Ordinary and Partial Differential Equations and Applications

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Lecture - 49
Laplace Equation - I

Hello friends. Welcome to my lecture on Laplace equation. This is first lecture on Laplace equation. We shall start with 2-dimensional Laplace equation.

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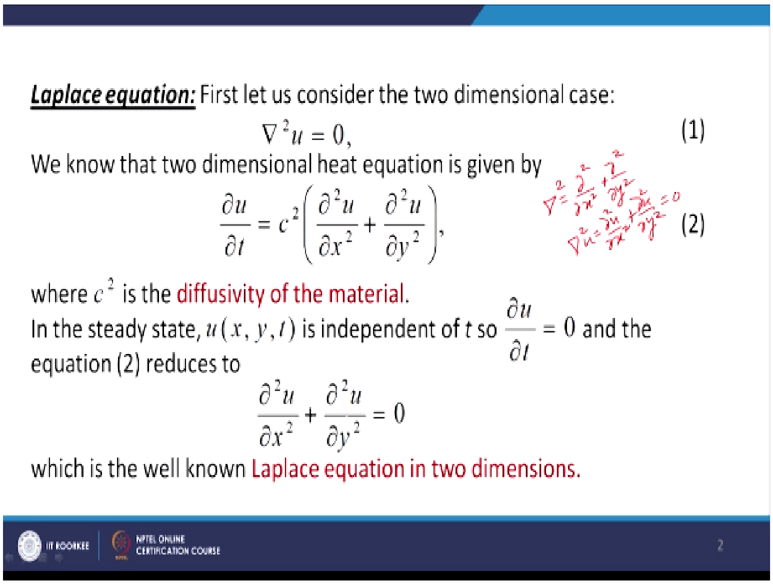

Let us consider the 2-dimensional case. Del square u=0, del square is the Laplacian operator defined as del square in the 2-dimensional case del square/del x square+ del square/del y square. So del square u= del square u/del x square+ del square u/del y square. Del square u=0 gives us this 2-dimensional Laplace equation and these 2-dimensional Laplace equations occur say for example in the case of 2-dimensional heat equation.

2-dimensional heat equation is given by del u/del t=c square del square u/del x square+del square u/del y square, c square is the diffusivity of the material. Now in the steady state case, the temperature u x, y, t no longer depends on time, so the partial derivative of u with respect to t is 0 and therefore this equation this 2-dimensional heat equation reduces to u/xx+u/yy=0 which is the well-known Laplace equation in 2 dimensions.

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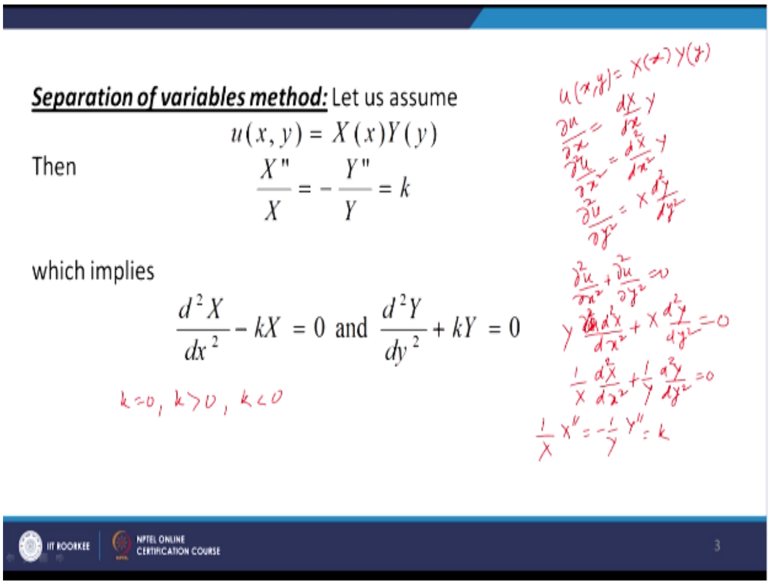

We will solve this 2-dimensional Laplace equation by using the separation of variables method u x, y is a function of X and Y. So we assume that u x, y can be expressed as a function of X*function of Y, this is called as separation of variables method. Now if X, Y is assumed as product of a function of x and product of end function of y then you find its partial derivative.

Partial derivative of u with respect to x is dx/dx*Y and then del square u/del x square=d square x/dx square*Y. Similarly, delta square u/del y square will be=x times d square y/dy square and now del square u/delta x square+del square u/delta y square=0 gives us Y times d square X/dx square+X times d square Y/dy square. We can divide this equation by X*Y and write it as 1/X d square x/dx square+1/Y d square Y/dy square=0.

Or we can write it as 1/X X double dash, X double dash denotes the derivative of X with respect to X of second order and similarly this is -1/Y Y double dash. Now left hand side is a function of X only because X is a function of x, its twice second order derivative will be a function of X/x so left hand side is a function of x, right hand side is a function of y, x and y are independent variables.

This can happen only when they are equal to a constant, so let us write it as some constant k so X double dash/X=-Y double dash/Y=some constant k. Now these equations X double dash/X=k and Y double dash/Y=-k, they give rise to 2 second order ordinary differential equations d square X/dx square-kX=0 and d square Y/dy square+kY=0. Now k is an arbitrary constant here. So there arise 3 cases, k=0, k>0 and k<0. Let us discuss these 3 cases one by one.

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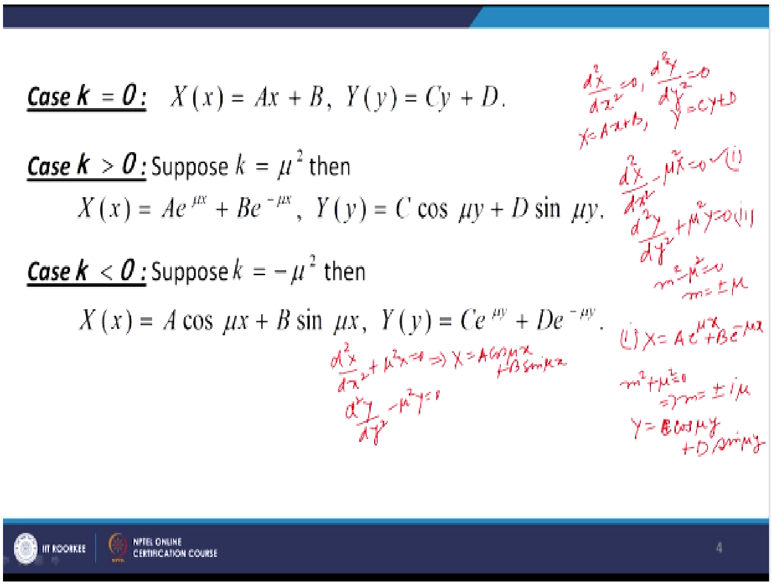

So in the first case when k=0, k=0 means d square X/dx square=0 and d square Y/dy square=0. So when second derivative of X with respect to x is 0, it means X is a linear function so Ax+B and similarly Y=Cy+D. So we have X x=Ax+B, Y y=Cy+D. Now in the case when k is>0 we can write k as square of some real number, so k=mu square. So when k=mu square in this case we have d square X/dx square-k square X=0 and for Y we have d square Y/dy square+k square Y=0.

