Ordinary and Partial Differential Equations and Applications

Dr. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 42
Second Order PDE with Variable Coefficients

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The second order linear PDEs occur in various physical problems such as the motion of a vibrating string, heat flow, electricity, magnetism and fluid dynamics.

Hello friends, welcome to my lecture on second order PDE with variable coefficients, the second order linear PDE is occur in various physical problems such as the motion of a vibrating string, heat flow, electricity, magnetism and fluid dynamics. Now, let us see some physical problems, where and see how they occur.

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The origin of second order equations:

Let $z = f(u) + g(v) + w$ (1)

where f and g are arbitrary functions of u and v, respectively, and u, v and w are prescribed functions of x and y. Then

$$p = f'(u)u_x + g'(v)v_x + w_x$$

$$q = f'(u)u_y + g'(v)v_y + w_y$$

and hence

$$r = z_{xx} = f''(u)u_x^2 + g''(v)v_x^2 + f'(u)u_{xx} + g'(v)v_{xx} + w_{xx}$$

$$s = z_{xy} = f''(u)u_x u_y + g''(v)v_x v_y + f'(u)u_{xy} + g'(v)v_{xy} + w_{xy}$$

$$t = z_{yy} = f''(u)u_y^2 + g''(v)v_y^2 + f'(u)u_{yy} + g'(v)v_{yy} + w_{yy}$$

So, let us consider the equation, j = fu = gv + w, where f and g are arbitrary functions of u and v respectively and u, v and w are prescribed functions of x and y, Then you differentiate this equation with respect to x partially, so that you get the derivative of z with respect to x, partial derivative of z, with respect to x, which is p, = f dash u * derivative of u with respect to x which is ux + g dash v * vx + wx.

And when we differentiate this equation partially with respect to y, we get the partial derivative of z with respect to y as q = f dash u * uy + g dash v * vy + wy. Now, we can find the second order partial derivatives, so let us differentiate this equation with respect to x partially, so then, the partial derivative of p with respect to x will be r because p is the partial derivative of z with respect to x.

So, partial derivative of q with respect to x will be zxx, so then this is the product of 2 functions of x and y, so we use the product formula for the derivative, so f double dash u * partial derivative of u with respect to x, so you get ux * ux that is ux square, then f dash u * derivative of ux with respect to x, so you get f dash u * uxx, so this is f dash u to uxx and g dash v * vx, when we differentiate with respect to x, what we get?

G double dash v * vx * vx, so vx square and then g dash v * vxx and this wx gives you wxx and similarly, when we differentiate this p with respect to y, what we get? P with respect to y gives zxy, so, s, this is = s. Now, zxy will be you differentiate this equation with respect to y, so f double dash u * uy, so f double dash u * ux uy and then u double dash v * vy, so g double dash v * vx vy and then f dash u * uxy.

So, f dash u * uxy g dash v * vxy and wxy and when we differentiate this equation, q = f dash u * uy + g dash v * vy + wy with respect to y, what we get is; zyy, so this is = zyy, so we get zyy = f double dash u * uy; uy * uy, uy square, g double dash v * vy * vy is vy square and then we get f dash u * uyy, so f dash u * uyy g dash v * vyy and wyy, so this is how we get the values of r, s and t.

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Eliminating the arbitrary quantities f', g', f'' and g'' from these five equations, we get

$$\begin{vmatrix} p-w_x & u_x & v_x & 0 & 0 \\ q-w_y & u_y & v_y & 0 & 0 \\ r-w_{xx} & u_{xx} & v_{xx} & u_x^2 & v_x^2 \\ s-w_{xy} & u_{xy} & v_{xy} & u_x u_y & v_x v_y \\ t-w_{yy} & u_{yy} & v_{yy} & u_y^2 & v_y^2 \end{vmatrix} = 0 \qquad (2)$$

which involves only the derivatives p, q, r, s, t and known functions of x and y. Hence it is a PDE of the second order. Further from (2) we have

Here, we are assuming that zxy = zyx that is the order of differentiation can be interchanged. Now, eliminating the arbitrary constants f dash g dash, f double dash and g double dash from the 5 equations, we have 5 equations here, this is 1 equation, second equation, third equation, fourth equation, fifth equation, so from these 5 equations, let us say, you need the 4 quantities f dash, g dash, f double dash and g double dash here.

We can write these equation as p - wx = this can be written as; p - wx – f dash u - ux – g dash v * vx, okay and similarly q – wy, this equation converting as q - wy = f dash u r – f dash u * uy – g dash v * vy = 0, so we can write these 5 equations in the known arbitrary functions f dash g dash f double dash g double dash eliminating those 4 quantities, f dash g dash, f double dash, g double dash, we arrive at this equation.

