1 that satisfy the initial condition term, so this we have discussed. Now, based on these existence and uniqueness theorem, let us discuss certain example, so, first example is this that given that y dash = t (1+ y) is the differential equation is given to us.

And the initial condition is given y0 = -1 and we want to find out the solution satisfying the condition y0 = -1 but if you look at the domain where this y0 = -1 is a; if you look at this 1 + y is not defined in that particular reason, so here we cannot directly solve this particular equation by dividing by 1 + y, it means generally if you look at this is a separable equation, in separable equation, we can solve like y dash = t (y + 1).

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$$y' = t(y+1) \qquad y(0) = -1$$

$$\frac{dy}{y+1} = t\,dt$$

$$\Rightarrow \ln(y+1) = \frac{t^2}{2} + c$$

$$y+1 = c\,e^{t^2/2}$$

$$y = c\,e^{t^2/2} - 1 \qquad y(0) = -1$$

$$c = 0 \qquad \Rightarrow y(t) = -1$$

And if you want to solve this, we simply write dy/y + 1 = tdt and we can solve like, this is ln y + 1 = t square/ 2+ some constant, so we can write y +1 = e to the power t square/ 2 and we can consider this constant as e to the power c, so you can write it like this and we have y = c e to the power t square/ 2 - 1 that is what we generally used to do and then put y0 = -1 then you can see that here this c comes out to be 0 and you can see that our solution yt is coming out to be -1 and (()) (04:29).

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Solution. Let $R : \{(a,b) \in \mathbb{R}^2 : |t - 0| \le a, |y + 1| \le b\}$ and let $f(t,y) = t(1+y)$. Then f is continuous on R and

$$\begin{aligned} |f(t,y_1) - f(t,y_2)| &= |t||y_1 - y_2| \\ &\le a|y_1 - y_2| \end{aligned}$$

i.e. f is Lipschitz on R so by existence and uniqueness theorem, there exists a unique solution of (4) on $|t| \le h = \min(a, b/M)$, where $M = \max_{(t,y)\in R}|f(t,y)|$ We may observe that $y(t) = -1$ is a solution of (4) satisfying the initial condition. So by existence and uniqueness Theorem, $y(t) = -1$ is the only solution of (4).

So, this is a constant solution given as -1, we can get this solution in a different way also, if you look at this t; this is fty is t * 1 + y and we can easily see that here, if we define a domain like this that R is a rectangle define as all those AB in R2 such that modulus of t is <= a and modulus of y+1 is <= b and we define f(t, y) as t * 1 + y, then we can easily see that this f is continuous for all t.

And we can also verify that this f (t, y) satisfies the Lipschitz condition that is ft y1 – ft y2 = modulus of t y1 –y2 and we can say that since modulus of t is <= a, we can say that this is <= a times modulus of y1 – y2, so it means that this f (t, y) satisfy the Lipschitz condition in terms of y, so it satisfy the Lipschitz condition, so we can say that f is Lipschitz on rectangle R, so by existence and uniqueness theorem, they exist a unique solution of the equation on modulus of t <= h.

Where h is the minimum of a and b/ M and how this M, we can define? M is the maximum of modulus of f (t, y), where t, y belongs to this region R, so we may observe that here if you look at M, M is what? Maximum of this quality and you look at that t is bounded by a and modulus of 1 + y is bounded by b, so we can say that this M is given by, you can say that it is given by ab, a unique solution exist in the interval modulus t <= h.

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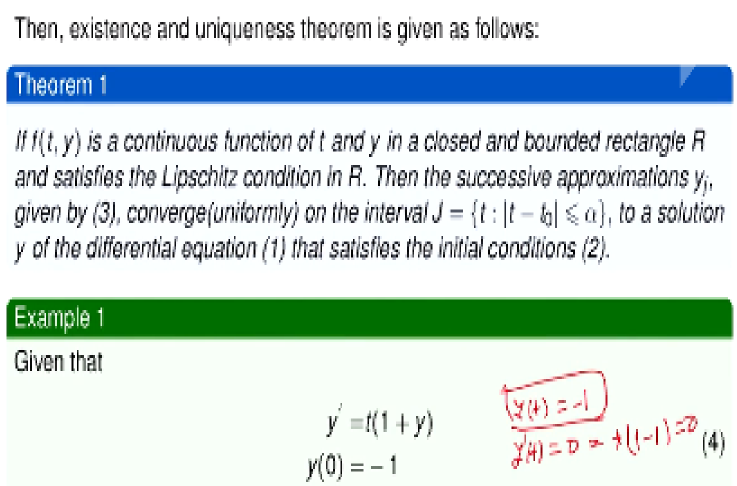