Now this is second order linear differential equation, its ordinary equation is m square-k square=0 so m=+-. This is k square, k is mu square so this is mu square. So this is d square/dx square-mu square*X=0 and this is mu square*Y=0 so m square-mu square=0 gives you m=+-mu and therefore this equation let me call it roman I, roman II so I gives us the solution as X=A e to the power mu x+B e to the power –mu x because there are 2 real roots +-mu.

In the other case second case, we have m square+mu square=0 so we get 2 complex roots m=+-i mu. So we get Y=C cos mu y+D sin mu y and in the case of k<0, k can be written as –mu square then we will have the equations d square X/dx square+mu square X=0 where the solution will be X=A cos mu x+B sin mu x. With respect to Y we will have d square Y/dy square-mu square Y=0.

So we will get Y y=C e to the power mu y+D e to the power –mu y and so u x, y will be product of X x and Y y for the equations k=0, k>0, k<0. Now which case will be suitable for our problem this will depend on the given boundary conditions.

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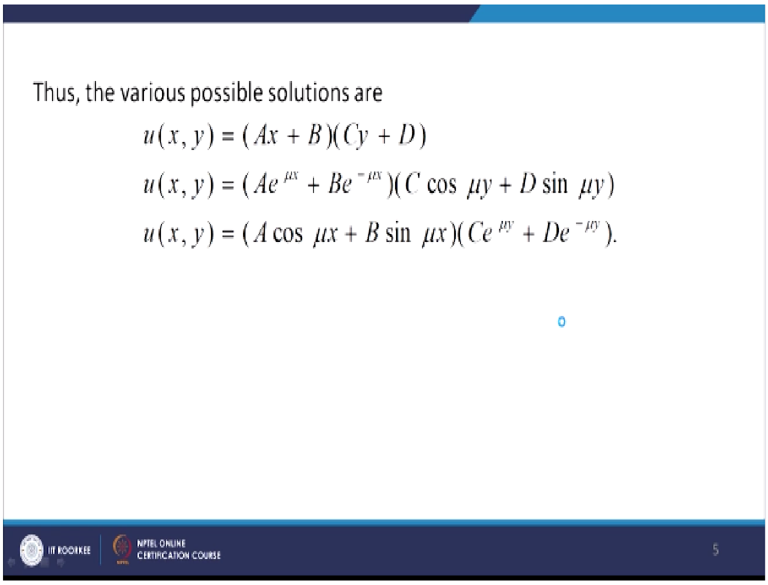

So let us see how we tackle for a given problem, so these are the various possibilities.

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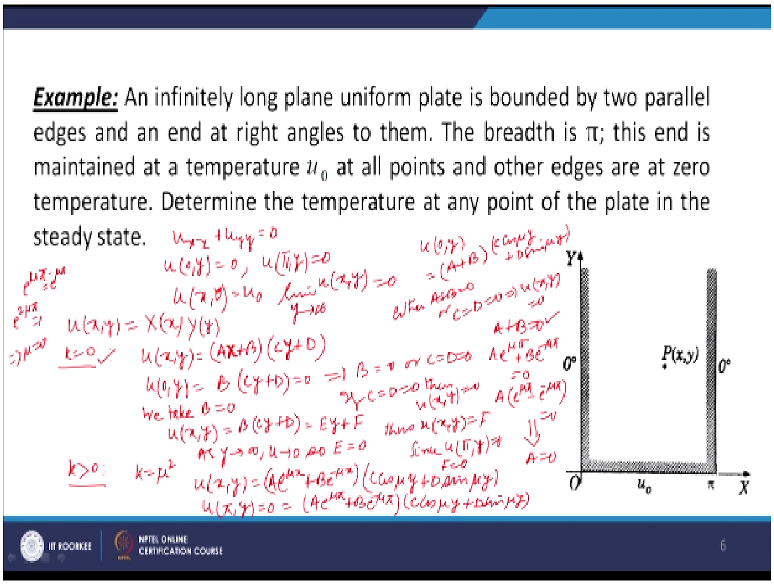

Now let us look at an infinitely long uniform plate, it is bounded by 2 parallel edges, these are 2 parallel edges and at an end at right angles to them this angle is at right end to them, this end is maintained at a temperature u0, this temperature u0 here and at all points and other edges are at temperature 0, so here the temperature is 0, here is the temperature 0. We have to find the temperature at any point of the plate in the steady state.

So we have to solve this equation u xx+u yy=0. The boundary conditions are on this y axis x=0 so u 0, y=0 for all y and on this axis, this is x=pi okay so we have u pi, y=0 for all y okay and for all y and for all y>0 and u here y=0 so u x, 0=u0 okay. So we are given and then what happens is that when y goes to infinity okay u goes to zero so limit u x, y y goes to infinity=0 okay.

So these are the boundary conditions. Let us solve the given equation u xx, so we assume that u x, y=X x*Y y. So we get the solutions k=0 the solution is u x, y=Ax+B*Cy+D okay now what happens when you take x=0 let us apply the boundary conditions so u 0, y=this is small x and this is small y okay so x=0 means B times Cy+D this=0 okay when x=0. Now this means that either B=0 or C=D=0.

If C=D=0 okay then y will be=0 then Cy+D will be=0, so then u x, y will be 0 for all x and y. So this is not possible okay so we take B=0 okay and then u x, y becomes B times Cy+D. I can write it as BC, BC is a new constant, I can write it as Ey+F okay. Now we are given another condition what we have is that u 0, y=0; u pi, y=0 and u x, 0=u0 okay, so what do we get, we will have u x, y=0 as y goes to infinity okay.

See as y goes to infinity okay u x, y goes to 0 so E must be 0 okay, as y goes to infinity u goes to 0 so E has to be 0 otherwise the temperature will not remain finite okay. So E=0 means u x, y=F, we have u z, y=F and we see that u x, y=F is not possible because we are given that u pi, y=0 okay. So since u pi, y=0 okay F must be 0 okay. So u x, y has to be 0 and therefore this case is not possible u x, y=Ax+B*Cy+D.