P - wx ux vx 00 q-wy uy vy 00 and r – wx, so here the 0, 0 occur because there are no terms containing f double dash and g double dash here in these 2 equations. Here, we have the terms in f dash u, g dash u, f double dash u, g double dash v, so that is why we have here these terms but here these 0's occur because these 2 equations; first 2 equations do not contain f double dash and g double dash.

Now, so this is how we come to this determinant =0, now which this determinant involves only the partial derivatives p, q, r, s and t and known functions of x and y, w is known to us, u is known to us, they are functions of x and y, so their partial derivatives ux, vx, uy, vy, wx, wy, then second order derivatives, uxx, vxx, uyy, vyy, wxx, wxy, uxy, vxy are all known to us. So, what we do is; we expand this determinant by the first column, okay.

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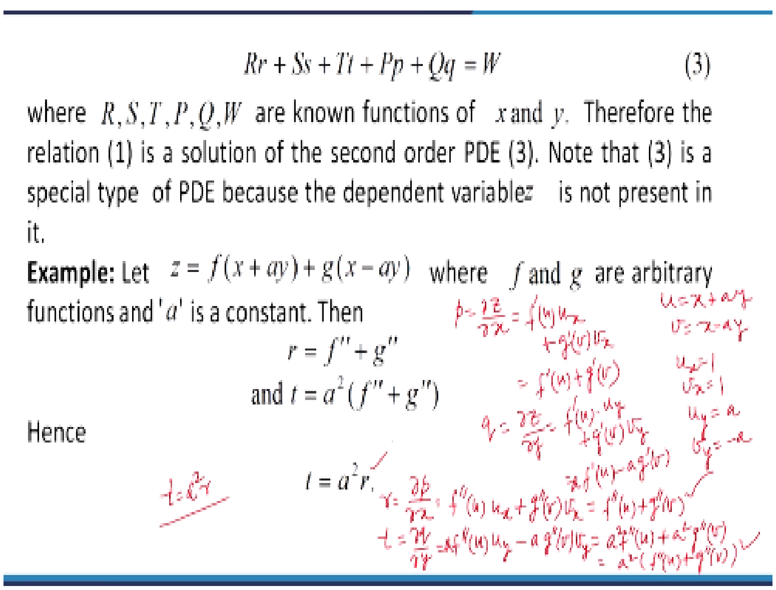

So, then what will happen is that when we expand this determinant by the first column, we will have this equation; Rr + Ss+ Tt + Pp + Qq = W, where, R, S, T, P, Q, and W, they are known functions of x and y and therefore, the relation one is just solution of the second order PDE 3, this is the PDE 3 and this is; this is you can see, this is second order partial differential equation because R is second order derivative of z with respect to S.

And S is second order partial derivative of Z with respect to Y and T is second order derivative of z with respect to y and p is partial derivative of z with respect to x, first order q is partial derivative of z with respect to y of first order, so this is the second order partial differential equation and it has been derived from z = fu + gv = w, so we can say that this relation, z = fu + gv + w is a solution of this second order PDE.

Now this PDE is of a particular type because the dependent variable z is not present in this equation, the R, S, T, P, Q, W, they are functions of x and y and rest are partial derivatives are r, s, t, p, q, so this is a partial differential equation of second order of a special type because the dependent variable z is not present in it. Now, let us take an example say, suppose we have this relation, z = f of x + y + g of x – ay.

Where f and g are arbitrary functions and a is some given constant, so then what we do is; let us try to find second order partial differential equation from here, we shall see, we will get a second order partial differential equation when we eliminate the arbitrary functions f and g, so let us differentiate this equation with respect to x, then we get, let me take u = x + ay and v = x – ay, so then, when we differentiate this relation with respect to x, w get f dash u * ux + g dash v * vx.

And this is = f dash u, now, u = x + ay, so ux = 1 and v = x- ay, so vx = 1, so we get f dash u + g dash v, now let us differentiate this equation, z = fx + ay + gx - ay with respect to y, so q = partial derivative of z with respect to y and we get f dash u * uy + g dash v * vy and what we get? Uy =; now u = x + ay, so uy = a, vy = -a, so we get here, f dash u - a times g dash v, okay, now what we do is; let us differentiate this equation, p = f dash u + g dash v once more with respect to x.

