Ordinary and Partial Differential Equations and Applications
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Lecture – 39
Compatible System of First Order Equations

Hello friends, welcome to this lecture, in this lecture we will discuss the concept of compatible system of first order partial differential equation and this concept is very, very useful in finding the so called very important method of Charpit method to solve; to find the complete solution of first order nonlinear PDE.

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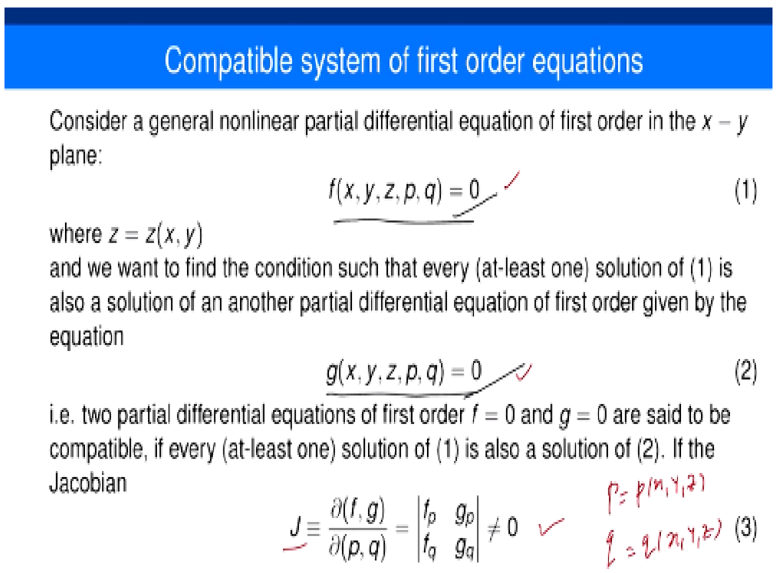

So, first let us start with the compatible system, so here you consider a general nonlinear partial differential equation of first order in xy plane that is f of x, y, z, p, q = 0, here we are assuming that z is a function of 2 independent variables that is x, y, so where z is = z of x, y and we want to find out the condition such that every solution of 1 is also a solution of another partial differential equation of first order given by the equation, g of x, y, z, p, q = 0.

So, means what we want to find out? We want to find out a conditions under which every solution of this equation is also satisfying the solution here and in this case, when we say that every solution of first equation is a solution of second equation, we call these 2 equations are compatible equations are we say that f and g are compatible to each other. Actually, here in the definition of compatibility, there are several literatures in which you find out this definition.

That compatible means a compatible partial differential equation mean that every solution is common but in some book, you will find out that compatibility means that they have at least one solution in common, so here, I will also prefer to have this definition that at least, they have at least one common solution, so it means that we may say that the partial differential equation f of x, y, z, p, q = 0 and the partial differential equation g x, y, z, p, q = 0 is compatible to each other if they have at least one solution in common.

And once, we say that they have one solution in common, then we just want to see that how we can find out that one that common solution, for that we assume that the Jacobian of p and q, so here we assume that if the Jacobian; J = dou of g upon pq which is given as fp gp fq gq is nonzero, so here to find out that common solution, we assume that the quantity J, which is nothing but Jacobian of fg with respect to pq is nonzero.

Actually, this condition enables us to find out this equation number 1 and 2 for p and q in terms of say, x, y and z. Similarly, we can write q as function of x y and z, so it means that to find out the common solution, we need to find out the expression for p and q from these 2 equations and once we have these expressions for p and q.

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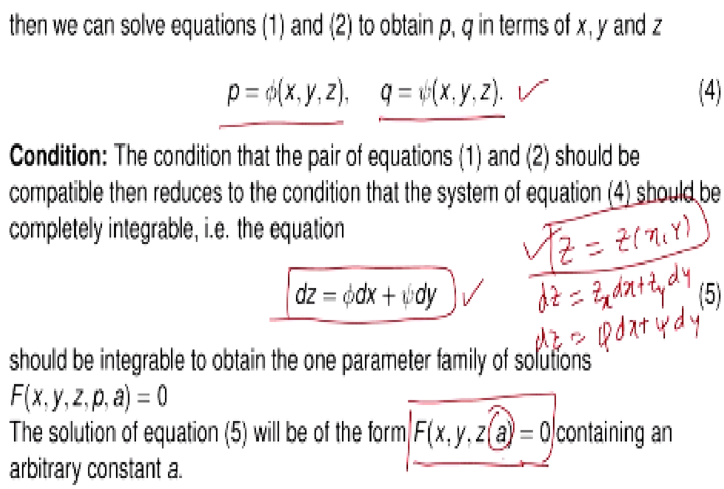
then we can solve equations (1) and (2) to obtain p, q in terms of x, y and z

$$p = \phi(x, y, z), \quad q = \psi(x, y, z). \tag{4}$$

Condition: The condition that the pair of equations (1) and (2) should be compatible then reduces to the condition that the system of equation (4) should be completely integrable, i.e. the equation

$$dz = \phi dx + \psi dy \tag{5}$$

should be integrable to obtain the one parameter family of solutions $F(x, y, z, p, a) = 0$

The solution of equation (5) will be of the form $F(x, y, z, a) = 0$ containing an arbitrary constant a.

