**Ordinary and Partial Differential Equations and Applications**
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**Lecture - 44**
**Classification and Canonical Form of Second Order PDE-II**

Hello friends welcome to my lecture on classification and canonical forms of second order PDE. We will discuss now the case 2, first case was where S square – 4 RT was strictly greater than 0, now we consider the case when S square – 4 RT = 0 then the roots lambda 1 lambda 2 of the lambda quadratic equation R lambda square + S lambda + T = 0 are real and equal.

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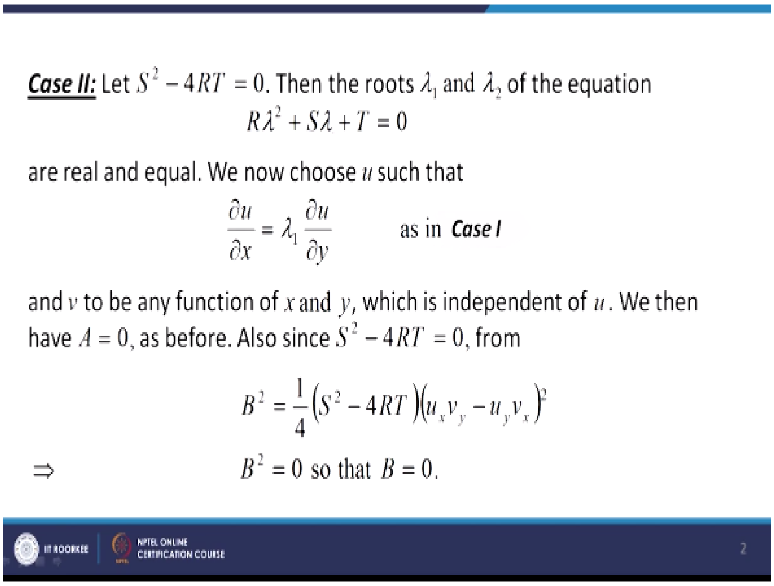


Now in this case what we do is we choose the function uxy in such a way that partial derivative of u with respect to x is = lambda 1 times partial derivative of u with respect to y exactly as in the case of 1, where S square – 4 RT was > 0, but the other function vxy can be taken to be any function of x and y which is independent of uxy. So that is the change in this case.

Now we then have a = 0 because the choice of ux = lambda 1 uy makes a = 0 as you have seen in the case 1. So a = 0 as before also since S square – 4 RT = 0 so B square is 1/4, S square - 4 RT * ux vy – uy vx, ux vy- uy vx is not equal to 0 because the Jacobian of u and v with respect to x and y is not 0 and therefore S square – 4 RT = 0 gives us v square = 0 which implies that v = 0. So we have a = 0, v = 0 here.

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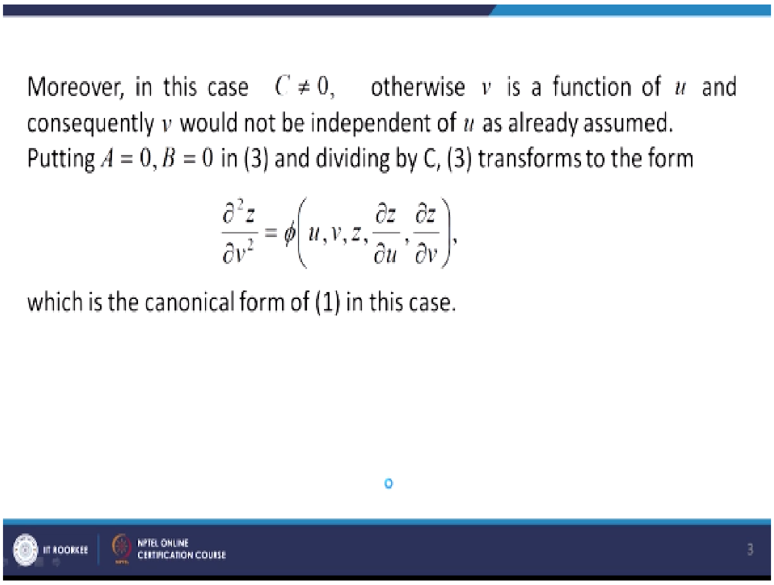


Further C cannot be 0 in this case because otherwise v will be a function of u, but we have already assumed that v is independent of u. So C cannot be = 0 and therefore putting A = 0 and B = 0 in the equation 3 and then dividing by C we get the equation zvv = phi of u, v, z, zu, zv which is the canonical form of 1 in this case and this can be then easily integrated.