Let us take then k>0, when k is>0 we can take k to be=mu square and then the solution u x, y is of the type we have A e to the power mu x+B e to the power –mu x*C cos mu y+D sin mu y. Now again let us put x=0, when we put x=0 we get u 0, y=A+B*C cos mu y+D sin mu y okay. This is 0 for all y okay so either A+B=0 or C=D must be 0 okay. C=D=0 gives u x, y=0, this gives you x, y=0 for all x and y so not possible so A+B=0 okay.

Now let us use the other condition, u pi, y=0=A e to the power mu pi+B e to the power –mu pi*C cos mu y+D sin mu y. So what happens either A e to the power mu pi+B e to power –mu pi is 0 or C=D=0. C=D=0 makes u x, y=0, so C=D=0 is not possible. So we have A e to the power mu pi+B e to the power –mu pi=0. Now from this equation B=-A so I can write it as A times e to the power mu pi-e to the power –mu pi=0.

Now what happens either A=0 or e to the power mu pi-e to power –mu pi=0. If A=0 then what will happen we have B=0, A=0 means B=0. When A=0, B=0 will make u x, y=0 which is not possible. So the other possibility is e to the power mu pi-e to the power –mu pi=0 and which makes e to the power mu pi=e to the power –mu pi or we can say e to the power 2 mu pi=1 which gives mu=0.

And mu=0 case we have already seen, mu=0 means k=0, k=0 case we have already discussed. It is not possible so here if e to the power mu pi=e to power –mu pi then we get the case k=0 which is not possible so this gives you A=0 and A=0 gives B=0 so we get u x, y=0. So this k>0 is also not possible.

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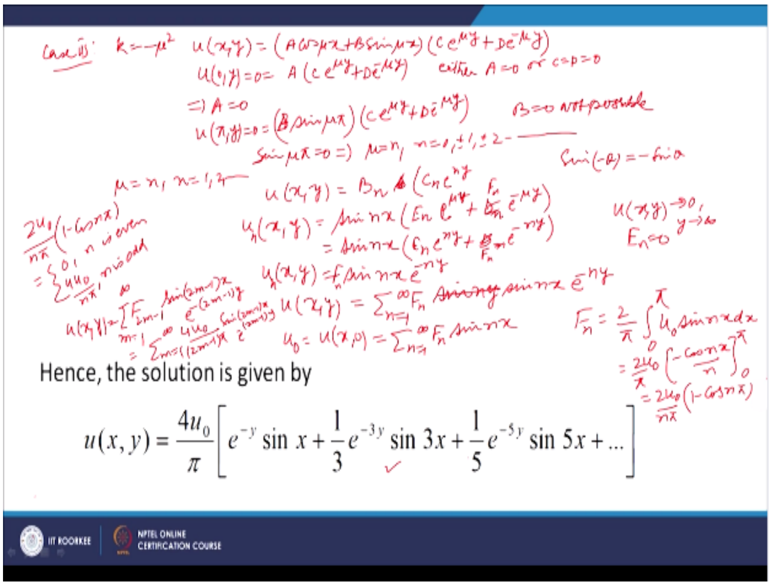

Let us go to the third case. In the third case, we have case III let us say k=-mu square, when k is –mu square u x, y=A times cos mu x+B times sin mu x*C e to the power mu y+D e to the power –mu y. Now u 0, y let us put 0, u 0, y=0 gives you A*1 and then this term become 0, then C e to the power mu y+D e to the power –mu y. So this is 0 means either A is 0 or C=D=0.

C=D=0 is not possible, so A=0 so this gives you A=0. Then, u pi, y=0, let us use this one, so this gives you A term is already 0 so we have B sin mu pi*C e to the power mu x+D e to the power –mu y. Now this is 0 means either B is 0, B=0 gives you A=0, B=0 gives you u x, y=0 so B=0 not possible okay. C e to power mu y+D e to power –mu y cannot be 0 because for that C and D both have to be 0s.

And when C and D both are 0s again u x, y is 0, so what will happen sin mu pi=0 which gives you mu=n, n taking values 0, +-1, +-2 and so on. Now here if your n=0 mu=0, mu=0 means k=0, k=0 is not possible so n=0 cannot be taken. When n is taking negative values then what will happen, we know that sin-theta=-sin theta so taking negative values of n will only add a negative sign to the expression.

And negative sign will be absorbed in the constant appearing here and therefore we can only take the positive values of n. So we will take mu=n where n=1, 2, 3, and so on okay and then what will happen, so we will have mu=n and then this will be u x, y=we had A=0, B will be absorbed inside there and we will get u x, y=sin nx, mu=n and then BC will become some constant say En then En e to the power mu y+Fn e to the power –mu.

So we will have this sin nx times En e to the power ny+Fn e to the power –ny. Now we have to see there are 2 constants En and Fn, so let us put Fn okay. En and Fn, let us see that as y goes to infinity u x, y goes to 0, u x, y goes to 0 as y goes to infinity, so this condition means that the En's must be 0s okay otherwise the solution will not remain finite. So this means that En will have to be 0 so we will have un x, y so un x, y=sin nx*Fn e to the power –ny okay.

Now these are solutions of the Laplace equation for every value of n=1, 2, 3 and so on but we have to still get the condition this one this condition we have. We have not still used the condition u x, 0=u0. So this condition has to be used, so what we do is for this condition to be used we will need the general solution u x, y=sigma n=1 to infinity Fn sin nx*e to the power –ny okay. So let me put y=0, when we put y=0, u x, 0=sigma n=1 to infinity Fn sin nx.

U x, 0 is given to be u0, so this is Fourier sine series for the function u0. So we will have Fn=2/pi half range Fourier sine series it is 0 to pi and we will have u0 sin nx dx okay. So we can integrate this, so 2 u0, u0 is constant, this is –cos nx/n. We have here pi, here 0, so we get 2 u0/n pi and we get 1-cos nx okay 0 to pi. So what we get, 2 u0/n pi 1-cos n pi this is=when n is even this is 0 and it is=4 u0/n pi when n is odd because cos and pi will be -1.