Then, we say that we can solve equation 1 and 2 to obtain p , q in terms of x, y, z and let us say that p; expression for p is = phi of x, y, z and q is = psi of x, y, z. Now, the condition that the pair of equation 1 and 2 should be compatible then reduces to the condition that the system of equation 4 should be completely integrable means, the equation dz = phi dx + psi dy should be integrable.

And once we say that is integrable, then we can find out one parameter family of solution given as f of x, y, z, p, a = 0, so this condition number 5, we can obtain from this that if z = z x y, then you can write dz as zx dx + zy dy and zx we are denoting as p, which is given as here, so we can write phi dx + psi dy, so if we can integrate then we can obtain our integral surface like z = z of xy.

So, what we want is that once we are able to solve equation number 1 and 2, then once we able to solve 1 and 2 in terms of p and q and after obtaining the expression for p and q, we put in this equation number 5 and this if we able to integrate this to find out our solution z = zxy, then we say that we are able to find out one common solution and the solution of equation 5 will be the form of f of xyz = a and that will contain an arbitrary constant a.

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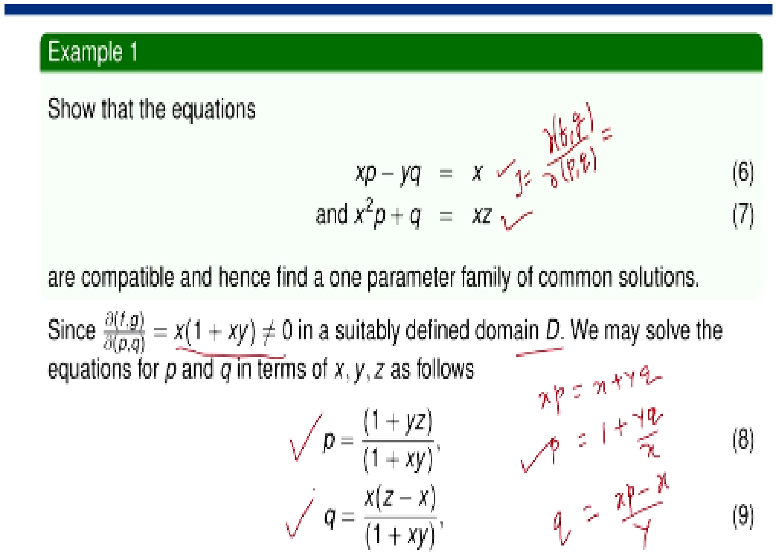

So, we say that this kind of a solution is one parameter family of solution here, so here let us take one example, so example 1 show that the equation xp – yq = x and x square p + q = xz are compatible to each other and hence find a one parameter family of common solution. So, first thing we want to show that if they are compatible to each other, so they must have at least one solution in common and let us find out that they have one solution in common or not.

So, for that first you will find out the expression for dou of g and if this quantity is nonzero then only we can proceed or we can say that then only we can say that 6 and 7 can be solved for p and q, so if you calculate this Jacobian, J = dou of g upon dou of pq and it is we have already calculated and the value is coming to be x * 1 + xy and we say that this will be a nonzero in a suitably defined domain D, so let us defined a domain D such that x * 1 + xy is nonzero.

In that domain, we can solve our equation number 6 and 7 for p and q, so for example I can use this xp – yq = x and we can simply say that xp is = x + yq and p is = say, 1 + yq upon x, so once we have p, then you can write down this as; you can use you, q here, q here is what? Q, I can say that xp - x divided by p here, so now, once we have the value of p here, you use equation number 7 to get the value of q.