And we want to find out the unique solution and we can observe here that yt = -1 is already a solution of this equation, if you look at that yt = -1, if you look at then here y dash will be; y dash t is = 0 because it is a constant solution and if you look at it is given by t times y +1 is given as 1-

1 that is 0 here, so it means that yt = -1 satisfy this equation and we already know that it has a unique solution, so it means that there is a solution which we are searching for, okay.

So, though we have this difficulty that 1 + y may not be defined at one particular point of the domain, it is still with the help of existence and uniqueness theorem, we can find out the solution and in this case, it is given as yt = -1.

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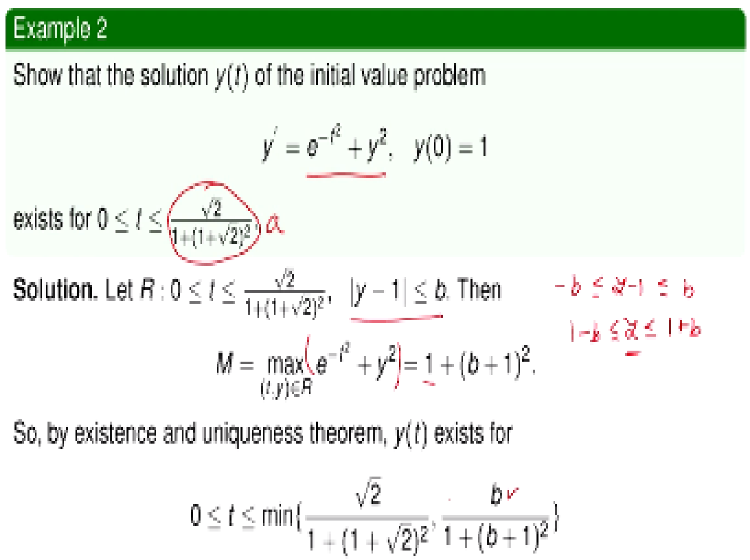

Now, coming on to example 2, here we look at the following differential equations, so that the solution yt of the initial value problem, y dash = e to the power –t square + y square, where y0 = 1 is given as the initial condition exists in this particular domain, the t is >= 0, <= root 2 upon 1 + 1 + root 2 whole square, so here actually we are looking only for the interval in which the solution exist.

So, first of all we find out that solution exists and then we try to find out in which interval this solution exist. so here let us defined the rectangle, so here the rectangle is defined in the sense that t is lying between 0 to this quantity; root 2 upon 1 + 1 + root 2 whole square and y is defined as y – y0; y0 is 1 here, so y; modulus of y – y0 <= b, so this is the rectangle R we have define.

And now, you utilise the existence and uniqueness theorem, we have proved, so first find out the maximum value of f (t. y), here f (f, y) is given as e to the power -t square + y square, so we want to find out the maximum of this over the rectangle R, so here maximum of ty belongs to R and e to the power – t square + y square, so we want to find out the maximum of this quantity. So, if you look at whatever t you will take, the maximum value of e to the power – t square is 1.

And maximum value of y square, if you look at this, this we can write it y -1 <= b; >= -b, so we can write y is <= 1 + b >= 1 – b, so we can say that y is bounded by 1 + b, so we can say that y square is bounded by 1 + b whole square, so we can say that maximum of e to the power – t square + y square is going to be 1 + b +1 whole square, so once we have M then we can utilise the existence and uniqueness theorem

And we say that solution exist in this interval, where alpha is minimum of the h, a given here like this, this is our a and b/ M, where M is defined like this, so first of all you please observe here that here, e to the power -t square + y square is a continuous function and it is defined in this rectangle R, which is a close rectangle, so maximum exist so and we have already find out the maximum M, it is given as this and it also satisfy the Lipschitz condition in this reason.