So 4 u0/n pi we have and therefore the solution u x, y will be=so we take here when n takes even values Fn is 0 and when n takes odd values Fn becomes 4 u0/n pi so u x, y will be=let me write nx 2m-1 so m=1 to infinity F2m-1 and sin 2m-1*x e to the power-2m-1*y okay and F2m-1=4 u0/2m-1 pi. So summation m=1 to infinity of 4 u0/2m-1*pi and then sin 2m-1x e to the power -2m-1*y, so we get this solution okay. You can take m=1, 2, 3, and so on you get these solutions.

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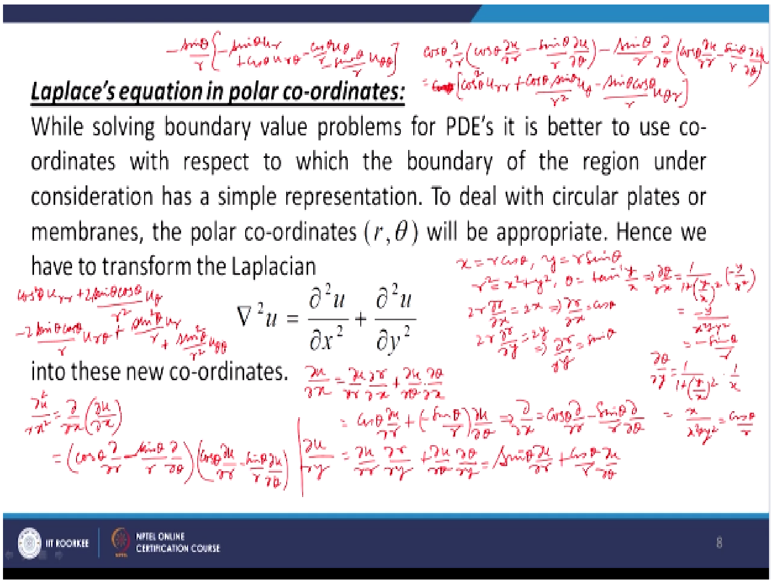

So Laplace equation in polar co-ordinates, while solving boundary value problems for partial differential equations, it is better to use co-ordinates with respect to which the boundary of the region under consideration has a simple representation. To deal with circular plates or circular membranes, the polar co-ordinates r, theta will be appropriate so we have to transform the Laplacian del square u to polar co-ordinates.

So let us see the representation of del square u in the polar co-ordinates. Let me see the relationship between Cartesian and polar co-ordinates z=r cos theta, y=r sin theta and this gives you r square=x square+y square and theta=tan inverse y/x. So this gives you 2r del r/del x=2x and 2r delta r/delta y=2y. This gives you delta r/delta x as x/r, x/r means cos theta and this gives you delta r/delta y=pi y r means sin theta and here this gives you delta theta/ delta x as 1/1+y/x whole square*-y/x square.

So this is = –y/x square+y square and this is –r sin theta/r square so –sin theta/r and similarly delta theta/delta y=1/1+y/x whole square*1/x. So we get here x/x square+y square which is r cos theta/r square. So we get cos theta/r. Let us now write the partial derivative of u with respect to x. So this is delta u/ delta r*delta r/delta x delta u/delta theta delta theta/delta x. I can put the value delta r/delta x is cos theta so cos theta delta/u/delta r+delta theta/delta x is –sin theta/r so –sin theta/r delta u/delta theta.

Similarly, delta u/delta y we can write=delta u/delta r delta r/delta y delta u/delta theta delta theta/delta y and I can write it as delta r/delta y is sin theta so sin theta delta u/delta r, delta theta/delta y is cos theta yr so cos theta yr delta u/delta theta. Now let us write delta square u/delta x square. We have to find value of this, so delta square u/delta x square=delta/delta x of delta u/delta x.

Let us notice that delta u/delta x is cos theta delta u/delta r –sin theta/r delta u/delta theta so this gives the differential operator delta/delta x as cos theta delta/delta r-sin theta/r delta/delta theta okay. So I can put it here so cos theta delta/delta r-sin theta/r delta/delta theta okay applied to delta u/delta x and delta u/delta x is cos theta delta u/delta r-sin theta/r*delta u/delta theta okay.

Now let us apply cos theta delta/delta r to this okay, so what we get cos theta delta/delta r of cos theta delta u/delta r-sin theta/r delta u/delta theta okay. So cos theta delta/delta r we applied to this then we apply –sin theta/r delta/delta theta to this expression, cos theta delta u/delta r-sin theta/r delta u/delta theta okay and let us see what we have, you see this is = cos theta delta/delta r of cos theta, r and theta are independent.

So delta/delta r of cos theta will be 0, so what we get cos theta times delta r of delta u/delta r that is u rr okay. Now we apply delta/delta r to sin theta which again r and theta are independent so –sin theta when we differentiate with respect to r it will be 0. So we differentiate 1/r and 1/r gives you -1/r square, --becomes + so we get cos theta*sin theta/r square*u theta and then we apply to u theta.

So –sin theta cos theta/r*u theta r okay, we are differentiating u theta with respect to r so u theta r; u theta, r and ur theta will be taken as m. We are assuming that second order partial derivatives are continuous. So this is the expression 1 okay and similarly we can find the other expression –sin theta/r times when we differentiate with respect to theta, so derivative of cos theta is –sin theta.

So we get –sin theta ur okay then we have cos theta times ur theta, so we have differentiated this term. Now we have multiplied cos theta, we have put here cos theta u rr and then I have multiplied cos theta inside, so I think this cos theta has to be multiplied inside here, cos square theta it should be actually okay so we have what cos square u rr we have. Now here what we have we have differentiated cos theta we got –sin theta*ur.

Then, we get cos theta*ur theta then we differentiate this sin theta what we get –cos theta/r so –cos theta/r and we get u theta and then 1/r when differentiated with respect to theta will give 0. So we get –sin theta/r and we get u theta, theta okay so what terms we now have cos square theta u rr okay, sin theta cos theta/r square u theta okay and here we will get –sin theta/r*-cos theta/r sin theta cos theta/r square u theta.

So we get 2 sin theta cos theta/r square u theta okay. Here we get –sin theta cos theta/r u r, theta and here what do we get –sin theta/r cos theta u r theta so –sin theta/r ur theta so we get -2sin theta cos theta/r ur theta okay. Now what do we have so we have combined this term and this term with the 2 terms here, this one and this one okay and we get sin square theta/r*ur okay, sin square theta/r*ur and we also have one more term sin square theta/r square theta/r square u theta theta okay.