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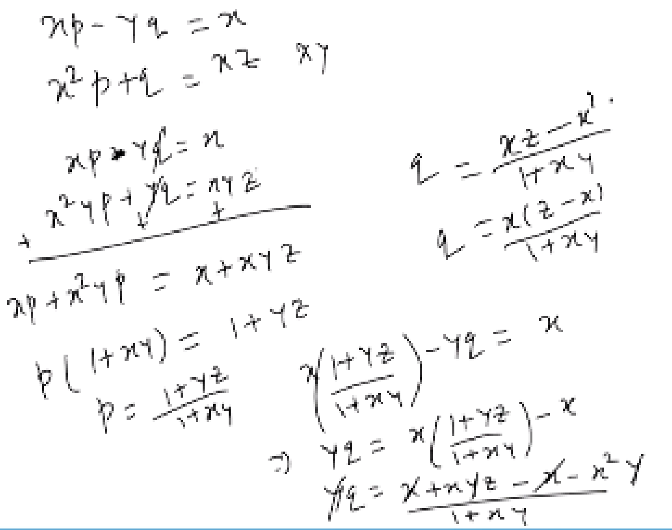

And once we have q, then you can solve equation number, we can have the value of p and q like this, so let us see whether we are able to solve this equation or not, so let us say that xp - yq here, so xp – yq = x here and x square p + q is = say, xz, so here we have this, so what we can do; here, we can multiply by y and what you will have? Xp + yq = x and we have x square yp + yq = xyz, this is minus sign, okay.

So, now add these 2, so what you will get here? Here, we have xp + x square yp and these will cancel out each other = x + xyz, so we can take it x common here, so we have p 1 + xy and here it is 1 + yz, so we can write p as 1 + yz upon 1 + xy, so that is what we have written here 1 + yz upon 1 + xy. So, once we have p, then you can find out q like this, x * 1 + yz and divided by 1 + xy – yq = x.

So, we can say that yq = x1 + yz divided by 1 + xy – x, so if we simplify this, you will get x + xyz divided by 1 + xy – x - square y here, so this will be cancel out here and here we have yq, so now if we take y common, then what we will get? This y will be cancelled out here and here, so what you will get? Here, you will get xz, here, I think, it is so x - x square y here, so what we will get; xz - x square upon 1 + xy.

So, here we can take out x z -x upon 1 + xy, so q =; so here, q = x * z – x; x * z - x upon 1 + xy, so which we have written here. So, what we have achieved here that since Jacobian is nonzero, so we solve equation number 6 and 7 for p and q, which we have given that how we can solve for p and q.

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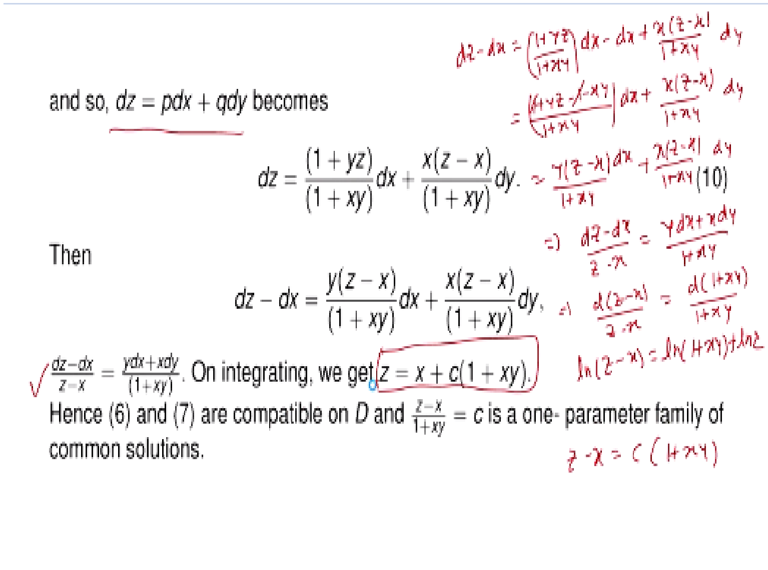

So, once we have p and 1, then our next aim is to solve this equation dz = pdx + qdy and we; if we see that if this is integrable, than we can say that the equation number 6 and 7 are compatible to each other, so you just put the value of p and q, then it is dz = 1 + yz upon 1 + xy dx + x * z = x upon 1 + xy dy and we want to see whether it is integrable or not. So, here you have to apply your experience and you say that here if you look at it is x * z – x.

And here, in dx, we have yz here and here also we have xz here, so we can simplify this and we can write this that if you subtract dx from dz, then we can write dz – dx =; so, let me write it here, dz - dx = 1 + yz upon 1 + xy dx – dx + x z – x divided by 1 + xy dy and if you simplify it is 1 + yz -1 - xy divided by 1 + xy dx + x * z - x upon 1 + xy dy and if you solve this, what you will get?

Here, you can take y common, so yz - x d of x divided by 1 + xy + x z - x 1 + xy dy, now you can divide by z – x, so here we have dz - dx divided by z - x and what you will have here is ydx + x dy divided by 1 + xy, so here you got this that dz - dx upon z – x = ydx + xdy upon 1 + xy. Now, you can integrate this directly in fact, you can say that it is what; d of z - x upon z - x and here you can write this as d of 1 + xy divided by 1 + xy.