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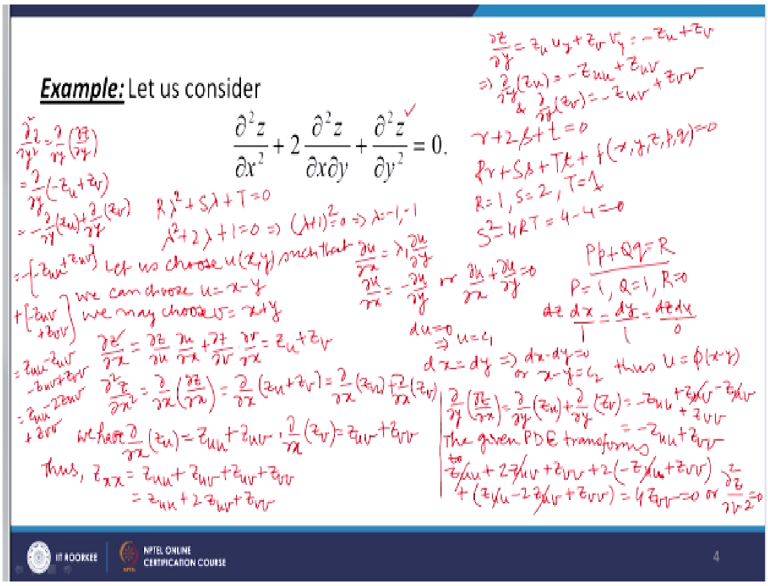


Now let us consider the example of this partial differential equation of second order zxx, so that means or I can write it as r + 2s + t = 0, R is zxx, S is zxy, T is zyy and when we compare it with this tendered form Rr + Ss + Tt + f xyz pq = 0 what we have R = 1, S = 2 and T = 1, which means that if you calculate S square –4 RT this is = S square is 4 -R and TR1 so 4-4 = 0 and so we have case II.

This partial differential equation belongs to the case II S square – 4 RT = 0 so what we do let us find the roots of the lambda quadratic. R lambda square + S lambda + t = 0, this is the lambda quadratic R = 1 so we have lambda square S = 2 so we have 2 lambda T =1 so we have lambda square + 2 lambda + 1 = 0 and this is lambda + 1 whole square = 0 which means that the 2 values of lambda are equal they are -1 -1.

Now we have to choose the functions uxy and vxy so let us choose u xy such that partial derivative of u with respect to x = lambda 1 times partial derivative of u with respect to y which gives you ux = -uy because lambda 1 = -1 or I can say partial derivative of u with respect to x + partial derivative of u with respect to y = 0. Now this is Lagrange's equation where of the form Pp + Qq = R, Lagrange's equation, okay.

So here P is partial derivative of u with respect to x, Q is partial derivative of u with respect to y, P =1, Q = 1 and R = 0 okay, now so we have the characteristic equations dx/1 dy/1 = du/0, because here the independent variable is u and therefore what we have du = 0 which implies u is the constant say C1 and dx = dy gives you dx-dy = 0 or we can say x-y = some constant C2. So C2 is the function of C1.

So we can say u is the function of x and y, thus u is the function of x-y okay, we can take because we have to choose u, so we can choose u = x-y okay. Now we have to choose v in such a way that v is independent of u. So we may choose v = x + y then v is independent of u. Now let us reduce the given partial differential equation to the canonical form.

So what we will get? We have to find partial derivative of z with respect to x, so this is partial derivative of z with respect to u * ux + partial derivative of z with respect to v * vx and this is = zu * ux = 1, okay and vx is also 1, so zu + zv and now let us find second order partial derivative of z with respect to x so we have this.

Okay, so this is zx we have already found okay so this is = now z is the function of u and v so zu is also function of u and v and therefore in order to find the partial derivative of zu with respect to x instead of z here we put z = zu, so we get, we have partial derivative of z with respect to x, zuu * ux, ux =1 then zu partial derivative of zu with respect to v so zuv and vx = 1, so this is what we get and similarly we can find partial derivative of zv.

We replace z/zv here so we have partial derivative of zv with respect to u so we get zuv and then ux is 1 and then partial derivative of zv with respect to v is zvv and vx = 1. So we have got the values of both the partial derivatives, partial derivative of zu with respect to x, partial derivative of zv with respect to x and thus zxx = we put the values here so zuu + zuv and then this value of this is zuv + zvv.