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Applying chain rule,

$$u_{xx} = \cos^2\theta\, u_{rr} + \frac{\sin^2\theta}{r} u_r - \frac{2\cos\theta\sin\theta}{r} u_{r\theta} + \frac{2\cos\theta\sin\theta}{r^2} u_\theta + \frac{\sin^2\theta}{r^2} u_{\theta\theta}.$$

Similarly,

$$u_{yy} = \sin^2\theta\, u_{rr} + \frac{\cos^2\theta}{r} u_r + \frac{2\cos\theta\sin\theta}{r} u_{r\theta} - \frac{2\cos\theta\sin\theta}{r^2} u_\theta + \frac{\cos^2\theta}{r^2} u_{\theta\theta}.$$

So this is what we get as u xx. We can check it in the next slide u xx=cos square theta u rr then we get 2 sin theta cos theta/r square*u theta, so 2 sin theta cos theta/r square*u theta and then we get -2 sin theta cos theta/r ur theta. So we get -2 sin theta cos theta/r*ur theta and then we get sin square theta/r ur, so we get that and then we get sin square theta/r square u theta theta so we get this.

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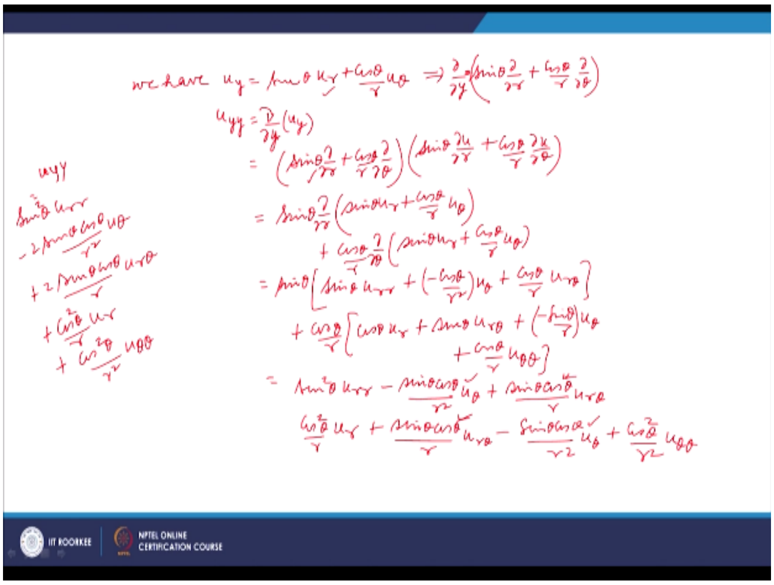

Now we go to u yy okay so we have uy, uy we found to be equal to let us see uy, uy=sin theta ur+cos theta/r u theta. So here also we can find u yy, u yy=del/del y of uy okay. Now this gives you the differential after del/del y, sin theta del/del r+cos theta/r del/del theta. This is del/del y okay. So what we have sin theta del/del r+cos theta/r delta/delta theta applied to sin theta delta u/delta r to this value okay+cos theta/r delta u/delta theta.

We apply this first term to both the terms here, so we get sin theta delta/delta r of sin theta ur+cos theta/r u theta and then we apply the second term cos theta delta/delta theta 2 sin theta ur+cos theta/r u theta, what we get sin theta r and theta are independent so derivative of sin theta with respect to r will be 0, so we get sin theta u rr. Then, we apply delta/delta r to cos theta, it is 0.

We apply to 1/r we get –cos theta/r square*u theta and then we get cos theta/r*ur theta okay. When we apply here we get cos theta, we get delta/delta theta if sin theta is cos theta ur then we apply to ur so sin theta ur theta then we have here delta/delta theta of cos theta will be –sin theta so –sin theta/r*u theta. Then, we have derivative of 1/r with respect to theta mu 0 so we get cos theta/r*u theta theta.

So we can get here sin square theta u rr-sin theta cos theta/r square u theta. Then, we have sin theta cos theta/r ur theta and then we have cos square theta ur. Then, we get sin theta cos theta ur theta, we have missed 1/r here I think yes sin theta delta/delta r cos theta/r, this cos theta/r we have missed so this cos theta/r we have missed. So we get cos square theta/r ur then sin theta cos theta/r ur theta and then –sin theta cos theta/r u theta okay.

This one –sin theta cos theta/r square u theta and then cos square theta/r square u theta theta okay. So what we get, sin square theta u rr-2 sin theta cos theta/r square mu theta okay. This one and this one and then we take this and this okay. So this one and this one, so we get 2 sin theta cos theta/r ur theta okay. We get cos square theta/r ur and then we get cos square theta/r square u theta theta.

So this is the value of u yy okay. Let us check this, so sin square theta u rr we have, cos square theta/r ur we have okay, 2 sin theta cos theta/r ur theta we have, -2cos theta sin theta/r square u theta we have and we have cos square theta/r square u theta theta okay. This one so now we add u xx and u yy and we can see easily that we get u rr+1/r ur+1/r square u theta theta.

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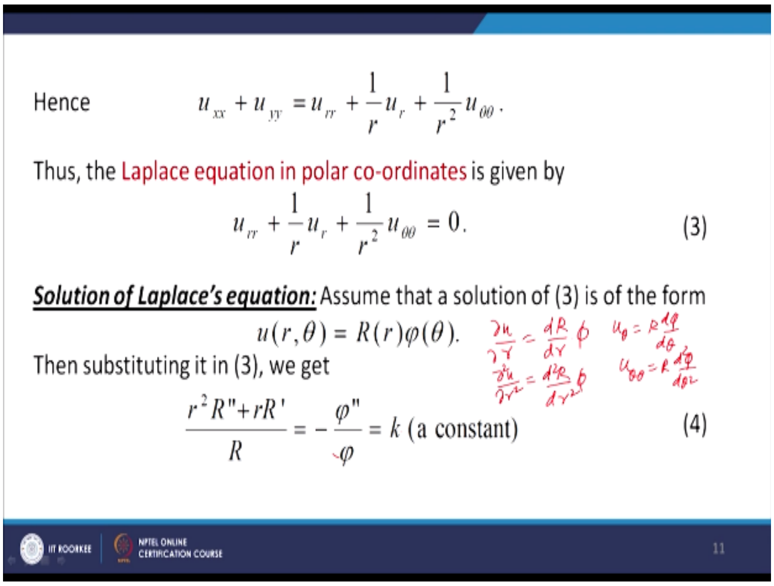

Thus, the Laplacian del square in polar co-ordinates is given by this expression, so del square u is u rr+1/r ur+1/r square u theta theta=0. Now let us solve this equation by again separation of variable method. We are having 2 independent variables r and theta, so let us take u r, theta=R r*phi theta. When you use this equation u rr=R r*phi theta in this equation what you get delta u/delta r=dR/dr*phi okay.