Now, when you integrate it, then you will get what? So, ln of z – x = ln of 1 + xy + some constant, let us take this as ln c or whatever, so here you can say that z –x = c times 1 + xy, so that is what we have written here that z = x + c times 1 + xy and we say that since we are able to integrate this dz = pdx + qdy, then we integrate means that we are able to find out the expression of z in terms of xy in volume 1 parameters.

So, we say that this is a common solution of both the equation and we say that this is a one parameter family of common solution and since they have one parameter family of common solution, so we say that our equation f and g, which we have the written here as xp - yq = x and x square p + q = xz are compatible to each other because they have one family; one parameter family of solution in common.

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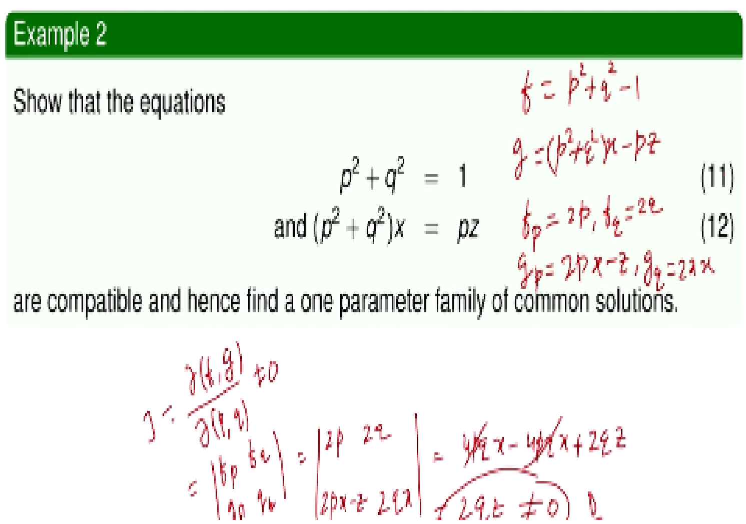

Similarly, let us consider one more example, show that the equation p square + q square = 1 and p square + q square x = pz are compatible to each other and then we want to find out a parameter family of common solution. So, here I asked you to provide that Jacobian that is dou fg upon dou pq is nonzero, so here you define f as p square + q square -1 and g, you can define as p square + q square x – pz.

So, here we have to calculate this quantity, so it is what? It is fp fq gp gq, so here you can calculate f of p; f of p is 2p, f of a is = 2q and g of p is = here, you can say that it is 2px - z and gq is = 2qx that is all, so we can say that this expression is going to be what? It is 2p and it is 2q here it is 2px – z, here we have 2qx here, so gq is only 2qx, so when you simplify this, it is what? 4pqx – 4pqx – 2qz, so that its + here.

So, 4pqx – 4pqx + 2qz, so we can say that these are cancel out, so we have 2qz is a value, so we say that this is not identically equal to 0, so let us define a domain in a way such that this quantity will remain nonzero, so we say that in this domain D, we say that Jacobian is nonzero, so it means that we can solve equation number 11 and 12 in terms of p and q as a function of x, y and z.

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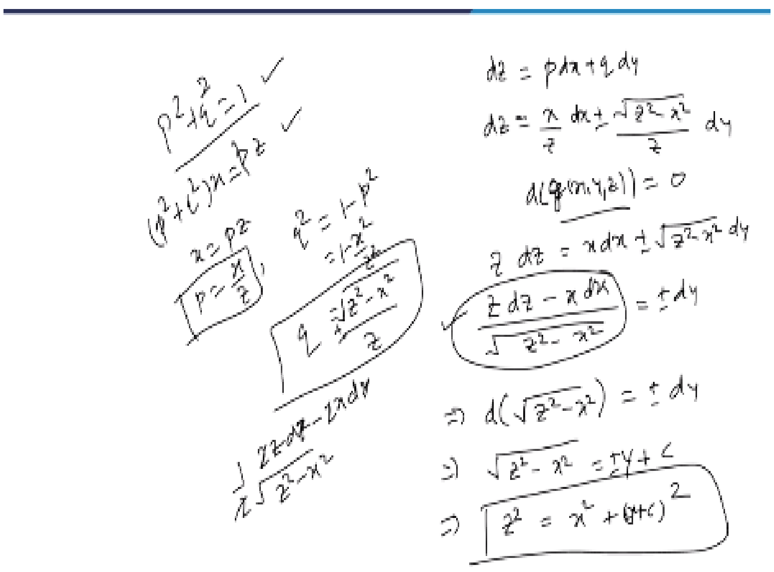

So, here let us solve this; let me write it here, here we have p square + q square = 1 and here we have p square q square x = let me write it here it is pz, so here we have pz, so if you use equation number 1, then we can write x = pz, so we can write p as x, y, z here, so p is x/z, then you can find out the value of q from 1, so we can write q square = 1 – p square that is 1 - x square upon z square or you can say that z square - x square under root divided by z, it is the value of q.