So this is zuu + 2 times zuv + zvv okay, now let us find this S, okay, so we have to differentiate this partial derivative of z with respect to x, y with respect to y, so what we get let us differentiate, so delta/delta y of delta z/delta x = partial derivative of zu with respect to y + partial derivative of zv with respect to y and these values can be found from here, we have to differentiate.

We have not differentiated so far z with respect to y, so let us write partial derivative of z with respect to y. So this is zu * uy + zv * vy and how much is that, uy uy = -1 so we get –zu and vy = 1 so we get zv. Okay, now we differentiate zu okay with respect to y so this gives you we can replace z by zu here and when we do that this is 1zuu + zuv and partial derivative of zv with respect to y can similarly be obtained.

Replace z by zv there so we get –zuv + zvv alright so this is = we can put the values now – zuu + zuv and then – zuv + zvv so this cancels with this and we get –zuu + zvv. We still have to find this term okay. So we have got the very derivative of z with respect to y. We can find now second derivative of z with respect to y. So delta square/delta y square z this is = this and this gives you zy = -zu + zv.

So this is minus this, okay, in order to find the partial derivative of zu with respect to y replace z by zu there, okay so we get –zuu okay, + zuv okay and then we find partial derivative of zv with respect to y, replace z by zv there so we get –zuv + zvv, okay and when you multiply this -1 inside what you get is zuu – zuv – zuv + zvv. So this gives you zuu – 2 zuv + zvv, okay. Now let us put the values in the give PDE okay.

So we have found the value of zxx, zxx is zuu, so the given PDE transforms to, okay, so zxx is zuu + 2 zuv + zvv alright + 2 times zxy, zxy we found here okay, partial derivative of zx with respect to y we found and it is –zuu + zvv and then + second order derivate of z with respect to y which is zuu – 2 zuv + zvv, let us see what we get, okay. So zuu + zu2 dash zuu which will cancel with -2zuu here okay and zvv will be = we have 2 zuv here and 2 zuv here.

So this term cancels with this term and this term and 2 zuv cancels with 2 zuv okay what we have zvv + 2 zvv + zvv so this is 4 times zvv = 0 okay, so this gives you r delta square z/delta v square = 0 okay, this is what we get, so this is the canonical form here.

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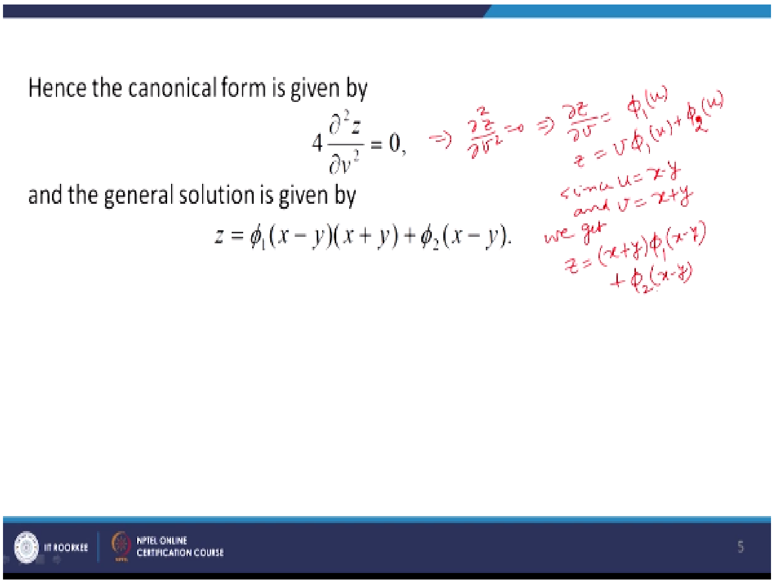


You can see 4 delta square z/delta v square = 0 now we can find the general solution also here because this is now simple we can easily integrate this so this gives you delta square z/delta v square = 0 and which implies that when we integrating with respect to v, okay keeping you as fixed we get delta z /delta v = some function of u so we get phi 1 u okay.

Because we are considering while integrating we are considering u as constant so the arbitrary function will depend on, this phi 1 will depend on u and then we will integrate it further with respect to v, so we get z = v times phi 1 u + phi 2 u okay, now let us see the values of u and v, okay, so value of u is what, u = x-y, v = x +y. So let us put that, so since u = x-y and v = x+y we get z = x+y * phi 1, x–y + phi 2 x-y. So this is the general solution of the given PDE here.