And delta square u/delta r square then gives you d square R/dr square*phi okay and when you find ur, ur we have already got, u theta theta similarly will give you u theta will be your r*d phi/d theta and u theta theta will be=r*d square phi/d theta square. So let us substitute these values in this equation and then what will happen we will bring all functions of r on one side, so we will have r square R double dash+rR dash/R=-phi double dash/phi we will have.

Now this is the function of r, left hand side is the function of r only, this right hand side is the function of theta only, so r and theta are independent variables so this is possible on even each is equal to a constant. Let us take the constant as k, so we have r square R double dash+rR dash/R=-phi double dash/phi=k. Now it will lead us to 2 differential equations of second order.

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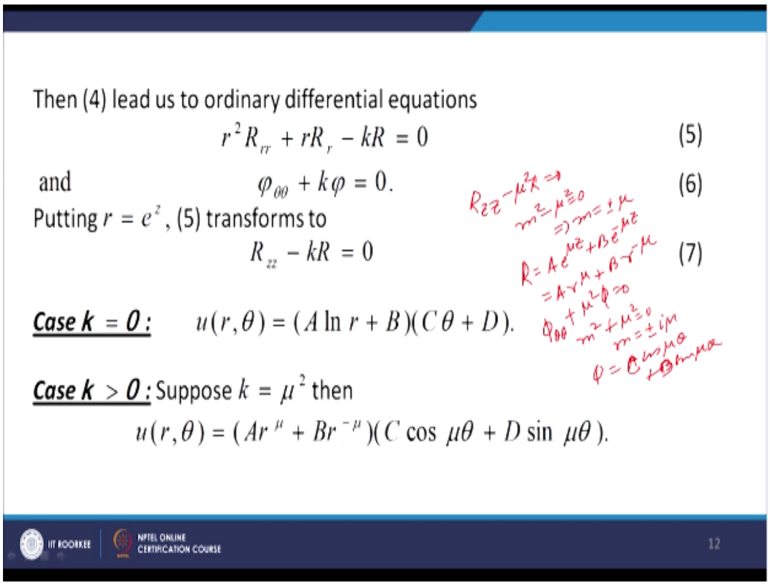

So we will have r square Rrr+rRr-kR=0, phi theta theta+k phi=0. Now this is Cauchy-Euler equation. In this standard method is that you would replace the independent variable r by say e to the power z, then it transforms to a second order differential equation with constant coefficients we get d square r/dz square-kR=0. So it is a linear differential equation with constant coefficient.

And the second equation is again a linear differential equation with constant coefficient. Now here what we have m square- so what will happen let us now take the various cases k=0, when k=0 phi theta theta=0, so phi will be = C theta+D, R zz=0 will give you R=Az+B but z=ln r so ln r+B you will have so u r, theta=A ln r+B*C theta+D. When k=mu square then what we will have, R zz-mu square r=0.

So it is a linear differential equation with constant coefficient, the auxiliary equation is m square-mu square=0. So m=+-mu and therefore r will be=A e to the power mu z+B e to the power –mu z but e to power z=r so we have A times r to the power mu+B times r to the power –mu and the other equation is phi theta theta+mu square*phi=0. So here we have the auxiliary equation m square+mu square=0.

So m=+-i mu, so phi=A cos mu theta+B sin mu theta because we have complex roots. So A we have already taken, so let me write them as C and D okay. So phi=C cos mu theta+D sin mu theta. So this is the solution in the case k is>0.

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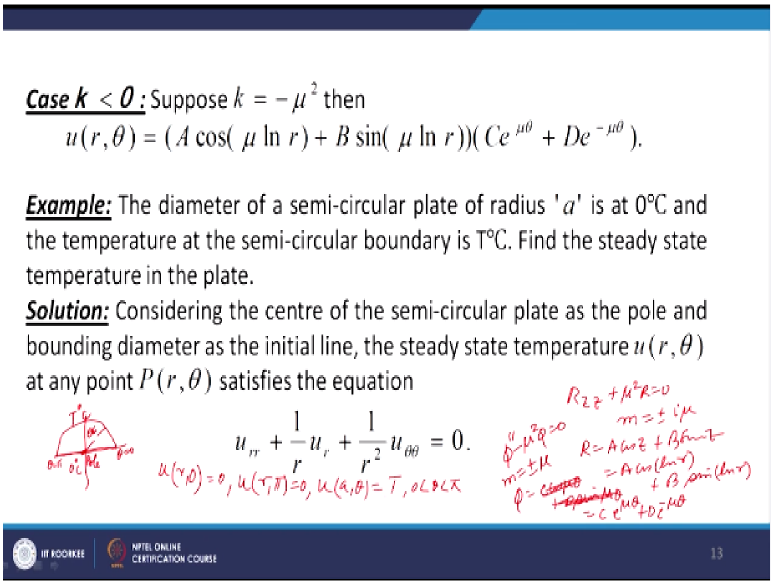

Now we take the case k<0 when k=-mu square. So when k=-mu square R zz+mu square R=0 we will get and therefore the auxiliary equation will have 2 complex roots +-i mu. So R will be=A cos z+B sin z and this will give you A cos ln r+B sin ln r because z is ln r and phi double dash+k=-mu square okay. So k=-mu square you will get yeah –mu square phi=0 so phi double dash-mu square phi=0.

So here we will real roots m=+- mu and therefore phi will be C cos mu theta+D sin mu theta okay. So we will get this solution, this will be +- mu so they are real roots so they are not complex so we will get C times e to the power mu theta+D times e to the power –mu theta okay. Now let us take the case of a physical problem. The diameter of a semi-circular plate of radius a is at 0 degree and the temperature at the semi-circular boundary is T degree.

Find the steady state temperature in the plate. Considering the center of the semi-circular plate as the pole, so let us draw the figure. This semi-circular plate, this pole is taken as the center of the semi-circular plate, initial line is this bounding diameter, this is bounding diameter this initial line, so here this is pole okay. Pole means r=0, here theta=0, here theta=pi.

Bounding diameter is taken and the radius is taken as a because we had given the radius as a okay and the temperature at the boundary is T degree centigrade, let the bounding diameter it is 0 degree centigrade okay, so we have solve this equation u rr+1/r ur+1/r square u theta theta=0. Now let us see what are the boundary conditions u r, theta when theta=0 okay we have temperature 0.

Then, when theta=pi we again have temperature 0 okay here and then u a, theta=T degree where theta is>0 but<pi. So we have the boundary conditions u r, 0=0; u r, pi=0' u a, theta=T okay.