So, we can say that the solution exist, we are just looking at the interval where the solution exist, so here we want to find out the minimum of this quantity; root 2 upon 1 + 1 + root 2 whole square and b upon 1 + b +1 whole square, so here if you look at; we have to find out this region for Rb tb, so we had to find out the maximum value of this, so how to find out of this maximum value?

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Clearly, the largest h that we can achieve is the maximum value of the function $\frac{b}{1+(b+1)^2}$. This maximum value is $\frac{\sqrt{2}}{1+(1+\sqrt{2})^2}$ obtained at $b = \sqrt{2}$. Hence, by existence and uniqueness theorem, $y(t)$ exists for $0 \leq t \leq \frac{\sqrt{2}}{1+(1+\sqrt{2})^2}$.

So, here we can easily observe that the largest h that we can achieve is the maximum value of the function, b upon 1 + b + 1 whole square and we can easily see that the maximum value is coming out to be root 2 upon 1+1+ root 2 whole square which is obtained at b is = root 2. So, how to find out the maximum value of this function?

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$$g(b) = \frac{b}{1+(b+1)^2}$$

$$g'(b) = \frac{[1+(b+1)^2] - b\,[2(b+1)]}{[1+(b+1)^2]^2}$$

$$= \frac{1+b^2+1+2b-2b^2-2b}{[1+(b+1)^2]^2}$$

$$= \frac{2-b^2}{\ } \quad \Rightarrow g'(b) = 0$$

$$2-b^2=0$$

$$\Rightarrow b = \pm\sqrt{2} \qquad b = +\sqrt{2}$$

g''

So, for that just look at; you look at your g of b, you consider this as the new function of b, so it is b, 1 + b + 1 whole square right and we can get; we consider this as a function of b and to find out the maximum value of gb with respect to b, we just differentiate this g dash b, so how to find out the g dash b? So, it is 1 + b +1 whole square and here, we will get what? 1 + b + 1 whole square - b and this is what differentiation of this that is 2 b +1, right.

So, if you simplify this, it is 1 + b square +1+ 2b – 2b square – 2b, so if you simplify; 2b is cancel out upon this quantity, 1 + b + 1 whole square and this is going to be 2 - b square divided by this quantity, so this implies that g dash b is going to be 0, when 2 - b square is = 0, so we can say that b is given as + - root 2. Now, we already assume that b is a positive quantity, so we will not consider this – of root 2, so we simply say that b is = + root 2 is a critical point of this.

And we left it to you to show that this actually maximise the value of g or b with the maximum minimum result, so here we have seen that this gb can be extremise or can be maximise for b = root 2 and if you look at the value of this quantity when b = root 2, is given as root 2 upon 1+1+ root 2 whole square and it means that the solution of this problem exists for this quantity, this also is coming out to be the same root 2 upon 1+1+ root 2 whole square.

So, we can just say that solution exist in the set interval that is t is >= 0, <= this quantity; root 2 upon 1+1+ root 2 whole square, so here we have shown that the solution yt of this initial value problem exists and it is unique and it is defined in this particular interval.

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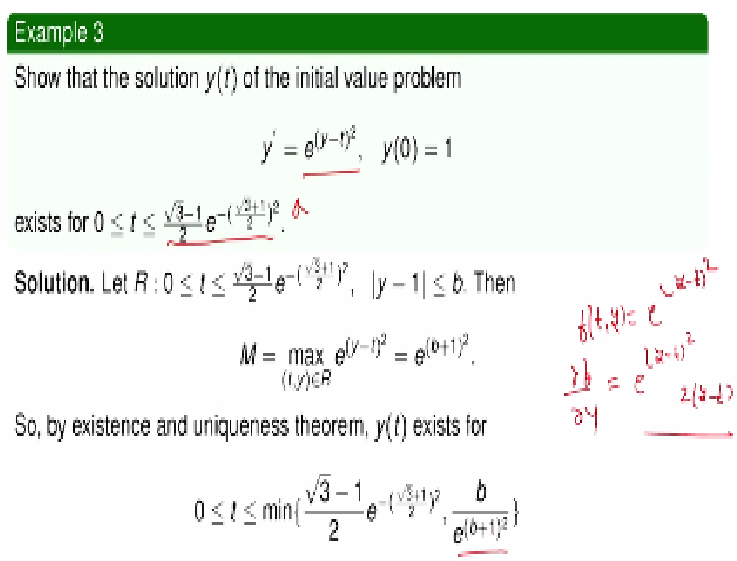