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***Case III:*** Let $S^2 - 4RT < 0$. Then, the roots $\lambda_1$ and $\lambda_2$ of (5) are complex. Hence, this case III is formally same as case I. Therefore, proceeding as in case I, we find that (1) reduces to (8) but that the variables $u, v$ instead of being real are now complex conjugates. To obtain a real canonical form, we make further transformation

$$u = \alpha + i\beta, \ v = \alpha - i\beta$$

so that

$$\alpha = \frac{u+v}{2}, \ \beta = \frac{-i(u-v)}{2}.$$

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Now let us consider the third case when S square – 4 RT is < 0. So when S square – 4 RT is < 0 then the roots lambda 1 and lambda 2 of the lambda quadratic are complex conjugates hence in this case the equation 3 formally same as equation 3, in case 1 we had taken lambda 1 and lambda 2 be real and distinct. Here lambda 1 and lambda 2 are distinct but they are complex conjugates of each other so this case is formally same as case 1.

Therefore, proceeding as in case 1 we find that 1 reduces to 8, but that the variables u and v instead of being real are now complex conjugates. The functions u and v are now not real functions they are rather complex conjugates of each other so we too obtained a real canonical form let us make further substitution.

Okay, since u is the complex function of x and y we can put u = alpha + i beta where alpha beta are function of x and y and v as alpha – I beta because B is the complex conjugate of u. So then u can find the values of u and alpha and beta, alpha = u + vy2 beta = - i times u-v/2. So we get the values of alpha and beta and then we use the transformation.

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Now,

$$z_u = z_\alpha \alpha_u + z_\beta \beta_u = \frac{1}{2}(z_\alpha - i z_\beta)$$

$$z_v = z_\alpha \alpha_v + z_\beta \beta_v = \frac{1}{2}(z_\alpha + i z_\beta)$$

$$z_{uv} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta}\right)\frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{1}{4}(z_{\alpha\alpha} + z_{\beta\beta})$$

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We can reduce the given form to this one, you see because 1 reduces to 8 and 8 is of the form zuv = function of xyz, zu zv so what we do is we get the value of zv in terms of the partial derivatives of z with respect to alpha and beta. So let us look at this transformation z is the function of u.

So partial derivative of z with respect to is partial derivative of z with respect to alpha * partial derivative of alpha with respect to u + partial derivative of z with respect to beta * partial derivative of beta with respect to u and since alpha = u + v/2 we have alpha = u+v/2, so what we get alpha u = 1/2, alpha v = 1/2 and beta = here 2 i beta = alpha u–v, u-v = i beta so beta = 1/2i * u-v.

So this means that 1/i is –i, so we get u-v, u-v = 2 i beta, so beta = 1/2i, i square = -1, so 1/i so i = -1/i, so 1/i is –i so this will be 1/i/2*v-u, 1/i is –i so 1/i/2 v-u and therefore beta u will be = -i/2, so we get here beta = i times v-u/2. So beta u = -i/2 and therefore what we have here 1/2 z alpha – i z beta, 1/2 times z alpha – i z beta, alpha v = 1/2 and beta v = i/2. So here z alpha alpha v = 1/2, beta v = i/2.

So we can get this okay, and we have zu like this. Then we differentiate zu with respect to v okay, so what we have zuv, we can use in order to find zuv we can use the differential operators this is = 1/2 delta/delta alpha – i times delta/delta beta of z. So this is the differential operator and here 1/2 delta/delta alpha + i times delta/delta beta, okay z in the form of differential operator.

So zuv = the partial derivative with respect to u * partial derivative of z with respect to v so this is okay, so zu is partial derivative of z with respect to u so this partial differential operator delta/delta u = 1/2 of this, so 1/2 delta/delta alpha-i delta/delta beta *1/2 delta z/delta alpha + i delta z/delta beta. Now when you apply delta/delta alpha to this, this is 1/2 * 1/2 is 1/4, so delta/delta alpha when applies to delta z/delta alpha gives you z alpha alpha.

Then delta/delta alpha applies to i delta z/delta beta gives you i times z alpha beta and then – i delta/delta beta applies to this gives you –z i times z alpha beta and then you get -i square, i square is -1 so –i square becomes +1 and then we get z beta beta, so this cancels and we get 1/4 z alpha alpha + z beta beta.