But, we get partial derivative of p with respect to x will be = f double dash u * ux + g double dash v * vx and we know, ux, vx are 1 each, so f double dash u = g double dash v, okay and when we differentiate q = f dash u – a times g dash v once more with respect to y, what we get; and we know this is = r; r = zxx, so partial derivative of p with respect to x and t is partial derivative of q with respect to y and this is = f double dash u * uy – a times g double dash v * vy.

Remember a is constant, so now here uy = a, okay and vy = -a, so we get a times f double dash u vy = -a, so we get a square g double dash, here when we differentiated q with respect to; here when we differentiated z with respect to y, we got f dash u * uy and uy =; I have not written a here, so this is a square, okay, so a square f double dash u + a square g double dash v and this is = a square times f double dash u + g double dash v.

Now, from this equation and this equation, this one, okay, we can eliminate the arbitrary functions f double dash and g double dash, r = f double dash + g double dash t = a square * f double dash + g double dash, so we can say that t = a square * r and we get a second order partial differential equation from the given equation; z = f of x + ay + g of x – ay.

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Similarly, if

$$z = \sum_{r=1}^{n} f_r(v_r),$$

where the functions f_r are arbitrary and v_r are known, we get a linear PDE of n^{th} order.

Second order equations in physics: Now we shall study how second order PDE arise in mathematical physics.

Let us consider the flow of electricity in a long insulated cable. Assuming the flow to be one dimensional, the potential drop in a linear element of length δx situated at the point x is given by

$$-\delta E = iR\,\delta x + L\,\delta x\frac{\partial i}{\partial t} \qquad (4)$$

Now, similarly if we have an equation, which involves n arbitrary functions; z = sigma r = 1 to n fr vr; fr is a function of vr, for r = 1 to n, so where the functions fr are arbitrary and vr are known functions, then we get a linear partial differential equation of nth other, we can follow the same procedure as we have done for the equation, z = f of x + ay + g of x – ay, we have 2 arbitrary functions of f and g and they are functions of x + ay and x – ay.

Now, second order equation in physics, let us see some physical phenomena and see how these second other equations originate in that, so suppose we have a long insulated cable and we are considering the flow of electricity in that, so assuming that the flow is one dimensional, the potential drop in a linear element of length delta x, okay. Suppose, you have a long insulated cable, okay you take an element delta nx in this, okay at the point x.

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where R is the series resistance per unit length and L is the inductance per unit length. If there is a capacitance C and a conductance G per unit length, then

$$-\delta i = GE\,\delta x + C\,\delta x\frac{\partial E}{\partial t} \qquad (5)$$

The equations (4) and (5) imply that

$$\frac{\partial E}{\partial x} + Ri + L\frac{\partial i}{\partial t} = 0 \qquad (6)$$

$$\frac{\partial i}{\partial x} + GE + C\frac{\partial E}{\partial t} = 0 \qquad (7)$$

So, then the potential drop okay, will be =; where r is the resistance in the cable, i is the current and l is the inductance per unit length, so then what will happen? We will get the; e is the voltage, so potential drop will be given by – delta e = ir * delta x + L delta x * delta i/ delta t and if there is a capacitance c and the conductors g per unit length then we have this equation – delta e = g delta x + c delta x * delta ei delta t.

Now, from the equations 4 and 5, what we do is; we divide the equation 4 by delta x and let delta x go to 0, then what we have; let delta t go to 0 then delta x also goes to 0, we have – delta e/ delta x, okay = IR + L delta i/ delta t, so that is this equation; delta e/ delta x ri + l delta i/ delta t and this equation will also be divide by delta x and let delta t goes to 0, so that delta x is also goes to 0, so we have delta i/ delta x + g c times delta e/ delta t.

Now, from the equations 4 and 5, okay, we have the partial derivative of e with respect to x + Ri + l times partial derivative of I with respect to t = 0 and partial derivative of I with respect to x + g + c times partial derivative of e with respect to t = 0, we can easily get these equations by dividing the fourth equation by delta x and letting delta t go to 0, one delta t goes to, delta x is also goes to 0 and so we have the equation number 6.

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From (6) we find

$$\frac{\partial^2 E}{\partial x^2} + R\frac{\partial i}{\partial x} + L\frac{\partial^2 i}{\partial x \partial t} = 0 \qquad (8)$$

$$\frac{\partial^2 i}{\partial t \partial x} + G\frac{\partial E}{\partial t} + C\frac{\partial^2 E}{\partial t^2} = 0 \qquad (9)$$

Eliminating $\frac{\partial i}{\partial x}$ and $\frac{\partial^2 i}{\partial t \partial x}$ from (7) ,(8) and (9) we obtain

$$\frac{\partial^2 E}{\partial x^2} = LC\frac{\partial^2 E}{\partial t^2} + (RC + LG)\frac{\partial E}{\partial t} + RGE \qquad (10)$$

Similarly, differentiating (6) with respect to t and (7) with respect to x, we get

And similarly, in the equation number 5, we divide by delta x and let delta t goes to 0, so that delta x is also goes to 0 and v arrive at the seventh equation. Now from 6, we can see that when we differentiate this sixth equation again with respect to x partially, then we have second derivative of e with respect to x + r times partial derivative of I with respect to x + L times partial derivative of I with respect to x and t.