So, q is coming out to be under root z square - x square upon z and p is coming out to be xyz, here you can say that we have 2 value of q that is +- this thing, so once we have p and q value here, now let us see that dz = pdx + qdy, is all are integrable or not, so let us put the value of p that is x/z dx + q is under root z square – x square divided by z dy = dz. Now, here we want to show that this equation is exactly integrable or we can say that this is written as d of z.

So, we want to say that it is written as g xyz = some 0, so we want to find out this expression, g xyz = 0, so for that you see that here you can write down z dz = x dx + under root z square - x square dy, so from you can see that I can write these z dz – x dx divided by z square - x square = dy and if you again you see that it is what? I can write that it is under root z square - x square = dy, so because if you integrate this, then what you will get?

You will get; if you differentiate this, what you will get? 1 upon 2 under root z square - x square and when you differentiate this, it is 2z dz – 2x dx, so 2, 2 will be cancel out and we have this expression, z dz – x dx divided by z square – x square, so this is written as d of under root z square – x square = dy. When you integrate, then what you will get? It is z square - x square = y + some integration constant that is c.

So, we can write here z square = x square + y + c whole square, so that is the solution you can say that you can write down the one parameter family of common solution like this, so we have obtained a solution z square = x square + y + c whole square and please note down here that I have taken only the plus value that is q = under root z square – x square upon z, so here I have taken only plus value.

If we can take the negative sign, I can write it here, z + - here, so here I will write +- here and here you can write +- here, so you can write +- here, so that is the only change when you consider the both the sign, so we can say that here, we are able to find out one parameter family of common solution of the equation p square + q square = 1 and p square + q square x = pz, so what we have achieved here, by having one solution in common, we say that these 2 equations are compatible to each other.

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Remark 1

For the compatibility of two pdes of first order $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$, it is not necessary that every solution of $f(x, y, z, p, q) = 0$ be a solution of $g(x, y, z, p, q) = 0$ or vice- versa.

As, in the example (1), $z = x(y + 1)$ is a solution of (6) but not of (7).

Similarly, in example (2), $z = \frac{(x+y)}{\sqrt{2}}$ is a solution of (11) but not of (12).

And the common solution is given by this, so now move back to our problem, now here we wanted to give one and very important remark because what we have given as a definition of compatible system that every solution of first PDE is a solution of second PDE and vice versa but here in both the example which we have discussed, it is not happening in fact, if you look at in the first problem, we can verify that in first problem what is this?

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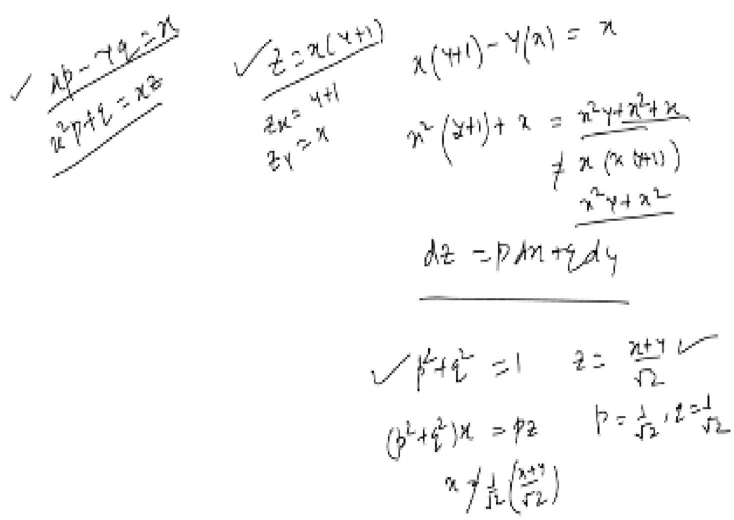

Let us look at the first problem, first problem is this; xp – yq = x, let me write it here, this is xp – yq = x and the other equation is x square + p x square p + q = xz, right, so we want to see that here this x = y, z = x y + 1 is a solution of first but not the second, let us see that z = x y + 1, so if we see that this will satisfy the first equation but not the second, you can see that here zx is = y +1 and zy is = x.