And now, moving onto next example which is similar to this and here we want to find out that show that the solution yt of the initial value problem; y dash = e to the power y –t whole square with the initial condition y0 = 1 exist in this set interval. So, it is again the similar problem and if you look at the f(t, y) is given as e to the power y – t whole square and here since it is a continuous function in the close rectangle which we are defined like this.

T is lying between 0 to root 3-1 upon 2 e to the power – root 3+1 / 2 whole square, y – y0; y0 is given as 1, <= b, so in this rectangle which is a close rectangle here, function f (t, y) is a continuous functions, so it is bounded, it will achieve the maximum value, so we say that by existence and uniqueness theorem, this has a solution and we want to find out the interval in which the solution exist.

So, here we say that since f (t, y) is e to the power y - t whole square then we can easily see that it satisfy the Lipchitz condition here, we can simply say that since f (t, y) is = e to the power y - t whole square, then we can easily check out dou f/ dou y exist and it is continuous, so you can simply say that e to the power y – t whole square and 2 of y – t will be there, so it is a continuous function and define on this close rectangle R, so it will be, it will be bounded.

And we have shown that if partial derivative exist and bounded, then f satisfy the Lipschitz condition. As if you look at the previous case, in this example when we look at f (t, y), here f (t, y) is what? Here, f (t, y) is e to the power t square + y square and we can easily check that it satisfy the Lipschitz condition by finding out f (t, y1) – f (t, y2), right, you just check that it is bounded by K times y1 – y2.

So, we can check that it is also satisfying the Lipschitz condition, so here we have shown that it satisfy the Lipschitz condition by showing that this partial derivative exist and it is bounded by some constant K, so it satisfy the all the condition of existence and uniqueness theorem, so we can say that by existence and uniqueness theorem, the solution of this initial value problem exist and in the exist in the interval, which is given as 0, <= t, <= minimum of the value a which is given here and b/ M.

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$$f(t, y) = e^{(y-t)^2}$$

$$0 \leq t \leq a, \quad |y-1| \leq b$$

$$1 - b \leq y \leq 1 + b$$

$$-a \leq -t \leq 0 \qquad 1 - b - a \leq y - t \leq 1 + b$$

$$(y-t)^2 \leq (1+b)^2$$

$$e^{(y-t)^2} \leq e^{(1+b)^2}$$

Here, b/ M is given by e to the power b + 1 whole square, so that you can easily check that how to find out this maximum value e to the power b + 1 square, if you look at here, so here your f (t, y) is what? F (t, y is e to the power y – t whole square and here, your t is given as that t is >= 0 to some a, where a, we have already define what is a and y -1 <= b, so we have already seen that this implies that y is given by 1 + b, 1 - b.

So, we can write that –t, I can write –t >= -a <= 0, so we can say that y - t is written is y – b - a < 1 + b, so we can say that y –t whole square is bounded by 1 + b whole square, so it means that e to the power y – t whole square is <= e to the power 1+ b whole square, so in this way we can find out the maximum value of f(t, y)and here we have the M is given as maximum of ty in rectangle R e to the power y – t whole square, this is given as e to the power b + 1 whole square.

So, we have to find out the interval in which the solution exist, so for that we have to find out the maximum value of this quantity, so again we can say that the; clearly the largest h that we can achieve is a maximum value of the function, b upon e to the power b + 1 whole square and how we can find out this maximum value and we can say that the maximum value is given by the value this and which is same as a.