And then this equation when we differentiate with respect to t, 7th equation we get partial derivative second order derivative of i with respect to t and x + g partial derivative of e with respect to t = c times ett that is delta square e/ delta t square = 0. Now, let us eliminate partial derivatives of I with respect to 2x and the second order partial derivative of I with respect to t and x from the equations 7, 8 and 9, okay.

In the seventh equation, we have partial derivative of I with respect to x and in the eighth equation, we have partial derivative of I with respect to x and second order partial derivative of I with respect to x and t and in the ninth equation also we have second order derivative of I with respect to x and t, so let us eliminate partial derivatives of I with respect to x and second order derivative of I with respect to x and t, from 7, 8 and 9, we will arrive at this equation which is a second order equation in the variable e.

So, second order derivative of e with respect to x square will be = lc ett = rc + lg et + rg, similarly you can differentiate 6 with respect t0 t, okay, when we differentiate 6 with respect to t and 7 with respect to x, okay, we differentiate 6 with respect to t and 7 with respect to x, we have this equation.

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$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (RC + LG)\frac{\partial i}{\partial t} + RGi, \qquad (11)$$

on eliminating $\frac{\partial^2 E}{\partial x \partial t}$ and $\frac{\partial E}{\partial x}$.

Equations (10) and (11) are called telegraphy equations. In the Special case where the leakage to the ground is small i.e., G is negligible and the frequencies are small i.e., L is negligible (for instance, a submarine cable) then the equations (10) and (11) reduce to

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{k}\frac{\partial E}{\partial t} \text{ and } \frac{\partial^2 i}{\partial x^2} = \frac{1}{k}\frac{\partial i}{\partial t}$$

where $k = (RC)^{-1}$.

Delta square I over delta x square = lc delta square i/ delta t square = rc + lg delta i/ delta t + rg I, when we eliminate the delta square e over delta x delta t and delta e over delta x, the procedure is the same; exactly same as we have done in this case to get the equation for the voltage E, we can get the equation for the current on eliminating the second order derivative delta square e/ delta x delta t and the first order derivative delta e/ delta x.

Now, the equations 10 and 11 are the same, the equation 10 and the equation 11, this one are called the telegraphy equations, in these special case, where the leakage to the ground is small that is g is negligible and the frequency are small that is l is negligible, for instance in the case of a submarine cable, then the equations 10 and 11, so taking g and l to be 0, these equations 10 and 11 reduced to these equations.

Delta square e/ delta x square + 1/k delta i/ delta t from this equations, let us put g and l =0; g if you put 0 and l = 0, so this term will vanish okay and this term will vanish, we will have delta square e / delta x square = Rc delta i/ delta t and in this equation, when you put g = 0 and l =0, what you get? This term vanishes, okay and g0 this term vanishes and this term vanishes, so we get delta square i/ delta x square = Rc times delta i/ delta t.

Now, let us take k to be; r and c are constant, so let us take k to be 1 over rc, when we take k = 1 over rc, then we get these equations; delta square e/ delta x square = 1/k delta e/delta t delta square i/ delta x square = 1/k = delta i/ delta t and these are referred to as one dimensional diffusion equations.

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These equations are referred to as one dimensional diffusion equations.
In the case where we deal with high frequency phenomena in a cable, G and R will be negligible and so (10) and (11) will reduce to

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} \text{ and } \frac{\partial^2 i}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 i}{\partial t^2}$$ — Radio equations

where $c = (LC)^{-1}$. These equations are referred to as one dimensional wave equations.

Poisson equation: Now let us derive a simple PDE of second order known as Poisson equation for the gravitational potential.
Let $\vec{F}$ be the gravitational field due to a point mass. Then from Newton's law of gravitation

So, we call them as; these 2 equations as one dimensional diffusion equations, now in the case where we deal with frequency; high frequency phenomena, for example in the case of radio cable, okay g and r will be negligible and so taking g and r to be 0 in these equations, if you take g = 0, r = 0, so this term will vanish, this term will vanish and we will get exx = lc * ett and here, what we will get?