So, when you put in equation number 1, so x pzx, so p is y + 1 - yq is zy that is x and when you simplify what you will get; is x, so it means that this satisfy this first PDE, so it means that z = xy = 1 is a solution of first but if you look at the second equation; second equation is what? x square, pi what? It is y + 1 + q; q is x here and when you simplify this, it is x square y + x square + x and it is != xz; xz means x y + 1.

So, when you simplify, it is what? x square y + x square, so here we say that it is not equal, so it means that z = x y + 1 is a solution of first PDE but not the second PDE, so it means that though we have shown that they are compatible to each other in the sense that they can be solve for p and q and dz = pdx + qdy is solvable and in fact, integrable and we have found the one parameter family of common solution.

So, we say that our solution; our definition that they are compatible to each other provided that every solution is common may not be say, correct or may not be taken in a precise manner, so similarly we can see one more example. If you look at the second example and we try to see that z = x + y/ root 2 is a solution of first equation but not the second equation, for that look att the equation.

The equation is p square + q square = 1, so let me write it here, p square + q square = 1 but p square + q square x = pz, so here we want to say z = x + y/ root 2, so we say that here if you calculate p is what? It is 1/ root 2, q is what? Q is 1/ root 2 and it satisfy the first equation but not the second equation because if you look at the second equation here, it will be x – 1/root 2 and z is x + y upon root 2.

So, here you can say that these 2 are not equal so, it means that this will satisfy the first equation but not the second equation, so here we have the say, problem in the sense that there are 2 things; first thing is the definition and the second thing is to show that the equations are compatible to each other. So, if we take the definition of compatibility as that every solution of first PDE is a solution of second PDE.

Then, we can say that then the method that they can be solvable for p and q and the corresponding equation dz = pdx + qdy is integrable, we can say that if we take this definition as that every solution is common, then we say that the next thing is a method to find out one solution in common because here, in both the example, what we have shown is that these 2 equations in both the example 1 and example 2, they have one solution in common.

So, if we take that definition as that every solution in common, so we simply say that they are not compatible to each other yet they are having one solution in common, so it means that either we take the definition as that every solution in common and the procedure to find out one solution; one common solution or we simply say that the compatibility means, they have one solution in common.

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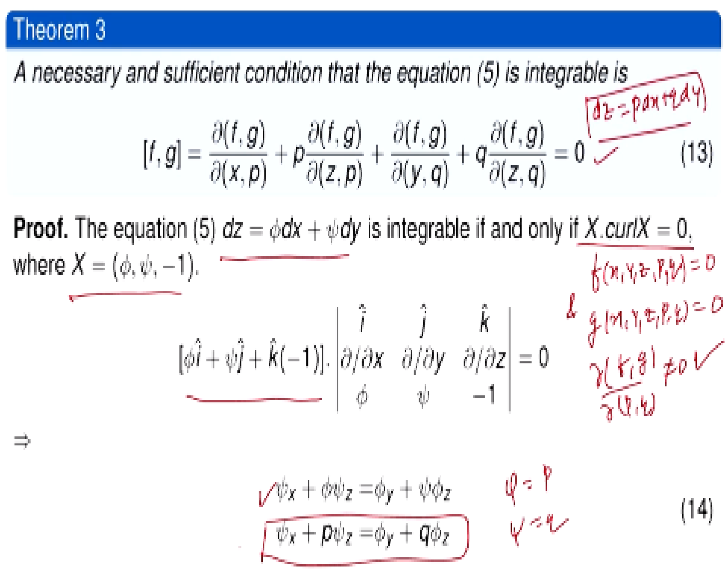

So, here we can have say, debate on this thing, so now let us move next because what we have shown here, we have shown that given in PDE, pf = f xyz pq = 0 and g x y z p q = 0, what we have shown that they are compatible to each other provided that we can solve f and g in terms of pq as a function of xyz and dz = pdx + qdy is = is integrable but many times it is not very easy to see that how we can solve.

So, is there any condition available by which without solving the 2 PDE's can we say that our equations are compatible to each other, it means that it is equivalent to say that we can solve f and g for p and q and dz = pdx + qdy is integrable. So, now we want to find out the condition in which we simply use the equation of PDE and by saying; by using the equation of PDE only, we say that these 2 equations are compatible to each other or not.

So, in theorem 3, we will give the necessary and sufficient condition that the 2 equations are compatible and the corresponding dz = pdx + qdy is integrable, so here this condition is to give that dz = pdx + qdy is integrable, so basically you can consider this as that it is a integrability condition provided that Jacobian is nonzero. So, let us say that look at the equation number 5, so we say that a necessary and sufficient condition that the equation 5 is integrable is that this bracket fg is = 0.