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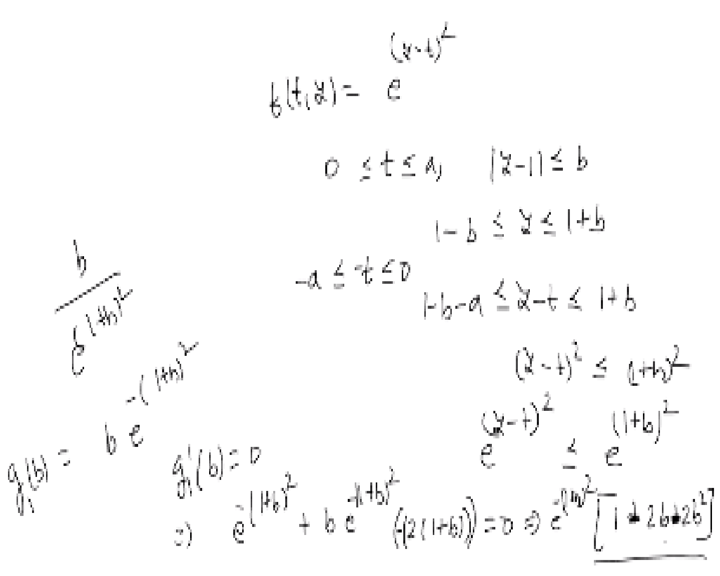

And which we can obtain by the b given as -1+ root 3 upon 2, how we can achieve? If you look at this, so we need to find out the maximum of b upon e to the power 1 + b whole square or you can write this as b * e to the power -1+ b whole square, so we consider again this as g1b, some function of b to find out the maximum value here, we look at g1 dash b as 0 and this implies that we have to look at this as e to the power -1+ b whole square + b e to the power -1 + b whole square * 2 of 1+ b and that is = 0.

So, we can take out this e to the power -1+ b square out and we can write this as e to the power -1+ b whole square and within bracket it is what? 1 + 2b + 2b square, is it okay, this one – sign here because of this, so we have a – sign here, so here we have this minus and this is minus, so here we can say that g dash, g1 dash b is = 0 provided that this quantity is 0. Now, what is this value?

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$$2b^2 + 2b - 1 = 0$$

$$b = \frac{-2 \pm \sqrt{4 - 4(2)(-1)}}{2 \times 2} \qquad (y-1) \le b$$

$$= \frac{-2 \pm \sqrt{4+8}}{4}$$

$$= \frac{-2 \pm 2\sqrt{3}}{4}$$

$$= \frac{-1 \pm \sqrt{3}}{2}, \qquad \frac{-1+\sqrt{3}}{2}, \quad \frac{-1-\sqrt{3}}{2}$$

So, here we have this 2b square + 2b -1 = 0, so it means that here, b will be a 0 of this provided that 2b square + 2b -1 is = 0, so it is a quadratic equation and we can easily find out the root of this and if you look at the root of this b is given as -b that is -2+ - 4 – 4 * 2 * -1 divided by 2 * 2, so it is -2+ -, it is 4 + 8 divided by 4 and we can write -2+ -, it is 12 root 12, so root 12 is given at 2 root 3 divided by 4, so it is -1+ - root 3 upon 2.

Now, here we have two values, -1+ root 3 upon 2 and -1 – root 3 upon 2 and since b satisfy this condition that y -1 <= b, so it means that b has to be positive, so if you consider this value then this value is going to be a negative value of b, so we have to take the value b, which is given as -1+ root 3 upon 2, I am leaving it to you to show that at this value, this g1b achieve the maximum value.

So, we say that in what we have shown here that for this value b, g1b extremise, now we can further check that it not only extremise, it actually maximise the function g1b, so that is what we wanted to show here, so here we have shown that the maximum value is achieved by b e to the power –b + 1 whole square, now b is given as root 3 –1/2 e to the power b + 1 is given by root 3+1/2 whole square.

So, here we have achieved the maximum value of b upon e +; e upon; b upon e to the power b + 1 whole square as this value and which is same as a, so we can say that to summarise all this thing, we say that by the existence and uniqueness theorem, yt exist for this interval 0, <= t, <= a, here a and b/ M is coming to be same. So, here we have proved that the solution yt of the initial value problem is exist and it is exist in this endeavour.