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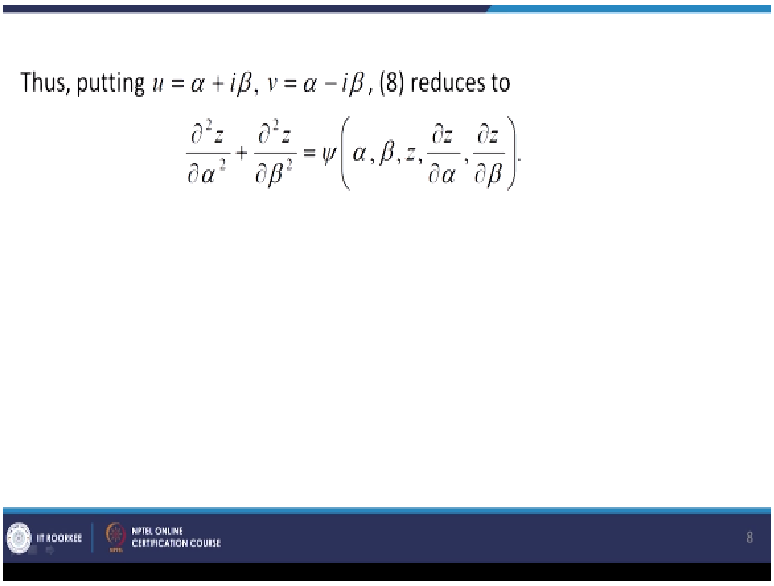


And therefore putting u = alpha + I beta, v = alpha – I beta the equation 8 reduces to z alpha alpha + z beta beta = a function of alpha beta z and then z alpha and z beta. Now this is the canonical form in the case of 3 where S square – 4 RT is < 0.

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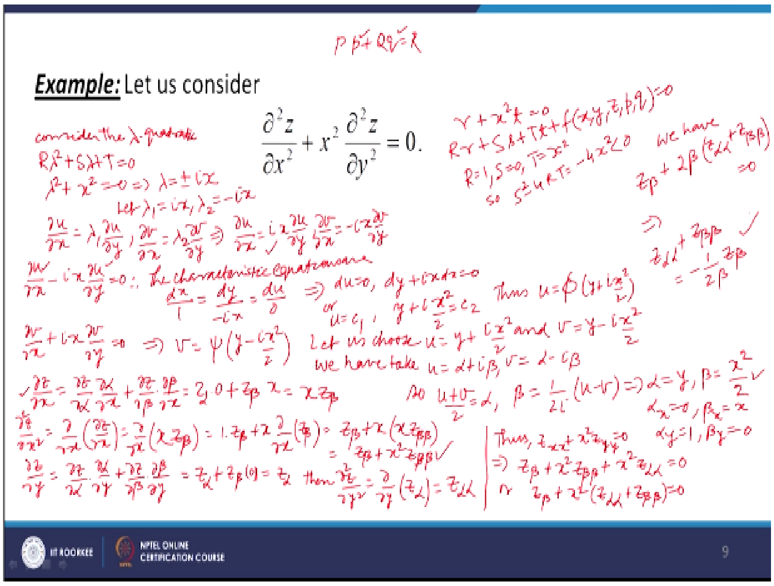


Now let us consider this problem and we will reduce it to the canonical form so this is a second order partial differential equation we have here r+x square * t = 0 so let us compare it will Rr + Ss + Tt + f xyz pq = 0 if we compare it with this what we get, R = 1, S = 0 and T = x square. So S square – 4 RT, S is 0 so this is 0 -4x square we get which is less than 0. So this partial differential equation belongs to the type 3, where S square – 4 RT is < 0.

So there the roots are complex conjugates, so lambda 1 and lambda 2 are complex conjugates. Let us find lambda 1 and lambda 2, so consider the lambda quadratic R lambda square + S lambda + T = 0. So R =1 so we get lambda square, S = 0, T is x square so lambda square + x square = 0 so lambda = +/- ix, so we can see that lambda 1 is ix, okay, so let lambda 1 be ix and lambda 2 be –ix.

So the roots lambda 1 and lambda 2 of the lambda quadratic are both complex so what we get is we proceed in this manner, delta u/delta x = lambda 1 delta u/delta y and delta v/delta x = lambda 2 * delta v/delta y. So they will give you delta u/delta x = i1 is ix, lambda 1 is ix. So ix * delta u/delta y and delta v/delta x = -ix delta v/delta y. Now so let us take this equation, okay, so this can be written as.