Where fg is written as a Jacobian of fg with respect to xp + p times Jacobian of fg with respect to zp + Jacobian of fg with respect to yq + q times Jacobian of fg with respect to zq and if you look at this expression that should come out to be 0, if this value is coming to be 0, then we say that this dz = PDX plus cutie why pdx + qdy is integrable and we can say that f x y z p q is = 0 and g x y z p q are compatible to each other provided that Jacobian of fg with respect to p and q is nonzero.

So, this condition along with this condition will simply say that this f and g are compatible to each other or we can say that they have at least one solution in common, so here we use one condition that dz = phi dx + psi dy is integrable if and only if X.curlX is = 0, so this is the result which we are using and you can find out this result in a study of a differential equation and you can see the book of (()) (29:37) to get more information about this condition that X.curl X = 0.

Where X is phi, psi and -1, so let us calculate the quantity that X.curl X = 0, so here X.curl X means is X phi i + psi j+ k * -1 * curl X; let us find out the curl X, this is I j k, dou/ dou x, dou/dou y, dou/ dou z phi psi -1 and you if you simplify this, this is coming to be psi x + phi psi z

= phi y + psi phi z, you can simplify this and you can have this value. Now, psi x + p psi z = phi y + q psi z, here we are just writing phi as p and psi as q.

So, we can write that our integrable condition is now reduced to 14 but if you look at this condition 14, it is still given in terms of p and q, so here if you want a condition in terms of f and g, then we had to get rid of the expression phi and psi here.

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Since $p = \phi(x, y, z, a)$, $q = \psi(x, y, z, a)$, the equation $f(x, y, z, p, q) = 0$ can be written as

$$f(x, y, z, \phi, \psi) = 0 \qquad (15)$$

By taking derivative of equation (15) with respect to x, we have

$$f_x + f_p\phi_x + f_q\psi_x = 0. \qquad (16)$$

Differentiating with respect to z, we get

$$f_z + f_p\phi_z + f_q\psi_z = 0. \qquad (17)$$

Adding ϕ times equation (17) to (16), we get

$$(f_x + \phi f_z) + f_p(\phi_x + \phi\phi_z) + f_q(\psi_x + \phi\psi_z) = 0 \qquad (18)$$

So, for that let us utilise the equation f and g here, so since p = phi x y z a and q = x y z that the equation f x y z p q = 0 can be written as f of x yz phi psi = 0. Now, what we can do here to find out the value here that is psi x + p psi z and phi y + q psi z, what we do? We use the equation f and g to find out this value separately and this value separately and then equate, so that we can eliminate this phi and psi, we can get a condition in terms of f and g purely.

So, for that you simply differentiate with respect to x and we have fx + fp phi x = fq psi x = 0 here, so here we are simply assuming that x y z phi psi are argument to each other and we are simply differentiating with respect to x provided that phi is written as a function of x y z here and psi as a function of x y and z here, so here I am not assuming that z is a function of x, so here we are writing fx + fp phi x + fq psi x = 0.

And then we again differentiate with respect to z here, so fz + fp phi z + fq psi z = 0 and with the help of this, you just multiply this equation number 17 by 5 and we can get this equation number 18. Now, here if you look at the equation number 18 carefully, then here the term which we want it to get the phi x + phi phi z and that is what we have written here, this sorry; this is psi x + phi psi z, so it is psi x + phi psi z.

So, this is the value we can obtain from this, so here we can get this value. Now, here if you see I have; when we differentiate with respect to x, I have taken z as a independent of x y but if you do not take; if we take that z is a function of x and y also then we can write here the following thing f of x y z phi and psi = 0. Now, here let us assume that z is also a function of x y and see what you will get?

So, in this case, we have f of x + f of z, now here we have z of x and here we have f of phi phi of x and here phi is a function of z, so phi of z, z of x + f of psi and then psi of x + psi z and zy zx = 0 and if you simplify what you will get? Here I am writing f of x + zx is p p of z + f of p and here it is what? phi x is, I am writing phi x + phi z and zx is p + f of q psi of x + psi z and this is q zx is p, so = 0.

And if you look at this is the exactly the equation number; fx + p phi z, I am writing here, fp phi x + phi phi z, I am writing p as phi, so fq psi x + phi psi z, so it is exactly the equation number 18, so whether you do it like this that first we will differentiate with respect to x keeping z as a independent variable, z is not depending on x and then you differentiate again with respect to z and then you suitably manage with equation number 16 and 17 to get this equation number 18.

Or you assume that z is a function of xy and you simply differentiate with respect to x and to get this expression that is given as equation number 18 and the same; similar expression is valid for equation number; for equation g x y z phi psi = 0.