If you look at so far we have discussed a few problems and we have not in the previous problem and this problem, we are not find out the solution, we are rather interested in finding the interval in which the solution exist, so here we know that solution exist and it exist in this particular interval.

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If in a region D, not necessarily a rectangle, the function f is satisfying Lipschitz continuity in y, then given any point (t_0, y_0) in D we can construct a rectangle R lying entirely in D with center at (t_0, y_0).

The hypotheses of Theorem 1 are then true in R, and we can apply Theorem 1 to find the existence of a solution $y(t)$ of $y' = f(t, y)$ through the point (t_0, y_0) on some interval about t_0. In fact, it may happen a solution may exists on a larger interval than the one constructed in the proof of the Theorem (1).

Example 1

Consider the following initial value problem $y' = y^2$ with $y(0) = 2$. The solution is given by $y(t) = \frac{2}{1-2t}$ and it exists on the interval $-\infty < t < \frac{1}{2}$. Here, we may define a rectangle $R = \{(t, y) : |t| \le a, |y - 2| \le b\}$, so $M = \max_{(t,y)\in R} y^2 = (2+b)^2$ and $\alpha = \min(a, \frac{b}{M}) = \frac{1}{8}$.

Thus, Theorem 1 gives the assurance of a unique solution in $|t| < \frac{1}{8}$, but we can easily observe that solution exists on a much larger interval.

Moving on next, we can write this, we can summarise this that if in a region D, not necessarily a rectangle, the function f is satisfying Lipschitz continuity in y, then given any point t0, y0 in D, we can construct a rectangle R lying entirely in domain D, so it means that you initially start with a region R, where this f (t, y) satisfies the Lipschitz condition, so in that region you can easily find out a rectangle containing the point t0, y0 such that that rectangle is purely contained in that region D.

And there, we can say that the hypothesis of theorem 1are then true in R and we can apply theorem 1 to find the existence of a solution yt of y dash = f (t, y) through the point t0, y0 on some interval about t0 and in fact, it may happen a solution may exist on a large interval, then the one constructed in the proof of the theorem 1, so it may happen that actually the solution exist in a larger interval but your theorem is given only the small interval in which the solution exist.

So, let us consider next example, where we can point out this problem, so if you look at this, consider the following initial value problem, y dash = y square with y0 = 2 and this; if you look at this is quite easy problem, it is y dash, this is separation variable problem, so we can simply say that how we can solve this.

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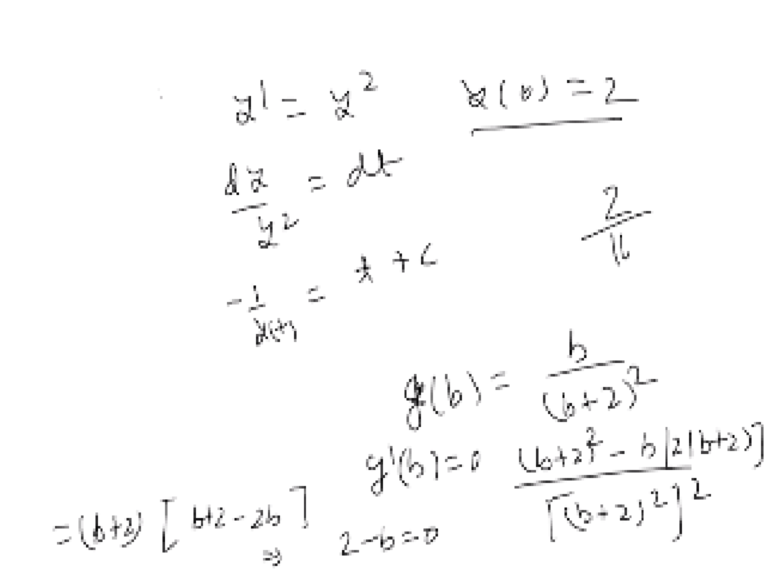

We have simply y dash = y square and y0 = 2, so we can simply say that so, dy/y square = dt and you can simply solve this by -1 upon yt is = t + c and you can find out the value of c and y0 = 2 and we can simply say that the solution of this problem exist as yt = 2 upon 1 – 2t and if you look at this solution, this solution exist in this interval, - infinity < t < 1/2 and but if we use the existence and uniqueness theorem which we have given.