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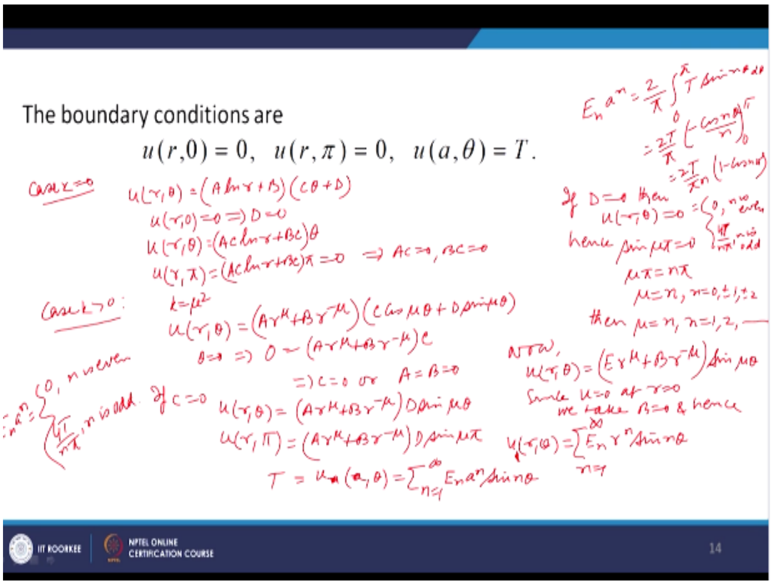

Let us go to these boundary conditions u r, 0=0; u r, pi=0; u a, theta=T. Now let us solve these equations, so we consider the case k=0, so when consider k=0 we go to the solution for k=0, k=0 the solution is u r, theta=A ln r+B*C theta+D okay. So let us write ur theta=A ln r+B*C theta+D okay. Put theta=0, so what do we get u r, 0=0 which implies that D=0 okay. Then, what will happen u r, theta will become A ln r+B*C theta.

So AC times ln r+BC times theta okay. Now we are given that when theta is pi okay theta is pi again u r, theta is 0 okay, so u r, pi=AC ln r+BC*pi okay, pi=0 so this implies that AC=0 and BC=0 which means that u r, theta=0. So this gives you because either C=0 or D is already 0, if C is 0 the solution will be 0, if C is not 0 then A and B both will be 0. So we will have again the 0 solution.

And therefore this is an interesting case so let us consider case k>0. In this case, what we have we have k=mu square and k=mu square when we have we have the solution as Ar to the power mu we have u r, theta=Ar to the power mu+B r to the power –mu and then C cos mu theta+D sin mu theta okay. Now here what happens when theta=0 okay we have u r, 0; u r, 0=0 so 0=Ar to the power mu+Br to the power –mu okay*theta is 0 means what this is C okay this is 0.

So this implies C=0 or A=B=0 okay. If C=0 okay then what will happen we will have u r, theta=Ar to the power mu+Br to the power –mu*D sin mu theta okay. D sin mu theta we will have and what will have happened when theta=2 pi okay u r, pi. We have Ar to the power mu+Br to the power –mu*D sin mu pi okay. If either sin mu pi is 0 or D is 0 or A=B=0. A=B=0 is not possible because u r, theta will be 0.

So either D is 0 or sin mu pi is 0. If D=0 then we get what will happen C=0 okay and D=0, so u r, theta will be=0 okay. Hence, sin mu pi must be 0 okay, so sin mu pi=0 means we will have here mu pi=n pi okay or mu=n. So n takes value 0, +-1, +-2 and so on okay. So when n=0 k=0 okay, k=0 is not possible right because k=0 we have already considered. When n=negative integer then sin –theta is –sin theta, so – will be absorbed in the arbitrary constants here.

So we can take n to the positive values, then mu=n and n takes values 1, 2, 3 and so on okay. So we have found this and D can be merged inside the constants A and B. We can write now u r, theta=D can be merged in A and we can write new constants say Er to the power mu+Br to the power –mu times sin u theta okay. Now let us look at this when r=0, r=0 means pole, at pole, pole lies on the initial line, on the initial line the temperature is 0.

So u is 0 when r=0, so if r=0 this r to the power mu will make it infinite so what will happen for this temperature to remain finite at r=0, we shall take B=0 so since u=0 at r=0 we have take B=0 and now let us replace mu/n, when we replace mu/n for each value of n we will get a constant E so we will write u and r theta=En r to the power n sin n theta okay and one more condition is there which is that at r=a that is the semi-circular boundary the temperature is T degree.

So for that we need to consider u and r theta as sigma n=1 to infinity En r to the power sin En theta. Now take r=a, so this will be now u okay so u r, theta, r=a so u a, theta will be=sigma n=1 to infinity En a raised to the power n r=a sin n theta. Now u a, theta=T okay, this is half range Fourier sine series. So En a to the power n okay can be now determined, this is 2/pi 0 to pi okay T sin n theta D theta okay.

So this is 2/pi 2T/pi, T is constant-cos n theta/n 0 to pi and we get 2T/pi n pi okay 1-cos n pi and cos n pi=1 when n is even. So this will be=0 when n is even and this will be 4T/n pi if n is odd because 1-cos n pi will become double okay. So we will have En okay. En a to the power n=2T/4 0 when n is even and 4T/n pi when n is odd and therefore the Fourier series so u r, theta okay.

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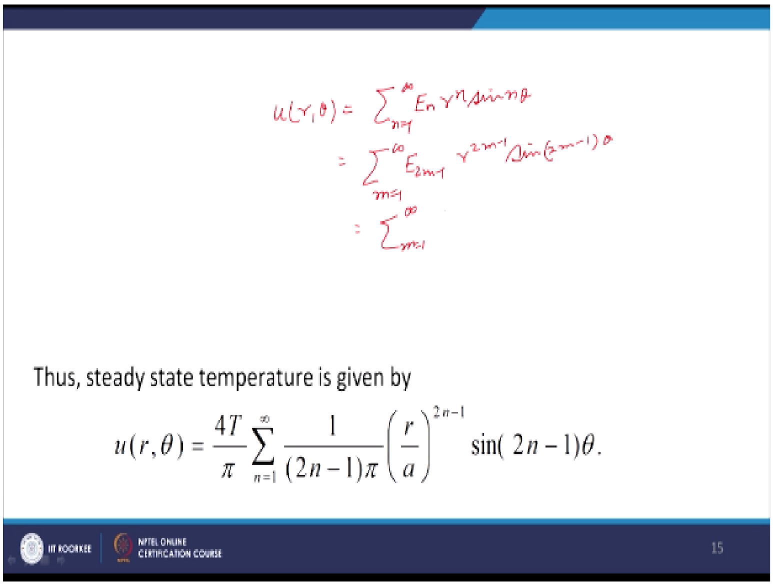

So now what is u r, theta, u r, theta=summation, n=1 to infinity En r to the power n sin n theta okay. The value of En is what En=this okay so we can put the value of En now, so we will get En s and m has to be taken odd okay so m=1 to infinity E to m-1 r to the power 2m-1 sin 2m-1 theta okay.