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Repeating the same process for g, we have

$$(g_x + \phi g_z) + g_p(\phi_x + \phi\phi_z) + g_q(\psi_x + \phi\psi_z) = 0 \qquad (19)$$

Solving the equations (18) and (19), for $\phi_x + \phi\phi_z$, we have

$$\psi_x + \phi\psi_z = \frac{(f_x g_p - f_p g_x) + \phi(f_z g_p - f_p g_z)}{f_p g_q - f_q g_p} = \frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)} + \frac{\phi\partial(f,g)}{\partial(z,p)}\right\} \qquad (20)$$

where J is the Jacobian defined as

$$J \equiv \frac{\partial(f,g)}{\partial(p,q)} = \begin{vmatrix} f_p & g_p \\ f_q & g_q \end{vmatrix} \neq 0 \qquad (21)$$

So, repeating the same process for g, we have gx + phi gz + gp phi x + phi phi z + gq psi x + phi psi z = 0 and as we said that we want to get expression for this, so what we do; here we multiply by fp and here we multiply by g of p and then subtract, so that this term will be cancel out here. So, when we do that solving the equation 18 and 19 for psi x + phi phi z means that is this thing. What we do here?

We and we can get psi x + phi psi z as follows; fx gp – fp gx + phi fz gp – fp gz divided by fp gq – fq gp here, so here this quantity if you remember this quantity is the Jacobian of fg with respect to p and q, so we are calling this quantity as J, so we can write this as; this quality is what? Jacobian of fg with respect to xp, I am writing here plus this is Jacobian of fg with respect to zp. So, I am writing psi x + phi z is 1 upon Jacobian of fg with respect to xp + 5 times Jacobian of fg with respect to zp.

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This gives an expression for the left hand side of equation (14). In a similar manner, differentiating f and g with respect to y, z and repeating the process, we obtain

$$\psi_y + \phi\psi_z = -\frac{(f_y g_p - f_p g_y) + \psi(f_z g_q - f_q g_z)}{f_p g_q - f_q g_p} = -\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(y,q)} + \psi\frac{\partial(f,g)}{\partial(z,q)}\right\} \quad (22)$$

So, using the expressions given in equations (20) and (22) into equation (14) and replacing ϕ and ψ by p and q respectively, we observe that the two conditions should be compatible such that $[f,g] = 0$, where

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0 \quad (23)$$

So, here we are able to find out the quantity psi x + phi psi z and here Jacobian is; we have utilise that Jacobian is non-zero, so we can divide by Jacobian. Now, this gives an expression for the left hand side of equation number 14 and in a similar way, we can differentiate f and g with respect to y and z or when we assume that z is not a function of xy but we can directly differentiate f s with respect to y, keeping z as a function of y.

Then, we can do the repeat the same procedure and we can get the value of psi y + phi psi z and it is written as - 1 upon Jacobian; - 1 upon Jacobian, Jacobian of fg with respect to yq + phi times Jacobian of fg with respect to zq, I am sorry, it is psi times Jacobian of fg with respect to zq, I am using the equation number 22 and the previous one that is 20 and equation number 14 where we equate these 2, so using this expression for this and this.

And we can write here that using this expression given in equation number 20 and 22 into equation number 14 and replacing phi and psi by p and q respectively, we observed that the 2 conditions should be compatible such that fg is = 0. When you equate these 2, we have the following thing that Jacobian of fg with respect to xp + p times Jacobian of fg with respect to zp + Jacobian of fg with respect to yq + q times Jacobian of fg with respect to zq is = 0.

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Now suppose that conditions (21) and (23) are satisfied. Then we can solve equations (1)and (2) for p and q and these obtained functions satisfy the condition: $p_x = q_y$.

This implies that there exist a function $z(x, y)$ such that $p = z_x$, $q = z_y$ and this function is an integral(common integral) of both the equations (1) and (2).

And this quantity we are taking as bracket fg, so we say that if bracket fg is 0, then f and g are compatible to each other and we say that suppose that equation condition 21and 23 are satisfy, then we can solve equation 1 and 2 for p and q and these obtained functions satisfy the condition that px + qy and this implies that they exist as function z = zxy such that p = zx, q = zy and this function is an integral of both the equation 1 and 2.

So, here the bracket fg = 0 is an integrabiliy condition that dz = pdx + qdy is exactly derivative of a function z = zxy and in this we have already assumed that Jacobian of fg with respect to pq is nonzero, so it means that we can; we are able to solve f and g with respect to p and q in terms of x y z, so that this equation is integrable. So, this is the method of this is basically an integrability condition, which help us to find out one common family of solution.

So, with this and this lecture and in next lecture, we will discuss some example based on this and we try to discuss a very important method that is Charpit method based on this compatibility concept. So, thank you very much, we will continue in next lecture, thank you.