G and r = 0, so again this term vanishes, this term vanishes and we will get ixx = lc * itt, now l and c are constants, so taking 1/lc = small c, we find these equations; exx = 1/ c square ett and ixx = 1/c square itt, these are referred to as one dimensional of wave equations and also called as radio equations. Now, let us take one more example where we are considering the Poisson equation for the gravitational potential.

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$$\vec{F}(r) = -\frac{GM\vec{e}_r}{r^2},$$

where $\vec{e}_r$ is a radial unit vector, r is the radial distance, and M is the mass of a point located at the origin.

Now let us suppose a spherical surface ∂V of radius r centred at the point mass M. Then the total flux of the gravitational field $\vec{F}$ over the closed surface ∂V is

$$\oint_{\partial V} \vec{F}.d\vec{s} = \oint_{\partial V} -\frac{GM}{r^2}\vec{e}_r.d\vec{s}$$

$$= -\frac{GM}{r^2}\oint_{\partial V} \vec{e}_r \cdot \vec{e}_r ds$$

$$= -\frac{GM}{r^2}\oint_{\partial V} ds = -\frac{GM}{r^2}(4\pi r^2) = -4\pi GM.$$

So, let us derive a simple partial differential equation of second order known as Poisson equation for the gravitational potential, let f be the gravitational potential due to a point mass, then from Newton's law of gravitation; f = - GM ER over r square, where ER is radial unit vector and r is the radial distance and M is the point mass, okay, mass of a point located at the origin. Now, let us suppose, a spherical surface delta V of radius r centred at the point mass M.

Then the total flux of the gravitational potential; gravitational field F over the closed surface del V is this integral, the integral over del V F. ds which is = integral over del V, F = - GM ER/ r square * .ds, so –GM over r square is a constant, we can write it outside the integral, so –GM over r square, the integral over del V ER dot ER ds; ER is the vector normal to element ds, now ER dot ER = 1, so we have – GM/ r square the integral over del V ds and this is a surface integral.

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Applying Divergence theorem

$$\oint_{\partial V} \vec{F} \cdot d\vec{S} = \int_V \nabla \cdot \vec{F} dV,$$

we get $$\int_V \nabla \cdot \vec{F} dV = -4\pi GM$$

Since $\vec{F} = -\nabla \phi$ and $M = \int_V \rho dV$ where ρ is mass density at each point we get

$$(\nabla^2 \phi)\frac{M}{\rho} = -4\pi GM$$

$$\nabla^2 \phi + 4\pi G\rho = 0$$

which is Poisson's equation.

$$\nabla \cdot \vec{F} = \nabla(-\nabla\phi) = -\nabla^2\phi$$

$$\int_V \nabla \vec{F} dv = \int_V (-\nabla^2\phi) dv = -\nabla^2\phi \cdot \int_V dv = -\nabla^2\phi \frac{M}{\rho}$$

So, the area; surface area of the; spherical surface is of radius R is 4 phi r square, so –GM over r square * 4 phi r square and we therefore have -4 phi * GM. Now, what we do is; let us apply the divergence theorem; the divergence theorem is a relation between the volume integral and surface integral, so, what we have here, the integral; this one; this integral over the spherical surface f.ds can be converted to the volume integral, divergence of fdv, okay.

And divergence of integral over fv, divergence of fdv = -; okay, so this integral we have already found, this surface integral we have already got, this is -4 phi GM, so this volume integral becomes -4 phi GM. Now, f = - grad phi and so M = integral over v rho dv, where rho is the mass density at each point, then we get rho square phi * M/ rho, we can see here this is integral over v del dot f dv, okay = integral over v; del dot f = - del square phi.

Del dot f = del dot = del phi, so what we get; - del square phi, so we will get here; - del square phi * dv and this will be = - del square phi * M over; so integral over v dv and this is volume, so - del square phi, this volume is = M over rho, so we get – M del square phi * M/ rho = - 4 phi GM and so, we get del square phi = 4 phi 0 which is the Poisson's equation.

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At points external to the distribution of mass, the net flux of the gravitational field is zero. Hence

$$\nabla^2 \phi = 0,$$

which is known as the Laplace equation.

So, now at points external to the distribution of mass, we can imagine an empty space centred at these points through which the net flux of the gravitational field is 0, hence we will have del square phi =0, net flux is 0 means, we will have 4 phi GM = 0, so we will get del square phi = 0 and del square phi = 0 is called as the Laplace equation. With that, I would like to end this lecture, thank you very much for your attention.