We may define our rectangle R such that the set of all ty, such that modulus of t <= a, modulus of y – y0, y0 is 2 here, modulus of y -2 <= b, so here we can say that the maximum value of y square is going to be 2 + b whole square, so M is given as maximum of ty in rectangle R y square and that is given as 2 + b whole square and we can say that alpha which is defined as minimum of a and b/ M is given as the maximum value of b/M.

So, here b/M, you can find out b upon 2 + b whole square and you can easily check that the solution, let me do it, so here we have say gb as b upon a b + 2 whole square, I think this is the thing we have given, it is b upon 2+ b whole square, so, g dash b will be 0 provided, we have b+2 whole square and here, we have b + 2 whole square – b 2b + 2, so if you simplify this what you will get?

We will get b +2, you can take it out, you had b + 2 – 2b and you can easily see that this is g dash b equal to 0 provided that 2 – b = 0, so we can get the maximum value at b = 2 and maximum values of gb is going to be 2 upon 2+2 whole square means 16, so it is 1/8, so it means that the interval in which the solution exist is going to be alpha is 1/8, so here the solution exist in modulus of t < 1/8.

But just now, we have shown that the solution actually exist in this interval – infinity to 1/2 but if you look at our theorem, this existence and uniqueness theorems assure the unique solution in this interval modulus t <= 1/8, it means that this existence and uniqueness theorem is a local existence theorem, means here solution is currently only in a small neighbourhood around the initial point.

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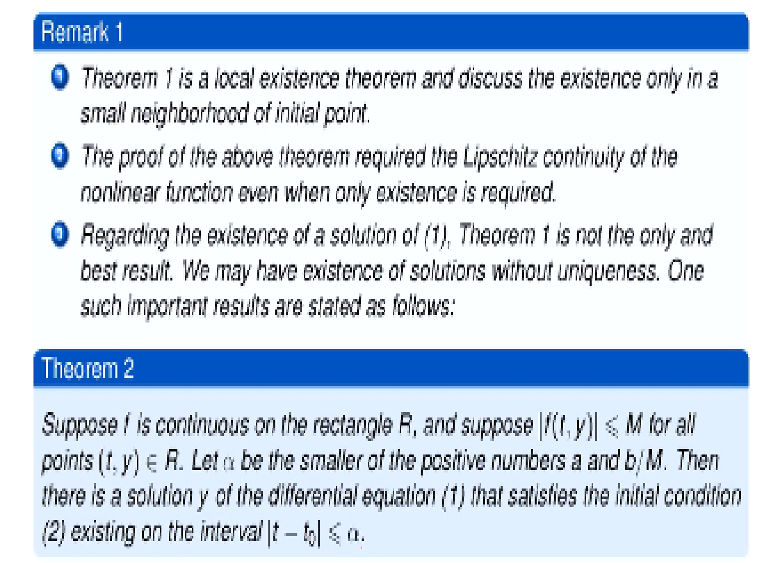

So, we can simply say that theorem 1 is a local existence theorem and discuss the existence only in a small neighbourhood of the initial point that we may observed from the previous example and the proof of this theorem required the Lipschitz continuity of the non-linear function even when we need only existence because we have shown that this Lipschitz continuity is required when we need an unique solution.

So, it means that regarding the existence of a solution 1, theorem 1 is not the only and best result, we may have existence of solution without uniqueness and one such important result are we can illustrate as follows. Here, we assume that suppose, f is continued on the rectangle R and suppose, f (t, y) is bounded by this constant M for all points t, y in R and let alpha b the smaller of the positive numbers a and b/M.

Then, there is a solution y of the differential equation 1 that satisfy the initial condition to existing on the interval t; modulus of the t – t0 <= alpha, if you look at here, here we have not observed, here we have not required that f (t, y) satisfy the Lipschitz continuity and is still the existence is guaranteed by this theorem. So, we will look at more point based on this theorem 2 in next class, so here we stop and we will continue in next lecture, thank you very much.