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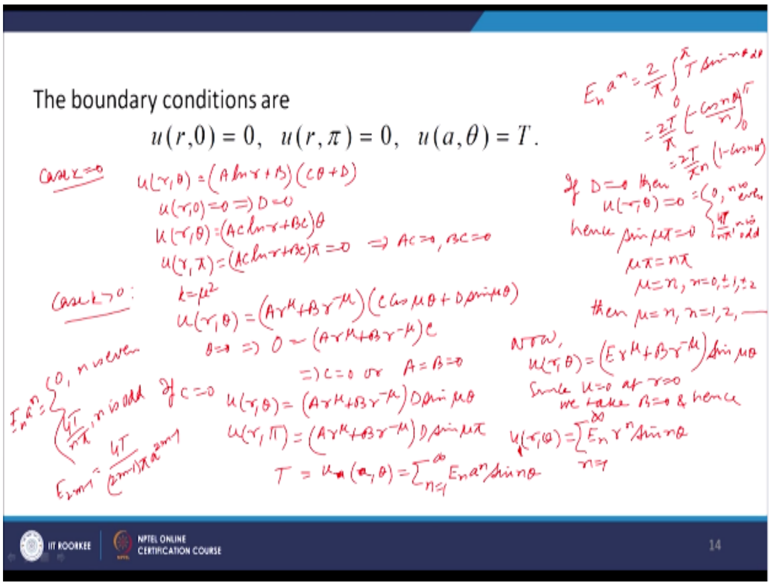

And this is summation m=1 to infinity e 2m-1 e 2m-1 from here, e to m-1 will be=4T/2m-1*pi and then a to the power 2m-1 okay.

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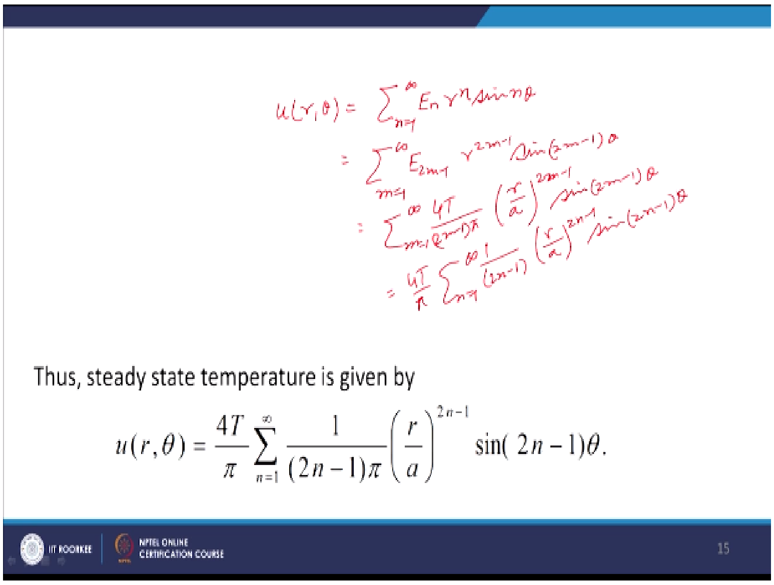

So we can write that so this is 4T/2m-1*pi and r/a to the power 2m-1 sin 2m-1 theta and that is nothing but this 4T/pi I can write summation n=1 to infinity 1/2n-1 r/a to the power 2n-1 sin 2n-1 theta okay. Now this case k=-mu square, this k=-mu square can be seen, it is not possible okay because what we have we have u r, 0=0 when theta=0 okay.

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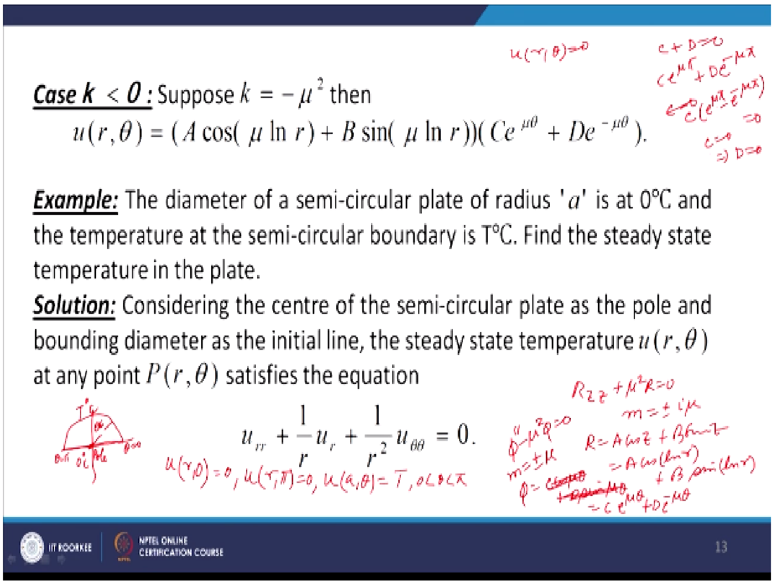

So what do we have, u r, 0=0 this means that C+D is 0 or A=B=0, A=B=0 will make your theta=0 so we have to take C+D=0 okay and when u r, pi=0 similarly we will have A cos ln mu ln r+B sin mu ln r C e to the power mu pi+D e to power mu pi and then what will happen this A and B cannot be 0 again, so C e to power mu pi+D e to the power mu pi will have to be taken 0 okay.

So that will make it C, you put D=-C so we get C times e to the power mu pi-e to the power –mu pi=0 okay. So if C is 0, D is 0 okay if C is not 0 then e to the power mu pi will be=e to power –mu pi, which means that e to the power 2 mu pi=1 which will mean that mu=0, mu=0 will make it k=0 which we have already discussed okay, so C has to be 0 and C has to be 0 will give you D=0.

So this case is not possible okay, so we have the case k=mu square which gives us the solution okay. So with that I would like to end my lecture. Thank you very much for your attention.