Now this is Lagrange's equation of the form Pp + Qq = R, okay where P and Q are this ux and uy okay. So P =1 and Q = -ix R = 0. So the characteristic equations are dx/1 = dy/-ix = du/0 and which imply that du = 0 and dy + ixdx = 0 okay, so du = 0 gives you u = some constant C1 and here what we get is y + i x square/2 = some constant C2. Now C2 is the function of C1 okay, so we can write y+u = thus u = a function of y + ix square/2 okay.

And in a similar manner other characteristic equation is this vx + ix vy = 0 implies that v = instead of –ix now we have +ix so this will get v = psi y-ix2/2. Okay let us choose u to be y + ix square/2 and v to be y - ix square/2, okay, then what will happen alpha = we have considered, we have taken u = alpha + i beta, v = alpha – i beta, okay. So 2 alpha = u+v, so u + v/2 = alpha and 2 I beta okay = u-v.

So beta = 1/2 i * u-v okay, so u+v = 2y okay so this implies alpha = y and beta = u-v, u-v is beta = x square/2, this is u-v = 2yx square/2 so beta = x square/2, okay, now what we have. So let us now convert this equation zxx + x square zyy = 0 2 it is a canonical form, okay, so let us find partial derivative of z with respect to x. So we have okay, so alpha x = 0 alpha y = 1 beta x = x and beta y = 0 okay.

So what do you get, alpha x = 0, so z alpha * 0 z beta * beta x that is x. So we get x * z beta. Let us find second derivative so delta square z/delta x square is delta/delta x of delta z/delta x which gives you x z beta. Okay, this is the product of functions of x so derivative with respect to x when we differentiate x with respect to x we get 1, so 1 * z beta + x times partial derivative of z beta with respect to x okay.

Now partial derivative of z beta with respect to x can be found from here, z is any function of alpha beta so z beta is also function of alpha beta so this will give you when you put z/z beta here we get here delta/delta x of z beta s, x times z beta beta. So what you get z beta + x square z beta beta okay. So this is what we get and then zy.

Let us find zy so delta alpha alpha y + z beta beta y, so this = delta alpha/delta y = 1, okay, so we get z alpha and then z beta y = 0 so 0 z beta * 0. So we get here z alpha only okay now let us find then delta square z/delta y square will be equal to delta/delta y of zy, but zy is z alpha. So z alpha here and when you find this derivate replace z/z alpha here so what do we get, z alpha alpha okay.

So now put the values so then, okay, z xx + x square zyy = 0 gives us zxx is how much, we found zxx here, zxx as this okay, this is zxx, so z beta + x square z beta beta + x square times z yy, zyy is z alpha alpha = 0, okay or z beta + x square times z alpha alpha + z beta beta = 0 and x square = 2 beta, okay, so what do you get, we have z beta + x square is 2 beta, so 2 beta times z alpha alpha + z beta beta = 0 and this gives you z alpha alpha + z beta beta = -1/2 beta z beta.

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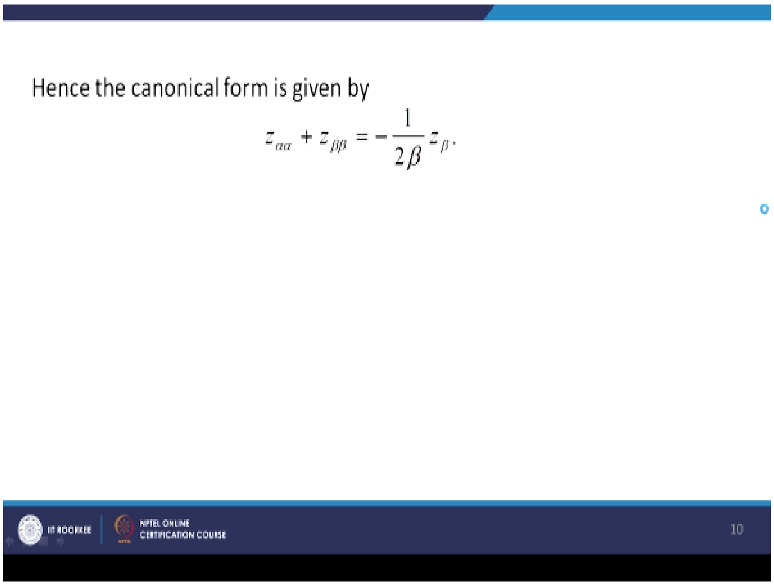


Okay, so this is the required canonical form okay, z alpha alpha + z beta beta = -1/2 beta z beta. That is all in this lecture thank you very much for your attention.